Thermal & Statistical Physics
Study Questions for the Fall 2022 Department Exam
May 25, 2022

1. Consider an extreme relativistic classical gas (ignore spin) with single particle energy

   $\varepsilon = \sqrt{p^2c^2 + m^2c^4} \cong pc$
   where p is momentum and c is the velocity of light.

   a. If this gas is confined to 1-dimension (length L) show that $\varepsilon = nhc/2L$ (n = 1, 2, 3, ...) and then find the partition function Z1 for a single particle.
   b. Now find F and show that PV = NkT. (4)

2. A sample of helium gas inside a cylinder terminated with a piston doubles its volume from Vi = $1 m^3$ to Vf = $2 m^3$. During this process the pressure and volume are related by $PV^{6/5} = A$ = constant.

   Assume that the product PV always equals $\frac{2}{3}U$, where U is the internal energy.

   a. What is the change in energy of the gas?
   b. What is the change in entropy of the gas?
   c. How much heat was added to or removed from the gas?

3. A system consists of three particles, each of which has three non-degenerate energy states--0, ε and 3ε. Determine the canonical partition function if the particles are

   a. classical particles. (note: classical particles are distinguishable)
   b. Fermions.
   c. Bosons.

4. Consider 3-d lattice of N atoms arranged in a box of side L.

   a. Briefly explain why there are only 3N independent phonon modes and show that ω_{max} (Debye frequency) is given by

   $$\omega_m = \left(\frac{\pi c}{L}\right)\left(\frac{6N}{\pi}\right)^{1/3}.$$

   b. Show that at high temperatures the internal energy $U = 3Nk_BT$ and that at low temperatures it is proportional to T^4. You can use the Bose-Einstein distribution to determine the occupation number of the states.

5. Calculate the magnetic susceptibility $\left(\frac{\partial M}{\partial H}\right|_{H=0} = X$ where M is the magnetic moment of the sample and H is the applied field) as a function of T of a gas of N permanent dipoles each of moment μ,

a. if any direction is allowed (classical spines).
b. if the dipole is only allowed to assume 2 directions, parallel and opposite to the applied field (Ising spins).

6. A <u>classical</u> monatomic ideal gas of N particles (each of mass m) is confined to a cylinder of radius r and infinite height. A gravitation field points along the axis of the cylinder downwards.

a. Determine the Helmholtz energy A.
b. Find the internal energy U and the specific heat C_v.
c. Why is $C_v \neq 3/2\ Nk$?

7. a. An ideal gas of N atoms of mass m is contained in a volume V at absolute temperature T. Calculate the chemical potential μ of this gas. You may use the classical approximation for the partition function, taking into account the indistinguishability of the particles.
b. A gas of N' such weakly interacting particles, absorbed on a surface of area A on which they are free to move, can form a two-dimensional ideal gas on such a surface. The energy of an absorbed molecule is then $\left(p^2/2m\right) - \varepsilon_0$ where p denotes its (two-component) momentum vector and ε_0 is the binding energy which holds a molecule on the surface. Calculate the chemical potential μ' of this adsorbed ideal gas. The partition function can again be evaluated in the classical approximation.
c. At the temperature T, the equilibrium condition between molecules adsorbed on the surface and molecules in the surrounding three-dimensional gas can be expressed in terms of the respective chemical potentials. Use this condition to find at temperature T the mean number n' of molecules adsorbed per unit area of the surface when the mean pressure of the surrounding gas is $\overline{p}$.

8. In a temperature range near absolute temperature T, the tension force F of a stretched plastic rod is related to its length L by the expression

$$F = aT^2(L - L_o)$$

where a and Lo are positive constants, Lo being the unstretched length of the rod. When L = Lo, the heat capacity CL of the rod (measured at constant length) is given by the relation CL = bT, where b is a constant.

a. Write down the fundamental thermodynamic relation for this system, expressing dS in terms of dE and dL.
b. The entropy S(T,L) of the rod is a function of T and L. Compute $\left(\partial S/\partial L\right)_T$.
c. Knowing S(To,Lo), find S(T,L) at any other temperature T and length L. (It is most convenient to calculate first the change of entropy with temperature at the length Lo where the heat capacity is known.)
d. If the rod is thermally insulated but stretched a small distance δL from equilibrium, find the change in T.
e. Calculate the heat capacity CL (L,T) of the rod when its length is L instead of Lo.

