Quantum Mechanics
Study Questions for the Spring 2016 Department Exam
December 1, 2015

1. a. If \( H = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \), what are the eigenvalues of this Hamiltonian?
   
b. Prove that \([H, \exp\{H\}] = 0\), that is, the two operators commute.
   
c. What are the eigenvalues of \( \exp\{H\} \)?
   
d. If a small perturbation \( H' = \lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), where \( \lambda \) is a small positive number, is applied to this system, calculate the change in the eigenenergy of the ground state, to first order in \( \lambda \).
   
e. Carry out the change in the ground state energy to second order in \( \lambda \).

2. Consider the potential barrier shown below.

\[ V(x) = 0 \quad x < -a \]

\[ = V_0 \quad -a < x < a \]

\[ = 0 \quad x > a \]

a. Write down the general expressions for the wave functions in the three regions.

b. Calculate the transmission coefficient from region I to III for \( E < V_0 \).

c. How does your result differ from a classical treatment?
3. Consider two different spin ½ particles whose Hamiltonian is completely specified as $H = c\vec{\sigma}(1) \cdot \vec{\sigma}(2)$ where $c$ is a real constant.

a. What are the constants of motion? Obtain the eigenvalues of $H$ for both singlet and triplet states.

b. Consider that at time $t = 0$ the spin of particle (1) along the $z$-axis is up and the spin of particle (2) along the $z$-axis is down. What is the wave function of the system at a later time $t$?

c. Calculate the probability that at a later time $t$ the spins of both particles are aligned as they were at $t = 0$.

Useful information.

\[
\chi_1^m = \begin{cases} 
\alpha(1) \alpha(2), & m = 1 \\
\beta(1) \beta(2), & m = -1 \\
\frac{1}{\sqrt{2}}[\alpha(1) \beta(2) + \beta(1) \alpha(2)], & m = 0
\end{cases}
\]

\[
\chi_0^0 = \frac{1}{\sqrt{2}}[\alpha(1) \beta(2) - \beta(1) \alpha(2)].
\]

4. A particle of mass $m$ is to be scattered by a one-dimension potential $-\gamma \delta(x - a)$, $\gamma > 0$. If the particle is incident from the right-hand side initially, and the particle has momentum $p$, calculate the probability that the particle can be found on the $+x$ side and on the $-x$ side of the potential at the end of the collision.

5. a. Ten protons are confined in a box of dimension $(a, 2a, a)$ on each side. Calculate the total energy of the ground state of these ten protons if we assume that the protons do not interact with each other.

b. If the ten protons are replaced by ten neutral hydrogen atoms in the ground state, what is the total energy resulting from the confinement. Again assume that the hydrogen atoms do not interact with each other. You can treat the mass of proton and hydrogen atom to be identical.

6. Show that for a system consisting of two identical particles with spin $I$, the ratio of the number of states symmetric under exchange of the two spins to the number of states antisymmetric under exchange of the two spins is equal to $(I+1)/I$. [Hint: If you do not know where to start, consider $I = \frac{1}{2}$ first.]

7. Two indistinguishable spin 1/2 particles of mass $m$ move in a one-dimensional infinite square well

\[
V = \begin{cases} 
0 & |x| < a \\
\infty & |x| > a
\end{cases}
\]

a. Find the eigenvalues of and the lowest two states of the system and their eigenfunctions.

b. If a perturbation $H_1 = V_0 \delta(x_1 - x_2)$ is introduced such that $V_0$ is positive, evaluate the first order shifts in energy of the states found in (a), and show how it removes any degeneracies encountered in (a). (Leave any integrals you cannot immediately evaluate in a form ready for a table of definite integrals.)
8. Suppose that $|\psi\rangle$ is a state with a well-defined value of $J_Z, J_Z |\psi\rangle = m\hbar |\psi\rangle$. Show that the average values of $J_x$ and $J_y$ are equal to zero. Hint: The $J_i$'s are angular momentum operators and they satisfy well-known commutation relations.

9. In an experiment you make repeated measurements of the energy of a system. You find that for one-fourth of the measurements you obtain the value $E_1$; for one-third of the measurements you obtain the energy value $E_2$; and for the remaining measurements you obtain the value $E_3$.

   a. Using only this information write as complete a time dependent wave function for this state as you can. For parts of the wave function which you cannot specify use general functions and indicate why you cannot specify it (them).
   
   b. Is this wave function uniquely determined by the experimental evidence? If so how? If not, are there further experiments which you could do to make a unique determination?

