

Thermal & Statistical Physics
Study Questions for the Spring 2022 Department Exam
December 3, 2021

1. Consider an extreme relativistic classical gas (ignore spin) with single particle energy

$$\varepsilon = \sqrt{p^2 c^2 + m^2 c^4} \cong pc$$

where p is momentum and c is the velocity of light.

- a. If this gas is confined to 1-dimension (length L) show that $\varepsilon = nhc/2L$ ($n = 1, 2, 3, \dots$) and then find the partition function Z_1 for a single particle.
 - b. Now find F and show that $PV = NkT$. (4)
- 2.
- a. Define the chemical potential μ . Show that two systems are in diffusive equilibrium if $\mu_1 = \mu_2$. You may start with $F = F_1 + F_2$ (free energy) and use the fact that $\mu_1 = \mu_2$ should minimize F .
 - b. Now consider a 1-d gas of length L and number of distinguishable and non-interacting particles N . Assume that temperature T is high enough that we are in the classical regime. Find the chemical potential μ in terms of T and other parameters of the problem.
 - c. Now find the free energy F and the specific heat at constant volume C_V .
3. A sample of helium gas inside a cylinder terminated with a piston doubles its volume from $V_i = 1\text{m}^3$ to $V_f = 2\text{m}^3$. During this process the pressure and volume are related by $PV^{6/5} = A = \text{constant}$. Assume that the product PV always equals $\frac{2}{3}U$, where U is the internal energy.
- a. What is the change in energy of the gas?
 - b. What is the change in entropy of the gas?
 - c. How much heat was added to or removed from the gas?
4. A system consists of three particles, each of which has three non-degenerate energy states-- 0 , ε and 3ε . Determine the canonical partition function if the particles are
- a. classical particles. (note: classical particles are distinguishable)
 - b. Fermions.
 - c. Bosons.

5. Consider 3-d lattice of N atoms arranged in a box of side L .
- a. Briefly explain why there are only $3N$ independent phonon modes and show that ω_{\max} (Debye frequency) is given by

$$\omega_m = \left(\frac{\pi c}{L} \right) \left(\frac{6N}{\pi} \right)^{1/3}.$$

- b. Show that at high temperatures the internal energy $U = 3Nk_B T$ and that at low temperatures it is proportional to T^4 . You can use the Bose-Einstein distribution to determine the occupation number of the states.
6. Consider two single-particle states and two particles. Find the entropy of this system when the following statistics apply:
- a. Maxwell-Boltzmann (classical)
 b. Fermi-Dirac
 c. Bose-Einstein
 d. Derive an approximate condition relating temperature and density to determine when or when not quantum statistics must be used.

7. Calculate the magnetic susceptibility $\left(\frac{\partial M}{\partial H} \right)_{H=0} = \chi$ (where M is the magnetic moment of the sample and H is the applied field) as a function of T of a gas of N permanent dipoles each of moment μ ,

- a. if any direction is allowed (classical spins).
 b. if the dipole is only allowed to assume 2 directions, parallel and opposite to the applied field (Ising spins).
8. In a mono-atomic crystalline solid each atom can occupy either a regular lattice site or an interstitial site. The energy of an atom at an interstitial site exceeds the energy of an atom at a lattice site by ϵ . Assume that the number of atoms, lattice sites, and interstitial sites are all equal in number.
- a. Calculate the entropy of the crystal in the state where exactly n of the N atoms are at interstitial sites.
 b. What is the temperature of the crystal in this state, if the crystal is at thermal equilibrium?
 c. If $\epsilon = 0.5$ eV and the temperature of the crystal is 273 K, what is the fraction of atoms at interstitial sites?

Hint: You have *two* choices to make in this problem – which atoms to put in interstitial sites, and which interstitial sites to put them in.