9. A lead bullet with mass m=10 grams, leaves a gun with speed v=500m/s and a temperature of 150ºC. It is shot into a large body of water at 25ºC. The specific heat of lead is 128 J/kg·K, and for water it is 4190 J/kg·K.

   a. Describe briefly what time-dependent temperature changes might occur in the bullet and in the water near it.
   b. Estimate the energy transferred to the water.
   c. Estimate the total entropy change after the bullet has cooled to the water temperature.

10. Consider a paramagnetic substance with the equation of state $M = AH/(T-T_0)$. Here M is the magnetization, H is the applied magnetic field, A and T_0 are constants, and T is the temperature. The equation of state is valid only for $T>T_0$. Show that C_M, the heat capacity at constant magnetization, is independent of M.

11. A zipper has N links; each link has a state in which it is closed with energy 0 and a state in which it is open with energy ε. We require, however, that the zipper can only unzip from the left end, and that the link number *s* can only open if all links to the left (1, 2, ----, *s* - 1) are already open.

   a. Show that the partition function can be summed in the form:

$$Q_N = \frac{1-\exp[-(N+1)\beta\varepsilon]}{1-\exp[-\beta\varepsilon]}.$$

   b. In the limit $\varepsilon >> kT$, find the average number of open links.

   The above model is a very simplified model of the unwinding of two-stranded DNA molecules.

12. Consider diatomic molecules adsorbed on a flat surface at temperature T. The molecules are free to move on the surface and to rotate within the plane of the surface – this is a non-interacting 2D gas with rotational motion about one axis. The rotational state of the molecules is given by a single quantum number m (m=0, ±1, ±2, ±3…) and the rotational energy is given by $\varepsilon_{\mathrm{m}} = \left(\hbar^2/2\mathrm{I}\right)\mathrm{m}^2$, where I is the moment of inertia of the molecule.

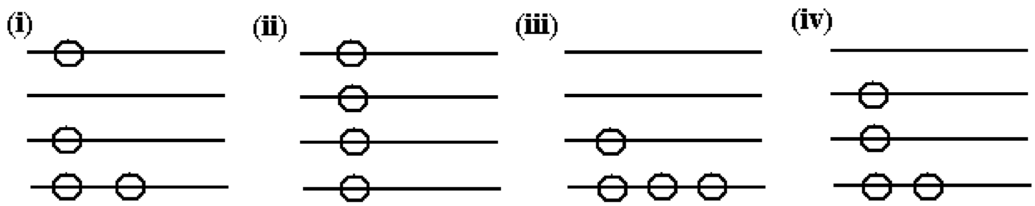
   a. Find an expression for the rotational partition function of a single molecule. You need not evaluate the infinite series.
   b. Find the ratio of the probabilities of finding a molecule in states m=3 and m=2.
   c. Find the probability that m=1 given that $\varepsilon \le \left(\hbar^2/2\mathrm{I}\right)$.
   d. Find the rotational contribution to the internal energy of the gas (N molecules) in the high temperature limit, where $\mathrm{kT} >> \hbar^2/2\mathrm{I}$.

13. Similar to the van der Waals equation of state is the Dieterici equation of state.

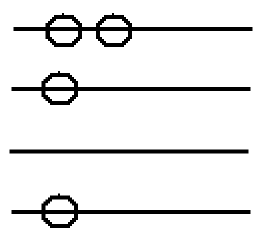
$$p(V-b) = RTe^{-a/RTV}$$

Find the critical constants pc, Vc and Tc in this model of a weakly interacting gas. This equation of state was proposed to account for the interaction of gas atoms with walls.

