Mechanics
Study Questions for the Spring 2022 Department Exam
October 7, 2021

1. Consider a planet in circular orbit around a star. Determine how the speed of the planet depends on how far away it is from the star. Does Pluto move faster or slower than Mercury around the sun?

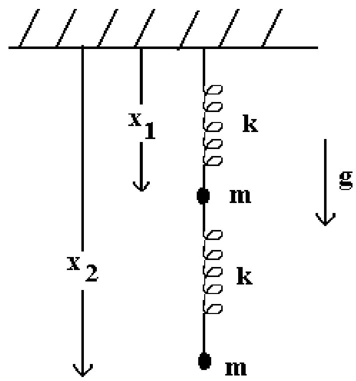
2. A satellite in circular orbit of radius R about the center of the earth is subject to a drag force of magnitude $F = a\text{v}^n$ (where v is the speed of the satellite) which results in the rate of change of the radial distance $\frac{dr}{dt} = -b$ (where $b > 0$ is small enough so that the loss of energy in an orbit is small compared to the total kinetic energy of the satellite). Determine a and n.

3. Consider a mass m in a circular orbit of radius R in a central-force potential $-am/r^n$, $a > 0$. Determine the range of $n > 0$ for which such a circular orbit is stable.

4. The gravitational acceleration on the surface of the moon is one-sixth of that on the surface of the earth. A woman on Earth lowers her center of mass by 40 cm by bending her knees. Exerting a constant force on the ground, she jumps straight up, raising her center of mass by 60 cm above that of her normal erect position. How much higher can she jump on the moon?

5. A uniform rigid rod of length L, mass m, and moment of inertia $mL^2/3$ about an end, is supported horizontally by props at each end. Find the force at the second prop right after the first one is removed.

6. Consider two identical masses m and two identical springs of spring constant k and unstretched length a.



a. What are the equilibrium positions x_{10} and x_{20}?

b. For small vertical oscillations about x_{10} and x_{20} what are the normal frequencies?

c. What are the motions associated with each of these frequencies?

7. Over a day, meteors deposit on the surface of the Earth a uniform layer of dust of thickness ΔR, small compared to the radius of the Earth R. If ρ_{dust} and ρ_{earth} are the densities of dust and the Earth, show that conservation of angular momentum results in the change of the length of the day by approximately $5\Delta R\ \rho_{dust}/(R\ \rho_{earth})$. The moment of inertia of a sphere of mass M and radius R about a central axis is $2MR^2/5$, and that of a thin spherical shell of mass M and radius R is $2MR^2/3$ about a central axis.

8. The free surface of a liquid is an isopotential surface. By considering the potential energy density of an incompressible liquid in a vessel rotating about a vertical axis at constant angular velocity ω, show that the free liquid surface is a paraboloid of revolution.

9. Two graduate students, each of mass m_g, stand at one end of a long flat car of mass m_c that has frictionless wheels. Either student can run to the other end of the cart and jump off with speed u (relative to the cart).

a. Find the recoil speed of the cart if both students run and jump off simultaneously.
b. What is the recoil speed of the cart if the second student starts running only after the first has jumped off? Is this less than or greater than that in (a)?

10. A uniform thin semicircular sheet of metal (of radius R) lies in the xy plan with its center at the origin and diameter lying along the x axis. Find the position of the center of mass.

11. Show that the force $\bar{F}$ on a charge q due to a fixed charge Q at the origin is conservative and find the corresponding potential energy V.

12. A small coin is placed on the top of frictionless sphere of radius R. The coin is given a tiny push and begins to slide down the sphere. At what vertical distance below the top of the sphere does the coin leave the surface of the sphere?

13. Consider a mass m moving in two dimensions with potential energy V(x, y)= $k\,(x^2 + y^2)/2$, $k > 0$. In the Cartesian (x,y) coordinate system, write down the Lagrangian and derive the two Lagrange equations of motion. Describe the solutions of the equations of motion.

14. A mass m is attached by a massless string of length l to the tip of a frictionless cone with vertex half-angle Θ. For the case when the mass moves at speed v in a horizontal circle on the surface of the cone, find: (a) the tension in the string; (b) the normal force on the mass from the cone; and (c) the maximum speed v for which the mass stays in contact with the cone.

15. A pendulum bob of mass m is suspended by a massless spring (of unextended length l) with spring constant k. Derive the Lagrangian equations of motion.

16. After a lengthy space flight, you have just landed on Planet X. Near the point where you are standing on the surface of Planet X, the gravitational force on a mass m is vertically down but has magnitude $m\gamma y^2$ where γ is a constant and y is the mass's height above the horizontal ground. (a) Find the work done by gravity on a mass m moving from $\mathbf{r}_1$ to $\mathbf{r}_2$, and use your answer to show that gravity on Planet X, although most unusual, is still conservative. (b) Still on the same planet, you thread a bead on a curved, frictionless, rigid wire, which extends from ground level to a height h above the ground. Show clearly the forces on the bead when it is somewhere on the wire. (Just name the forces so it is clear what they are; don't worry about their magnitude.) Which of the forces are conservative and which are not? (c) If you release the bead from rest at height h, how fast will it be going when it reaches the ground?

17. The potential energy of two atoms in a molecule can sometimes be approximated by the Morse function

$$U(r) = A\left[\left(e^{(R-r)/S} - 1\right)^2 - 1\right]$$

where r is the distance between the two atoms and A, R, and S are positive constants with $S << R$. Sketch the function for $0 < r < \infty$. Find the equilibrium separation r_0, at which $U(r)$is a minimum. Now write $r = r_0 + x$ so that x is the displacement from equilibrium, and show that, for small displacements, U has approximate form $U = constant + kx^2/2$. That is, Hooke's law applies. What is the force constant k?

