Fall 2022 Departmental Exams

Quantum Mechanics Study Guide

1. Let
$$|\psi\rangle = \sqrt{\frac{2}{5}} Y_{11}(\theta,\phi) \left|\frac{1}{2}\right\rangle - i\sqrt{\frac{3}{5}} Y_{32}(\theta,\phi) \left|-\frac{1}{2}\right\rangle,$$
where $\left|\pm\frac{1}{2}\right\rangle$ are the spin-up and spin-down states and Y_{lm} are spherical harmonics.
   (a) If you measure S_z, what is the probability that you will get $\hbar/2$?
   (b) If you do get $\hbar/2$ in (a), what is the new state after the measurement?
   (c) If you measure L_z after (b), what is the probability that you will get $-\hbar$?

2. As a mechanism for downward transitions from excited states of atoms and molecules, spontaneous emission competes with thermally stimulated emission (stimulated emission for which blackbody emission is the source). Show that at room temperature thermal stimulation dominates for transition frequencies well below 5 THz, whereas spontaneous emission dominates for frequencies well above 5 THz.

   Planck's formula for the energy density of thermal radiation is
$$\rho(\omega) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\hbar\omega/k_B T} - 1} \tag{1}$$
   and Einstein's A and B coefficients are related as $A = \dfrac{\omega_0^3 \hbar}{\pi^2 c^3} B$.

3. Consider a simple harmonic oscillator in one dimension. The Hamiltonian is
$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}.$$
   At time $t = 0$, the wavefunction is
$$\psi(x,0) = \frac{3}{5}\psi_0(x) + \frac{4}{5}\psi_2(x),$$
   where $\psi_n(x)$ denotes the eigenstate of the oscillator with energy $E_n = \hbar\omega(n + 1/2)$.
   (a) Write $\psi(x,t)$ for $t \geq 0$ in terms of $\psi_0(x)$ and $\psi_2(x)$.
   (b) What is the parity of $\psi(x,t)$? Does it change with time?
   (c) Find the expectation value of the energy. Does it change with time?
   (d) Find $\langle\psi|x|\psi\rangle$ for all times.

4. The result of a measurement shows that the electron spin is along the $+x$ direction at $t = 0$. For $t > 0$, the electron enters in a uniform magnetic field that is parallel to the $+z$ direction. Calculate the quantum mechanical probability as a function of time for finding the electron in each of the following states:

   (a) $S_x = 1/2$

   (b) $S_x = -1/2$

   (c) $S_z = 1/2$

   (d) $S_z = -1/2$

   The Pauli matrices are

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

   and the magnetic moment $\vec{\mu}_B$ of an electron is proportional to its spin ($\vec{\mu}_B = \gamma \vec{S}$, where γ is a constant called the gyromagnetic ratio).

5. Consider a helium-like ion. The two electrons are in the $2p$ and $3p$ states. The energy levels will have definite total angular momentum ($\vec{J} = \vec{L} + \vec{S}$).

   (a) What J-values can occur and how many energy levels of each J can there be?

   (b) How do your answers change if the electrons are both in $3p$ states?

6. Consider a physical system that has two stationary states $|1\rangle$ and $|2\rangle$, with energies $E_1 < E_2$ and Hamiltonian H_0

$$H_0 \, |i\rangle = E_i \, |i\rangle \qquad i = 0, \, 1.$$

   The system is modified by a time-independent, **real-valued** perturbation V, and the modified eigenenergies and stationary states are given by

$$(H_0 + V) \, |\psi_\pm\rangle = E_\pm \, |\psi_\pm\rangle \, .$$

   (a) Assuming $\langle 1|V|1\rangle = \langle 2|V|2\rangle = 0$ and $\langle 1|V|2\rangle = V_{12}$, deteremine the energy eigenvalues $E_\pm$ and normalized eigenstates $\psi_\pm$ of the perturbed system.

   (b) See what happens to $E_\pm$ as $V_{12} \to 0$; does the result make sense? Explain!

   (c) Find the normalized eigenfunctions in terms of E_1, E_2, and V_{12}.

7. A pair of electrons occupies two $3p$ levels (they are in the $3p^2$ configuration).

   (a) Using LS-coupling, what terms written as ${}^{2S+1}L_J$ are possible for the two electrons if the Pauli exclusion principle is not applied?

   (b) Which terms are possible after the Pauli exclusion principle is applied? Explain.

8. The potential for the isotropic harmonic oscillator (in 3D) is given by $V(r) = m\omega^2 r^2/2$.

   (a) Show that the eigenfunctions of the Hamiltonian can be expressed in the form $R_{nl}Y_{lm}$. In spherical coordinates, the Laplacian (acting on placeholder t) is

$$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial^2\phi}$$

   (b) The problem is separable in Cartesian coordinates. Use this to show that the eigenenergies can be expressed as $E_n = (n + \frac{3}{2})\hbar\omega$. You may use the 1D harmonic oscillator eigenenergy expression without derivation.

   (c) Find the degeneracy of the lowest three energy eigenvalues.