10. A polarized proton travels at velocity $v$ toward a magnetic field as shown.

   \[ \vec{B} = B_0 \hat{z} \]

   a. If the polarization before entering the field is $\langle S_x \rangle = h/2$, $\langle S_y \rangle = \langle S_z \rangle = 0$, find the polarization when the proton leaves the field.
   
   b. If one wants to use this device to change the polarization to $\langle S_x \rangle = -h/2$, $\langle S_y \rangle = \langle S_z \rangle = 0$, how long should the $B$-field region be?

11. Given the following matrix representations of operators ($\hbar = 1$)

   \[ \hat{L}_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{L}_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \hat{L}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \]

   a. The system is in the state with $L_z=1$, and $L_x$ is measured. What are the possible outcomes and what is the expectation value?
   
   b. Consider the state

   \[ \psi = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{pmatrix} \] in the $L_z$ basis.

   i) If $L_z^2$ is measured and the result is $+1$, what is the state after this measurement? (No need to specify the state vector.) How probable is the result?
   
   ii) If $L_z$ is measured, what are the possible outcomes and respective probabilities?
12. Consider the isotropic harmonic oscillator where the potential is given by

\[ V(r) = \frac{1}{2} m \omega^2 r^2. \]

a. Show that the eigenfunctions can be expressed in the form of \( R_n(r) Y_m \).

b. The problem is separable in Cartesian coordinates, show that the eigenenergies can be expressed as

\[ E_n = (n + 3/2) \hbar \omega. \]

c. Find the degeneracy \( d(n) \) of \( E_n \) for the first 4 values of energies. You obtain the degeneracy from the solutions in Cartesian coordinates. For each energy, identify the value(s) of orbital angular momentum (or momenta if more than one) for each eigenenergy.

13. The scattering amplitude for a proton on a target atom is \( f(\theta) \). That is, if the target is at \( \vec{r}_T = 0 \) then far from the target the wavefunction is

\[ \psi(r) = e^{i \vec{k} \cdot \vec{r}} + f(\theta) \frac{e^{ikr}}{r}. \]

a. Suppose that there are \( N(N >> 1) \) target atoms located at the points \( \vec{r}_1 = A, \vec{r}_2 = 2A, \vec{r}_3 = 3A \) and so on.

What is \( \psi(\vec{r}) \) for positions \( \vec{r} \) far from the line? That is, find \( \psi(\vec{r}) \) for \( |\vec{r}| >> N|A| \). It will have the form

\[ \psi(\vec{r}) = e^{i \vec{k} \cdot \vec{r}} + F(\theta) \frac{e^{ikr}}{r}. \]

Find \( F(\theta) \) and \( d\sigma/d\Omega = |F(\theta)|^2 \). Is \( d\sigma/d\Omega \sim N \)?

b. Suppose now that the \( N(N >> 1) \) target atoms are located at random positions, but all inside a small finite ball of radius \( R \). Calculate \( F(\theta) \) and \( d\sigma/d\Omega \). Is \( d\sigma/d\Omega \sim N \)?
14. a. Consider an incident particle being scattering from a central potential. What is the general form of the scattered wave function in terms of the scattering amplitude? Define the differential scattering cross section in terms of the scattering amplitude, and also in terms of the transition probability per unit time and the incident flux.

b. The scattering amplitude of two identical charged particles (of charge q) without spin in the center of mass system is of the form

\[
f(\theta) = \left( \frac{q^2}{4E} \right) \frac{e^{-i\Delta}}{\sin(\theta/2)^2}
\]

E is the relative kinetic energy and \( \Delta \) depends on E, \( \theta \) and q.

Consider proton-proton elastic scattering. Calculate the differential elastic scattering cross section in the center of mass system both in classical mechanics and in quantum mechanics. Assume that the incident proton is unpolarized. Indicate approximate value of the angle where there is a significant difference between the classical and the quantum results.