9. a. Write down the Boltzmann equation for a gas. Explain the meaning of each of the quantities appearing in it. Make sure you write the right-hand-side (collision integral) in its general form as an integration over distribution functions.
- b. What assumptions have gone into this equation? Quantify them if you can. When would you expect these assumptions to be valid? Indicate where these assumptions are used in the derivation of the Boltzmann equation.
- c. Consider a dilute gas in a steady state with a concentration gradient $(\partial n(x)/\partial s)$. Show that within the relaxation time τ and to first order, the particle flux may be written

$$J_x = n\bar{V}_x = -D \frac{\partial n(x)}{\partial x}; \quad D = \frac{kT\tau}{m}$$

where \bar{V}_x is the average value of v_x . Assume that locally the particles follow a Maxwell-Boltzmann distribution

$$f_0(v, x) = n(x) \frac{m}{(2\pi kT)^{3/2}} \exp\left(-mv^2/2kT\right).$$

10. A thin-walled vessel of volume V is kept at a constant temperature T . A gas leaks slowly out of the vessel through a small hole of area A into surrounding vacuum. Find the time required for the pressure in the vessel to drop to half of its original value.
11. A classical monatomic ideal gas of N particles (each of mass m) is confined to a cylinder of radius r and infinite height. A gravitation field points along the axis of the cylinder downwards.
- a. Determine the Helmholtz energy A .
- b. Find the internal energy U and the specific heat C_v .
- c. Why is $C_v \neq 3/2 Nk$?
12. a. An ideal gas of N atoms of mass m is contained in a volume V at absolute temperature T . Calculate the chemical potential μ of this gas. You may use the classical approximation for the partition function, taking into account the indistinguishability of the particles.
- b. A gas of N' such weakly interacting particles, adsorbed on a surface of area A on which they are free to move, can form a two-dimensional ideal gas on such a surface. The energy of an adsorbed molecule is then $\left(p^2/2m\right) - \varepsilon_0$ where p denotes its (two-component) momentum vector and ε_0 is the binding energy which holds a molecule on the surface. Calculate the chemical potential μ' of this adsorbed ideal gas. The partition function can again be evaluated in the classical approximation.
- c. At the temperature T , the equilibrium condition between molecules adsorbed on the surface and molecules in the surrounding three-dimensional gas can be expressed in terms of the respective chemical potentials. Use this condition to find at temperature T the mean number n' of molecules adsorbed per unit area of the surface when the mean pressure of the surrounding gas is \bar{p} .

13. Consider a Carnot engine operating between thermal reservoirs at temperatures T_h and T_c ($T_h > T_c$) with thermal radiation (“photon gas”) as its working medium. Calculate the work done and heat exchanged during each step of the Carnot cycle and show that the efficiency of the engine is $1 - T_c/T_h$ as expected for a Carnot engine.
14. Consider a Fermi gas of N extremely relativistic electrons, with energies $\epsilon \gg mc^2$, in a cube of volume $V=L^3$. The energy levels are $\epsilon = pc$ where p is the momentum.
- Determine the Fermi energy ($T=0$) in terms of the electron density $n=N/V$.
 - Find the total energy of this gas in the ground state.

15. In a temperature range near absolute temperature T , the tension force F of a stretched plastic rod is related to its length L by the expression

$$F = aT^2(L - L_0)$$

where a and L_0 are positive constants, L_0 being the unstretched length of the rod. When $L = L_0$, the heat capacity CL of the rod (measured at constant length) is given by the relation $CL = bT$, where b is a constant.

- Write down the fundamental thermodynamic relation for this system, expressing dS in terms of dE and dL .
 - The entropy $S(T,L)$ of the rod is a function of T and L . Compute $(\partial S / \partial L)_T$.
 - Knowing $S(T_0, L_0)$, find $S(T,L)$ at any other temperature T and length L . (It is most convenient to calculate first the change of entropy with temperature at the length L_0 where the heat capacity is known.)
 - If the rod is thermally insulated but stretched a small distance δL from equilibrium, find the change in T .
 - Calculate the heat capacity $CL(L,T)$ of the rod when its length is L instead of L_0 .
16. A lead bullet with mass $m=10$ grams, leaves a gun with speed $v=500$ m/s and a temperature of 150°C . It is shot into a large body of water at 25°C . The specific heat of lead is 128 J/kg·K, and for water it is 4190 J/kg·K.
- Describe briefly what time-dependent temperature changes might occur in the bullet and in the water near it.
 - Estimate the energy transferred to the water.
 - Estimate the total entropy change after the bullet has cooled to the water temperature.
17. Consider a paramagnetic substance with the equation of state $M = AH/(T-T_0)$. Here M is the magnetization, H is the applied magnetic field, A and T_0 are constants, and T is the temperature. The equation of state is valid only for $T > T_0$. Show that C_M , the heat capacity at constant magnetization, is independent of M .