18. The force on a mass m at position x on the x axis is $F = F_0 \sinh(\alpha x)$, where F_0 and α are positive constants. Find the potential energy $U(x)$, and give an approximation for $U(x)$ suitable for small oscillations. What is the angular frequency of such oscillations?

19. You are told that, at known positions x_1 and x_2, an oscillating mass m has speeds v_1 and v_2. What are the amplitude and angular frequency of the oscillations?

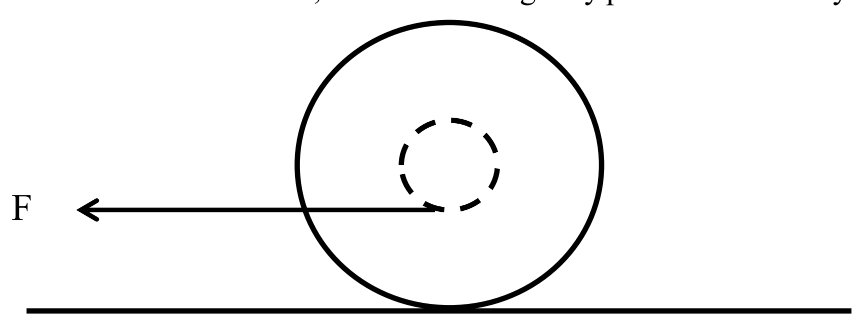
20. Find and describe the path y = y(x) for which the integral $\int_{x_1}^{x_2} \sqrt{x}\sqrt{1+y'^2}\,dx$ is stationary.

21. A mass m_1 rests on a frictionless horizontal table and is attached to a massless string. The string runs horizontally to the edge of the table, where it passes over a massless, frictionless pulley and then hangs vertically down. A second mass m_2 is attached to the bottom of the string. Write down the Lagrangian for the system. Find the Lagrange equation of motion, and solve it for the acceleration of the blocks. For your generalized coordinate, use the distance x of the second mass below the tabletop.

22. The center of a long frictionless rod is pivoted at the origin, and the rod is forced to rotate in a horizontal plane with constant angular velocity ω. Write down the Lagrangian for a bead of mass m threaded on the rod, using r as the generalized coordinate, where r,ϕ are the polar coordinates of the bead. (Notice that ϕ is not an independent variable since it is fixed by the rotation of the rod to be ϕ= ωt.) Solve Lagrange's equation for $r(t)$. What happens if the bead is initially at rest at the origin? If it is released from any point $r_0 > 0$ show that $r(t)$ eventually grows exponentially. Explain your results in terms of the centrifugal force $m\omega^2 r$.

23. The momentum $\boldsymbol{p}$ conjugate to the relative position **r** is defined with components $p_x = \partial L/\partial \dot{x}$ and so on. Prove that $\boldsymbol{p} = \mu\dot{\boldsymbol{r}}$. Prove also that in the center of mass frame, $\boldsymbol{p}$ is the same as $\boldsymbol{p}_1$the momentum of particle 1 (and also $-\boldsymbol{p}_2$).

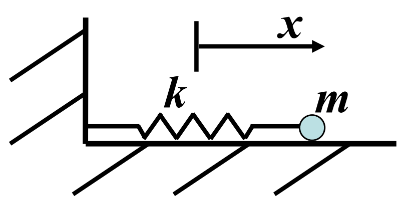
24. A yo-yo rests on a level surface, and the cord is gently pulled horizontally as shown below:



Which way does it roll? Why?

25. What would become of the earth's orbit (which you may consider to be a circle) if half of the sun's mass were to suddenly disappear? Would the earth remain bound to the sun? [Hints: Consider what would happen to the earth's KE and PE at the moment of the great disappearance. The virial theorem for the circular orbit helps with this one.] Treat the sun (or what remains of it) as fixed.

26. A mass m is attached to a spring of constant k. Letting $x(t)$ be displacement of the mass from equilibrium, the mass experiences a force from the spring ($\mathbf{F} = -kx\,\hat{x}$) and a linear resistance force $\mathbf{F}_r = b\mathbf{v}\,\hat{v}$, where $\sqrt{k/m} > b/2m$. The initial conditions are $x(0) = 0$ and $\dot{x}(0) = v_0$.

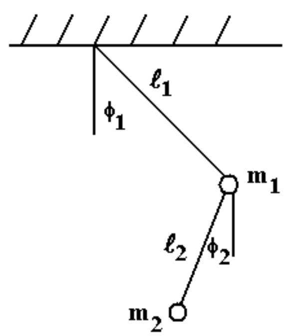


a. The mass will be at $x = 0$ at time $t = t_0$. Find an expression for t_0.

b. Find the velocity of the oscillator at $t = t_0$.

c. What is the work done by the resistive force $\mathbf{F}_r$ from time $t = 0$ to t_0?

27. Obtain the Lagrangian and equations of motion for the double pendulum shown in the figure, where the lengths of the pendula are ℓ_1 and ℓ_2 with corresponding masses m_1 and m_2.



28. The A monkey of mass m jumps on the outside edge of a freely rotating merry-go-round of rotational inertia I and radius R. By what ratio does the angular velocity change?

29. Consider a star in the disk of the Milky Way Galaxy, in circular orbit around the center of the Milky Way. Derive an expression for the mass of the Milky Way interior to the orbit of the star, in terms of the radius of the orbit and the speed of the star. Observations indicate that the speed of the most distant stars are independent of their distance from the center of the Milky Way.

30. Two masses m_1 and m_2 in a plane have an interaction potential energy $v(r) = \frac{1}{2}\,kr^2$ where $\vec{r}$ is their relative position. Write the Lagrangian in terms of $\vec{r}$ and the center of mass position $\vec{R}$. Find the equations of motion and describe the motion. What is the frequency of the relative motion?