15. A free particle of mass \( m \) is represented by a Gaussian wave packet, the momentum space representation of which at time \( t = 0 \) is given by

\[
\psi(p) = (2\pi\sigma)^{-1/4} \exp \left[ -\frac{(p-p_0)^2}{4\sigma^2} - \frac{ipx_0}{\hbar} \right].
\]

a. Find the wave function in coordinate space, \( \psi(x,t) \), for \( t \geq 0 \).

b. Calculate the expectation values \( \langle x \rangle \) and \( \langle x^2 \rangle \).

c. Determine the uncertainty \( \Delta x = \left( \langle x^2 \rangle - \langle x \rangle^2 \right)^{1/2} \) and comment on its time evolution.

Hint:

\[
\int dX x^{2n} e^{-x^2} = (-1)^n \left. \frac{d^n}{d\beta^n} \int dX e^{-\beta x^2} \right|_{\beta=1} = (-1)^n \frac{d^n}{d\beta^n} \sqrt{\frac{\pi}{\beta}} \bigg|_{\beta=1}
\]

16. Let \( |\alpha\rangle \) be a stationary state of the harmonic oscillator and that

\[
\alpha_- |\alpha\rangle = \alpha |\alpha\rangle.
\]

Where \( \alpha \) is a complex number. Calculate \( \langle x \rangle \), \( \langle p \rangle \), \( \langle x^2 \rangle \) and \( \langle p^2 \rangle \) where the expectation values are defined with respect to the state \( |\alpha\rangle \).

Also show that \( \sigma_x\sigma_p = \hbar/2 \). Note that \( |\alpha\rangle \) is called a coherent state. It is a general minimum uncertainty wave packet.

Hint: Remember that \( \alpha_- \) is the Hermitian conjugate of \( \alpha_+ \). Do not assume that \( \alpha \) is real.
Consider a hydrogen atom in an electric field for which \( H' = eEz = eEr \cos \theta \). The energies of \( 2s \) and \( 2p \) are degenerate without the presence of the electric field.

a. What matrix elements of \( H' \) are nonzero between the degenerate states? Give your reasons for your answer.

b. Consider now only those states where \( H' \) removes the degeneracy. Find the energies of these states.

\[
\psi_{2s} = \frac{1}{2\sqrt{2\pi}} \frac{1}{a_0^{3/2}} \left(1 - \frac{r}{2a_0}\right) e^{-r/2a_0}
\]

\[
\psi_{2p0} = \frac{1}{4\sqrt{2\pi}} \frac{1}{a_0^{3/2}} \left(\frac{r}{a_0}\right) e^{-r/2a_0} \cos \theta
\]

Note:

\[
\int_0^\infty e^{-cr} r^n \, dr = \frac{n!}{c^{n+1}}
\]

18. Consider the potential:

\[
V(x) = V_0 \left[ \left(\frac{x}{\alpha}\right)^2 - 1\right]^2
\]

where \( \alpha \) and \( V_0 \) are real, positive constants. Assume the particle has mass \( m \). A potential of this form is a good model for the position \( x \) of the nitrogen atom in ammonia, NH\(_3\). The shape of ammonia can be pictured as a pyramid with a triangular base --- the H's lie at the corners of the base, and the N at the top. There is no reason, however, for the nitrogen to stay above the H's, and it can tunnel through the base to be on bottom.

a. Sketch the potential. What symmetries does it possess?

b. For a particle localized on one side of the barrier, the potential appears nearly harmonic near the minimum. Find the effective harmonic frequency \( \omega_{\text{eff}} \) for such a particle.

c. Construct approximate ground and first excited state wave functions for the full potential \( V(x) \) as a linear combination of the ground states \( \psi_R(x) \) and \( \psi_L(x) \) of the right- and left-side effective harmonic oscillator potentials from (b). With these as trial functions, use the variational principle to estimate the energies of the ground and first excited state of the full potential. You may use the approximations:

\[
H\psi_L(x) \approx \frac{1}{2} \hbar \omega_{\text{eff}} \psi_L(x)
\]

\[
H\psi_R(x) \approx \frac{1}{2} \hbar \omega_{\text{eff}} \psi_R(x),
\]

where \( H \) is the full Hamiltonian of the system.

d. How will the energies you estimated in (c) change with \( V_0 \)? Explain your answer.

e. Estimate the time it takes for a particle initially localized to the left of the barrier to appear fully localized on the right side. You may assume that the approximate eigenstates from (c)
will evolve in time with a phase $e^{-iEt/h}$ where $E$ are the energies you calculated in (c). How does this time depend on $V_0$? Explain why your answer to this question makes sense.