18. a. Consider the liquid-gas co-existence curve (called the vapor-pressure curve or the condensation curve) in the P-T plane of a single component system. In order for equilibrium between phases, we need the chemical potentials of the two phases to be equal, i.e., all along the condensation curve $\mu_L(P, T) = \mu_G(P, T)$. Starting from this basic information, obtain an expression for the slope of the co-existence curve in terms of the latent heat of vaporization L, the temperature T and the difference in the molar volumes of the liquid and the gas phase. This is the so-called Clausius-Clapeyron equation.

[Hint: It may be useful to recall that the chemical potential μ is related to the Gibbs potential as $G(T, P) = N\mu(T, P)$.]

- b. Consider the vapor pressure of cesium in equilibrium with liquid cesium. Cesium vapor pressure is so low that cesium vapor is approximately an ideal gas. Also, the molar volume of the vapor is much larger than the molar volume of the liquid; and the latent heat of vaporization varies with temperature approximately as $L = a(T_c - T)^2$. Start from the Clausius-Clapeyron equation and use the above information to obtain an expression for the vapor pressure of cesium at a temperature T.
19. A zipper has N links; each link has a state in which it is closed with energy 0 and a state in which it is open with energy ε . We require, however, that the zipper can only unzip from the left end, and that the link number s can only open if all links to the left (1, 2, ..., $s - 1$) are already open.
- a. Show that the partition function can be summed in the form:

$$Q_N = \frac{1 - \exp[-(N+1)\beta\varepsilon]}{1 - \exp[-\beta\varepsilon]}.$$

- b. In the limit $\varepsilon \gg kT$, find the average number of open links.

The above model is a very simplified model of the unwinding of two-stranded DNA molecules.

20. Consider diatomic molecules adsorbed on a flat surface at temperature T. The molecules are free to move on the surface and to rotate within the plane of the surface – this is a non-interacting 2D gas with rotational motion about one axis. The rotational state of the molecules is given by a single quantum number m ($m=0, \pm 1, \pm 2, \pm 3, \dots$) and the rotational energy is given by $\varepsilon_m = \left(\hbar^2 / 2I\right)m^2$, where I is the moment of inertia of the molecule.
- a. Find an expression for the rotational partition function of a single molecule. You need not evaluate the infinite series.
- b. Find the ratio of the probabilities of finding a molecule in states $m=3$ and $m=2$.
- c. Find the probability that $m=1$ given that $\varepsilon \leq \left(\hbar^2 / 2I\right)$.
- d. Find the rotational contribution to the internal energy of the gas (N molecules) in the high temperature limit, where $kT \gg \hbar^2 / 2I$.

21. Similar to the van der Waals equation of state is the Dieterici equation of state.

$$p(V - b) = RTe^{-a/RTV}$$

Find the critical constants p_c , V_c and T_c in this model of a weakly interacting gas. This equation of state was proposed to account for the interaction of gas atoms with walls.

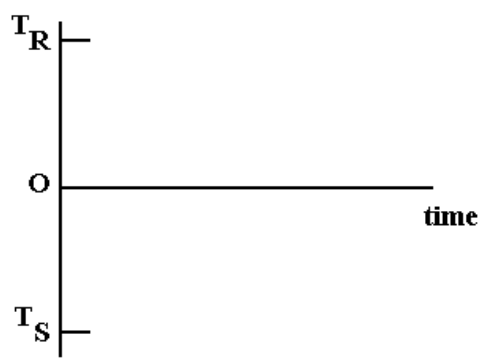
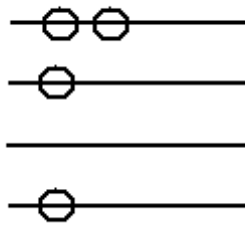
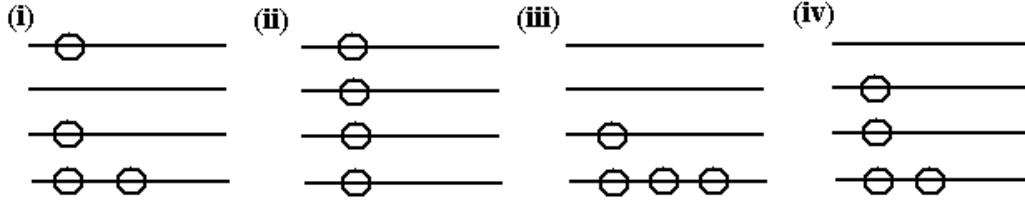
22. The surface temperature of the sun is T_o ($= 5800^\circ\text{K}$); its radius is R ($= 7 \times 10^8$ m) while the radius of the earth is r ($= 6.37 \times 10^6$ m). The mean distance between the sun and the earth is L ($= 1.5 \times 10^{13}$ m). In first approximation one can assume that both the sun and the earth absorb all electromagnetic radiation incident upon them.