9. In non-relativistic mechanics, the relation between the kinetic energy T and the momentum p of a particle is $T = p^2/2m$. This relation is modified by the theory of relativity, but for sufficiently low speeds, it is sufficient to keep the lowest order correction and T can be written as

$$T = \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} \tag{2}$$

   The second term acts a perturbation to the non-relativistic Hamiltonian (since $H = T + V$). Use time-independent perturation theory to determine the lowest order correction to the energy levels of the one-dimensional simple harmic oscillatordue to this perturbation.

   Hint: You do not need the wave functions of the harmonic oscillator. Use the ladder operators

$$\hat{a}_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(\mp i\hat{p} + m\omega\hat{x}) \tag{3}$$

   and recall that ($|n\rangle$ are the eigenstates of the non-relativistic simple harmonic oscillator)

   (a) the ladder operators act on $|n\rangle$ as

$$\hat{a}_+ |n\rangle = \sqrt{n+1}\, |n+1\rangle \tag{4}$$
$$\hat{a}_- |n\rangle = \sqrt{n}\, |n-1\rangle \tag{5}$$

   (b) The commutator of the ladder operators is $[\hat{a}_-, \hat{a}_+] = 1$

   (c) The non-relativistic simple harmonic oscillator Hamiltonian is $H = \hbar\omega(\hat{a}_+\hat{a}_- + \frac{1}{2})$

10. Neutral K-mesons are created in one of two states: $|K^0\rangle$ or $|\bar{K}^0\rangle$. These states are **not** eigenstates of the Hamiltonian H that describes the system. Instead

$$H|K^0\rangle = m|K^0\rangle + \epsilon|\bar{K}^0\rangle \tag{6}$$
$$H|\bar{K}^0\rangle = \epsilon|K^0\rangle + m|\bar{K}^0\rangle \tag{7}$$

Show that if the state $|K^0\rangle$ is created at time t=0, then the probability that the state is $|K^0\rangle$ at a later time oscillates between 1 and 0.

11. Consider two non-interacting spin-1/2 particles described by the Hamiltonian for the 2D infinite-square-well potential

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(x_1) + V(x_2) \tag{8}$$

where $V(x_i) = \infty$ for $x_i < 0$ and $x_i > a$ and $V(x_i) = 0$ for $0 < x_i < a$, with $i = 1, 2$. Except for part (c) below, assume that the electrons are in the spin state in which the total spin S is zero.

(a) Derive the eigenfunctions and eigenenergies of this Hamiltonian.

(b) Find the energy and wavefunction of the ground state.

(c) Find the energy and wavefunction of the lowest state with $S = 1$.

(d) Show that the energy of the second $S = 0$ state is the same as in part (c). Find the wavefunction of this state.

12. A particle is in the ground state of an infinite square well ($V(x) = 0$ for $0 \le x \le L$, and $V(x) = \infty$ everywhere else). At $t = 0$, the wall at $x = L$ is *suddenly* moved to $x = 2L$.

(a) Determine the eigenstates and eigenvalues of the Hamiltonian before $t = 0$.

(b) Calculate the probability that, long time after $t = 0$, the system is in the ground state of the new potential.

(c) Identify the *lowest excited state* that has a non-zero probability of being occupied long after $t = 0$.
Hint: $\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$

13. Calculate the reflection coefficient for a particle of mass m incident on a finite square well of depth V_0 and width a. For what value(s) of incident energy is the reflection coefficient zero?

14. In an experiment you make repeated measurements of the energy of a system. You find that for one-fourth of the measurements you obtain the value E_1; for one-third of the measurements you obtain the energy value E_2; and for the remaining measurements you obtain the value E_3.

    (a) Using only this information write as complete a time dependent wave function for this state as you can. For parts of the wave function that you cannot specify use general functions and indicate why you cannot specify it (them).

    (b) Is this wave function uniquely determined by the experimental evidence? If so, how? If not, are there further experiments which you could do to make a unique determination?

15. Let $|\alpha\rangle$ be a state of the harmonic oscillator such that

$$\hat{a}_- |\alpha\rangle = \alpha |\alpha\rangle ,$$

where $\hat{a}_-$ is the lowering operator (its Hermitian conjugate, $\hat{a}_+$, is the raising operator) and α is a *complex* number. $|\alpha\rangle$ is called a *coherent state* of the harmonic oscillator.

    (a) Calculate the expectation values $\langle x\rangle$, $\langle x^2\rangle$, , $\langle p\rangle$ and $\langle p^2\rangle$ for the state $|\alpha\rangle$.

    (b) Show that $\sigma_x\sigma_p = \hbar/2$. That is, the uncertainty product takes its smallest possible value. The coherent state $|\alpha\rangle$ is a *minimum uncertainty* state.