Potentially useful information:

$$\int_{-\infty}^{\infty} dx \, x^n e^{-x^2} = (-1)^n \frac{d^n}{d\beta^n} \int_{-\infty}^{\infty} dx \, e^{-\beta x^2} \bigg|_{\beta=1} = (-1)^n \frac{d^n}{d\beta^n} \sqrt{\pi} \bigg|_{\beta=1}. $$

19. Consider an isotropic 3D spherical square well. What are the conditions on its depth $V_0$ and radius $a$ such that a particle of mass $m$ has exactly one bound state in this well? Show your reasoning and calculations.

20. A particle of mass $m$ experiences the potential

$$V(x) = -V_0 \ell \left[ \delta(x-a) + \delta(x+a) \right]$$

with $\ell$ and $a$ positive, real constants with units of length and $V_0$ a positive, real constant with units of energy.

a. What symmetries does $V(x)$ possess?

b. Can you construct simultaneous eigenstates of energy and the symmetries you identified in (a)? Why or why not?

c. For a single $\delta$-function potential, you know there is only a single bound state, so it should be plausible that for $V(x)$ above there are at most two bound states. Find the transcendental equations that determine the energies of these states. Sketch the corresponding wave functions and point out the symmetries from (a) and (b).

d. What are the energies of each state in the limits $a \to 0$ and $a \to \infty$? Explain why your results make sense physically.

21. Consider the vector space of angular momentum eigenstates for two spin $-\frac{1}{2}$ particles. One possible basis is given by the four product states

$$|\pm, \pm \rangle = |\pm \rangle_{(1)} \otimes |\pm \rangle_{(2)}$$

where $|\pm \rangle_{(1)}$ and $|\pm \rangle_{(2)}$ are eigenstates of $\vec{S}^2_{(1)}$, $S_z_{(1)}$ and $\vec{S}^2_{(2)}$, $S_z_{(2)}$, respectively, such that (in atomic units, $\hbar = 1$):

$$\vec{S}^2_{(i)} = \frac{3}{4} \, |\pm \rangle_{(i)}$$

$$S_z_{(i)} \, |\pm \rangle_{(i)} = \pm \frac{1}{2} \, |\pm \rangle_{(i)}, \quad i = 1, 2.$$

a. In this vector space, find simultaneous eigenstates of the total spin operators,

$$\vec{S}^2 = (\vec{S}^2_{(1)} + \vec{S}^2_{(2)})^2,$$

$$S_z = S_z_{(1)} + S_z_{(2)},$$

and the corresponding eigenvalues.

b. Assume that $|x \pm \rangle_{(i)}$ are simultaneous eigenstates of $\vec{S}^2_{(i)}$ and $S_z_{(i)}$, $i = 1, 2$.

Expand $|x \pm \rangle_{(i)}$ in terms of the basis states $|\pm \rangle_{(i)}$. 


c. In the $|x^\pm\rangle_{(i)}$ basis, calculate the eigenstate to the total spin operator $\hat{S}^2$ with eigenvalue $(= \text{total spin}) \ 0$.

22. A particle is in the ground state of an infinite square well. That is

$$V(x) = \begin{cases} 0 & 0 \leq x \leq L \\ \infty & \text{otherwise} \end{cases}$$

At $t=0$ the wall at $x=L$ is suddenly moved to $x=2L$--this happens very fast, approximately instantaneously.

a. Calculate the probability that long after $t=0$ the system is in the ground state of the new potential.

b. How fast must the change take place for this "instantaneous" assumption to be good?

23. A particle of mass $m$ is subject to the following potential:

$$V(x) = V_0 \left( \frac{x}{x_0} \right)^6$$

where $V_0$ and $x_0$ are positive, real constants.

a. Sketch the trial function $\psi_1(x)$

$$\psi_1(x) = e^{-\frac{(x/a)^2}{2}},$$

and indicate which property the variational parameter $a$ controls. Estimate the ground state energy of $V(x)$ using the variational principle with $\psi_1(x)$ as your trial function.

b. Sketch the trial function $\psi_2(x)$,

$$\psi_2(x) = \begin{cases} (x^2 - a^2)^2 & |x| \leq a \\ 0 & |x| \geq a \end{cases},$$

and indicate which property the variational parameter $a$ controls. Estimate the ground state energy of $V(x)$ using the variational principle with $\psi_2(x)$ as your trial function.

c. Which trial function is a better approximation to the ground state? Explain your answer.

d. Based on the function you identified in (c), suggest a trial function suitable for approximating the first excited state.