Assume the earth has reached a steady state at some temperature T .

- Find an approximate expression for the temperature T of the earth in terms of the astronomical parameters mentioned above.
 - Calculate the temperature T numerically.
23. An ideal diatomic gas has rotational energy levels given by $\epsilon_j = \frac{h^2}{8\pi^2 I} j(j+1)$, with degeneracies $g_j = 2j+1$.

- For oxygen, with $\theta_{\text{rot}} = \frac{h^2}{8\pi^2 I k_B} = 2\text{K}$, what fraction of the molecules is in the lowest rotational energy state at $T=50\text{K}$?
- Repeat for hydrogen, with $\theta_{\text{rot}} = 85\text{K}$.

24. a. Compare the 4 level systems below. More than one particle may occupy a level. Which system has
 highest temperature _____ lowest temperature _____
 lowest specific heat _____ highest entropy _____



25. A certain solid contains N identical nuclei. Each nucleus can be in any one of three spin states labeled by the quantum number m , where $m = 1, 0, \text{ or } -1$. Each nucleus has zero energy in the $m = 0$ state, and has energy ϵ in both the $m = 1$ and the $m = -1$ states.
- Find an expression, as a function of temperature T , for the nuclear contribution to the average energy $\langle E \rangle$ of the solid. Assume that the N identical nuclei are located in distinct positions in the solid where the fields are identical.
 - Find an expression, as a function of T , of the contribution of these N nuclei to the entropy S of the solid.
 - By considering the total number of accessible states, calculate the nuclear contribution to the entropy of the solid in the $\epsilon \ll k_B T$ limit. Compare this with your answer in part (b).
 - Calculate the nuclear contribution to the heat capacity CV of the solid.
26. Consider a three dimensional cubic lattice of N atoms arranged in a box of side L . The modes of this system are called phonons.
- What is the total number of modes of this system? What would the total number have been if we were thinking of photons in a 3D box?
 - What is the maximum frequency that the phonons can have in the Debye model?
 - Calculate the total thermal energy of the phonons in the Debye model. Your answer may involve an integral.

$$\int_0^{x_D} dx \frac{x^3}{e^x - 1}$$

- Calculate the low temperature heat capacity of the lattice in the Debye model.
27. Consider a system made out of one single classical oscillator in contact with a heat reservoir at a temperature T . The potential energy of this *one-dimensional* oscillator is $V(x) = \epsilon_0 |x/a|^n$ where ϵ_0 and a are constants. What is the thermal energy of the oscillator?

(Note: Don't get scared by the integral that you will face; only powers of $k_B T$ in the partition function would contribute to the thermal energy.)

28. Equipartition. A classical harmonic oscillator

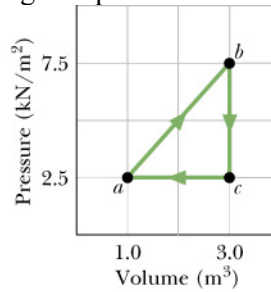
$$H = \frac{p^2}{2m} + \frac{Kq^2}{2}$$

is in thermal contact with a heat bath at temperature T . Calculate the partition function for the oscillator in the canonical ensemble and show explicitly that

$$\langle E \rangle = k_B T, \text{ and}$$

$$\langle (E - \langle E \rangle)^2 \rangle = (k_B T)^2.$$

29. For a gas of molecules with diameter d , number density n and at a temperature T , find
- the mean free path,
 - their average speed, and
 - the pressure of the gas using kinetic arguments.
30. A sample of ideal gas is taken through the cyclic process $abca$ shown in the figure. At point a , $T = 300\text{ K}$.
- What are the temperatures of the gas at points b and c ?



ΔE is the change in internal energy of the gas.

	Q	W	ΔE
$a \rightarrow b$			
$b \rightarrow c$			
$c \rightarrow a$			