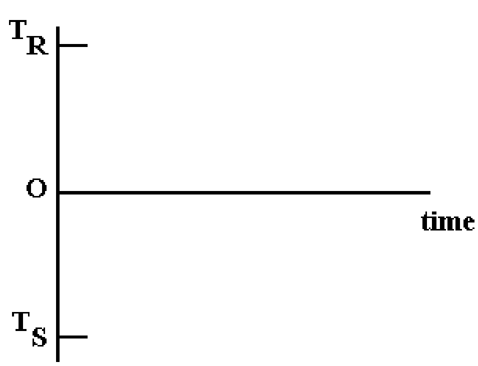
14. a. Compare the 4 level systems below. More than one particle may occupy a level. Which system has
highest temperature ________ lowest temperature ________
lowest specific heat _______ highest entropy_________



b. Consider the system below. Explain why it can be thought of as having a <u>negative</u> temperature. (Hint: Consider the Boltzmann factor $e^{-E/kT}$).



c. If this system (of part (b)) is brought into contact with a large reservoir at temperature TR (positive) draw a graph indicating how its temperature will change as a function of time as it comes to equilibrium with the reservoir. (Take the initial temperature of the system to be TS a negative number).



d. Explain why no problems with the third law are encountered in part (c).

15. Consider a three dimensional cubic lattice of N atoms arranged in a box of side L. The modes of this system are called phonons.

a. What is the total number of modes of this system? What would the total number have been if we were thinking of photons in a 3D box?
b. What is the maximum frequency that the phonons can have in the Debye model?
c. Calculate the total thermal energy of the phonons in the Debye model. Your answer may involve an integral.

$$\int_0^{x_D} dx \frac{x^3}{e^x - 1}$$

d. Calculate the low temperature heat capacity of the lattice in the Debye model.

16. Equipartition. A classical harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{Kq^2}{2}$$

is in thermal contact with a heat bath at temperature T. Calculate the partition function for the oscillator in the canonical ensemble and show explicitly that

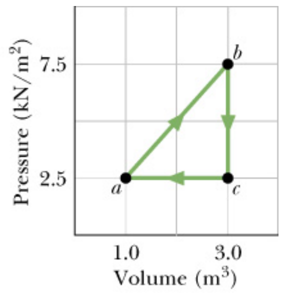
$$\langle E \rangle = k_B T, \quad \text{and}$$
$$\left\langle \left(E - \langle E \rangle\right)^2 \right\rangle = \left(k_B T\right)^2.$$

17. For a gas of molecules with diameter d, number density n and at a temperature T, find

a. the mean free path,
b. their average speed, and
c. the pressure of the gas using kinetic arguments.

18. A sample of ideal gas is taken through the cyclic process *abca* shown in the figure. At point *a*, *T = 300 K*.

a. What are the temperatures of the gas at points *b* and *c*?



b. Complete the table by inserting a plus sign, a minus sign, or a zero in each indicated cell. Note that Q is positive when heat is absorbed by the gas and W is positive when work is done by the gas. ΔE is the change in internal energy of the gas.

	Q	W	ΔE
$a \rightarrow b$			
$b \rightarrow c$			
$c \rightarrow a$			

19. A quantum harmonic oscillator has energy levels $E_n = (n + 1/2)\hbar\omega_0$; n = 0, 1, 2,... Treat this single oscillator to be a small system coupled to a heat bath at temperature T. What is the probability then of finding the oscillator in its nth quantum state?

20. A 1.00 kg block of ice at -20.0ºC is left to melt in the ocean, which is at +30.0ºC. The specific heat of ice is 2220 J/kg·K, of water is 4186 J/kg·K, and the heat of fusion of water is 333 kJ/kg.

a. Calculate the entropy change of the ocean.
b. Calculate the total entropy change of (ocean plus block of ice).