Hint: The ladder operators are

$$\hat{a}_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(\mp i\hat{p} + m\omega\hat{x}). \tag{9}$$

Recall that ($|n\rangle$ are the eigenstates of the non-relativistic simple harmonic oscillator)

    (a) the ladder operators act on $|n\rangle$ as

$$\hat{a}_+ |n\rangle = \sqrt{n+1}\ |n+1\rangle \tag{10}$$
$$\hat{a}_- |n\rangle = \sqrt{n}\ |n-1\rangle \tag{11}$$

    (b) The commutator of the ladder operators is $[\hat{a}_-, \hat{a}_+] = 1$

    (c) The non-relativistic simple harmonic oscillator Hamiltonian is $H = \hbar\omega(\hat{a}_+\hat{a}_- + \frac{1}{2})$

16. A two-dimensional harmonic oscillator has the potential

$$V(x, y) = \frac{1}{2}m\omega^2(x^2 + 4y^2).$$

    (a) Calculate the energies of the first three lowest states, and identify the degrees of degeneracy for each energy.

    (b) If there is an additional small coupling term $W(x, y) = axy$ present, where a is a small constant. Calculate the first-order correction to the energy of each of the three states.

17. Consider particles of energy E and mass m incident from the left striking a δ-function potential at $x = 0$ given by

$$V(x) = \frac{\hbar^2 \Omega}{m}\delta(x),$$

where Ω is a positive constant.

   (a) Obtain the general forms of the wave functions for $x < 0$ and $x > 0$.

   (b) What are the boundary conditions for the wave function and its first derivative at $x = 0$?

   (c) Obtain expressions for the reflection and transmission coefficients.

18. Matrix representations of angular momentum operators for $L = 1$ are

$$\hat{L}_x = \hbar \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \hat{L}_y = \hbar \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \qquad \hat{L}_z = \hbar \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

   (a) The system is prepared in the state with $L_z = 1$ and then L_x is measured. Determine the possible outcomes and their probabilities, and the expectation value of the measurement.

   (b) Conside the state

$$|\psi\rangle = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{bmatrix} \tag{12}$$

   in the L_z basis.

      i. If L_z^2 is measured and the result is $+1$, what is the state after this measurement? How probable is the result?

      ii. If L_z is measured, what are the possible outcomes and respective probabilities?

19. Suppose we put a delta-function bump in the center of the infinite square well ($V(x)$ is zero for $0 < x < a$, and infinite everywhere else):

$$H' = \alpha\delta(x - a/2) \tag{13}$$

   (a) Find the eigenstates and eigenvalues of the unperturbed Hamiltonian. Leave the normalization constants as A_n, where n labels the states.

   (b) Find the first order correction to the allowed energies. Are there any states for which this correction in zero?

   (c) Using

$$\psi_n^1 = \sum_{m \neq n} \frac{\langle \psi_m^0 | H' | \psi_n^0 \rangle}{E_n^0 - E_m^0} \psi_m^0 \tag{14}$$

   find the first three non-zero terms in the expansion of the correction to the ground state wavefunction.

20. Any state $|\psi\rangle$ of a two-level system can be represented as $|\psi\rangle = a|0\rangle + b|1\rangle$, where a and b are complex numbers and $|0\rangle$ and $|1\rangle$ are an orthonormal basis in the Hilbert space of the system. Note that $c|\psi\rangle$, where c is any finite complex number (other than 0) represents the same state in the sense that the computed outcomes of all measurements (possible values obtained and their probabilities) are identical for $|\psi\rangle$ and $c|\psi\rangle$. We can choose the complex number c to normalize the vector and to set its phase.

   (a) Show that any arbitrary state of the system can be written in the form

   $$|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)\mathrm{e}^{i\phi}|1\rangle,$$

   where $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$.

   (b) Using a three dimensional Cartesian coordinate system (x, y, z) and the spherical coordinates θ, ϕ, show on a figure where the following states lie on a unit sphere:

      i. $|0\rangle$ and $|1\rangle$
      ii. $|+\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$ and $|-\rangle = 1/\sqrt{2}(|0\rangle - |1\rangle)$
      iii. $|\mathbf{i}\rangle = 1/\sqrt{2}(|0\rangle + i|1\rangle)$ and $|-\mathbf{i}\rangle = 1/\sqrt{2}(|0\rangle - i|1\rangle)$

   (c) Show that any pair of diametrically opposite points on the unit sphere represent orthonormal states and therefore can serve as a basis.

   (d) Start with an arbitrary point $(\theta,\ \phi)$ on the unit sphere and describe the geometric effect of applying the following operators. (*Hint: Start with* $\cos(\theta/2)|0\rangle + \sin(\theta/2)\mathrm{e}^{i\phi}|1\rangle$, *apply the transform and the rewrite the transformed vector in the same form, but with* θ', ϕ'. *How is this new point on the sphere related to the old one, in terms of their locations on the sphere?*)

      i. $\hat{Z} = |0\rangle\langle 0| - |1\rangle\langle 1|$
      ii. $\hat{X} = |1\rangle\langle 0| + |0\rangle\langle 1|$
      iii. $\hat{Y} = |1\rangle\langle 0| - |0\rangle\langle 1|$

   In optics, the sphere represents all possible polarizations of a transverse electromagnetic wave and is called the Poincaré sphere, and in quantum optics it represents all possible states of a two-level system and is called the Bloch sphere.