Potentially useful information:

$$\int_{-\infty}^{\infty} dx \ x^n e^{-x^2} = (-1)^n \frac{d^n}{d\beta^n} \int_{-\infty}^{\infty} dx \ e^{-\beta x^2} \bigg|_{\beta=1} = (-1)^n \frac{d^n}{d\beta^n} \sqrt{\frac{\pi}{\beta}} \bigg|_{\beta=1}.$$
24. a. Consider the trial wave functions \( \phi(x) = A e^{-\lambda^2 x^2} \) where \( \lambda \) is an adjustable parameter.
   Determine the normalization constant \( A(\lambda) \).

   b. Use the trial functions of part a) to obtain the best approximation to the energy of the ground state of the Hamiltonian
   \[ H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + x^4 \]
   describing the motion of a particle of mass \( m \) in 1D.

   **Useful integrals:**
   \[ \int_0^\infty dx \, x^{2n} e^{-x^2/a^2} = \sqrt{\pi} \frac{(2n)!}{n!} \left( \frac{a}{2} \right)^{2n+1} \]

25. If the spin state of an electron is prepared such that it has spin \( \hbar/2 \) along the x-direction, what is the probability of finding this spin to have eigenvalue \( \hbar/2 \) if the spin is measured along the y-direction? Show the steps of your calculation. The Pauli spin matrices are:

   \[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

26. Consider a quantum mechanical system with only two stationary states \( |1> \) and \( |2> \) and energies \( E_o \) and \( 3E_o \) respectively. At \( t=0 \) the system is in the ground state and a constant perturbation \( <1|V|2> = <2|V|1> = E_o \) is switched on. Calculate the probability of finding the system in the state \( |2> \).

27. Is \( Y_{11} \alpha \) an eigenstate of \( J^2 \), of \( J_z \)?
   What is the expectation value of \( J_x \) for this state?

28. Let
   \[ |\psi\rangle = \frac{1}{\sqrt{3}} Y_{10} \alpha + \frac{1}{\sqrt{3}} Y_{11} \beta \]

   a. If you measure \( S_y \), what is the probability that you will get \( \hbar/2 \)?
   b. If indeed you do get \( \hbar/2 \) in (a), what is the new state (or wavefunction) after the measurement?
   c. If you measure \( L_z \) after (b), what is the probability that you will get \( -\hbar \)?
   (Note: \( \alpha \) for spin up and \( \beta \) for spin down, and \( Y \)'s are the spherical harmonics.)
29. Consider two spin 1/2 particles described by the Hamiltonian

\[ H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(x_1) + V(x_2) \]

where \( V(x) = \infty \) for \( x < 0 \) and for \( x > a \); \( V(x) = 0 \) for \( 0 < x < a \). Assume that the electrons are in the opposite spin state, that is, the total \( S = 0 \).

a. Write down the spin wave function(s). Use the standard notations: \( \alpha \) for spin up, and \( \beta \) for spin down.

b. Find the energy and wavefunction of the ground state of this Hamiltonian.

c. Find the energy and wavefunction of the lowest state for \( S = 1 \).

d. Find the energy and wavefunction of the second \( S = 0 \) state. Show that the energy is the same as in (c).

e. If the two particles have a small interaction \( W(x_1, x_2) = b x_1 x_2 \) where \( b \) is small and positive, show that the degeneracy in (c) and (d) is removed. Which one has the lower energy?

30. a. A two-dimensional harmonic oscillator has the potential \( V(x, y) = \frac{1}{2} m \omega^2 (x^2 + 4 y^2) \).

Calculate the energies of the first three lowest states, and identify the degrees of degeneracy for each energy.

b. If there is an additional small coupling term \( W(x, y) = ax^2 y \) present, where \( a \) is a small constant. Calculate the first-order correction to each of the three levels.