Appendix A

Adams-Bashforth-Moulton 4th Order Predictor-Corrector Method

The numerical integration method used for the driven ferromagnet and the antiferromagnet kink simulations is briefly described here. It is a standard 4th order (in time) method for integrating coupled ordinary diferential equations from some initial time t_0 to some final time t_1.

Let the variable y(t) be the unknown solution of a first order differential equation,

$$\frac{dy}{dt} = f(y(t), t) \quad . \tag{A-1}$$

Generally y can be assumed to be a vector function, as in these magnet simulations. For a system of N spins, y will be a vector with 3N components. The integration scheme necessarily requires the time to be discretized in units of some time step h, and solutions for y will be produced at times $t_n = nh$. The solutions at these times are denoted $y_n = y(t_n)$. Similarly the given function f at these times is denoted

$$f_n = f(y_n, t_n) \quad . \tag{A-2}$$

The method estimates y_{n+1} from assumed already known values of y at the four previous output times. It is a three step method, consisting of a predictor step, a corrector step and a mop-up step, as follows:

Predictor

$$y_p = y_n + h(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3})/24 \tag{A-3}$$

Corrector

$$y_c = y_n + h(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2})/24 \tag{A-4}$$

where

$$f_{n+1} = f(y_p, t_n + h) \tag{A-5}$$

is the predicted derivative at the next time step.

Mop-up

$$y_{n+1} = y_c + \frac{19}{270}(y_p - y_c) \tag{A-6}$$

The mop-up step is designed to minimize the errors introduced by the predictor and corrector steps. Since the solution at four previous times is necessary to obtain the solution at the next output time, the method is not self-starting. Therefore a one-step fourth order Runge-Kutta method has been used on the first four time steps to initiate the integration. This Runge-Kutta method is as follows:

Calculate

$$k_1 = f(y_n, t_n)$$

$$k_2 = f(y_n + \tfrac{1}{2}k_1, t_n + \tfrac{1}{2}h)$$

$$k_3 = f(y_n + \tfrac{1}{2}k_2, t_n + \tfrac{1}{2}h)$$

$$k_4 = f(y_n + k_3, t_n + h) \quad , \tag{A-7}$$

Then

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad . \tag{A-8}$$

See Ceschino and Kuntzmann (1966) for further details on these methods.

Appendix B

Evaluation of the Lagrangian for the Antiferromagnet Ansatz

First the evaluation of the Hamiltonian is sketched out. Hyperbolic tangent and secant functions occur frequently so a shorthand notation is used,

$$s = \text{sech } x$$

$$t = \tanh x \qquad \text{(B-1)}$$

where

$$x = (z-vt)/w \quad . \qquad \text{(B-2)}$$

To begin the Ansatz is written out. The factors f_A and f_B (equation 6-23) can be approximated as

$$f_A(x) = 1 + \frac{\beta^2}{32} \cos^2\theta_A \, s^2$$

$$f_B(x) = 1 + \frac{\beta^2}{32} \cos^2\theta_B \, s^2 \quad . \qquad \text{(B-3)}$$

Fourth order and higher terms in β are dropped. Then

$$\sigma_{Ax'} = (\tfrac{1}{4}\beta \cos\theta_A \cdot t^2 + r_N s) f_A \quad , \quad \sigma_{Bx''} = (\tfrac{1}{4}\beta \cos\theta_B \cdot t^2 - r_N s) f_B$$

$$\sigma_{Ay'} = (-\tfrac{1}{4}\beta \cos\theta_A \cdot st + r_N t) f_A \quad , \quad \sigma_{By''} = (-\tfrac{1}{4}\beta \cos\theta_B \cdot st - r_N t) f_B$$

$$\sigma_{Az'} = -\tfrac{1}{4}\beta \sin\theta_A \quad , \qquad \sigma_{Bz''} = -\tfrac{1}{4}\beta \sin\theta_B \quad . \qquad \text{(B-4)}$$

These are next rotated to the original xyz coordinate system, for instance, for the A sublattice,

$$y = x'\cos\theta_A - z' \sin\theta_A$$

$$y = y'$$

$$x = x' \sin\theta_A + z' \cos\theta_A \quad . \tag{B-5}$$

xyz represent spin components of $\hat{\sigma}_A$. Applying these rotations leads to

$$\sigma_{Ax} = \frac{1}{4}\beta + s\Gamma_A \cos\theta_A$$

$$\sigma_{Ay} = t\Gamma_A$$

$$\sigma_{Az} = s\Gamma_A \sin\theta_A$$

$$\sigma_{bx} = \frac{1}{4}\beta + s\Gamma_B \cos\theta_B$$

$$\sigma_{By} = t\Gamma_B$$

$$\sigma_{Bz} = s\Gamma_B \sin\theta_B \tag{B-6}$$

where

$$\Gamma_A = r_N - \frac{1}{4}\beta \cos\theta_A s + r_N \frac{\beta^2}{32} \cos^2\theta_A \, s^2$$

$$\Gamma_B = -r_N - \frac{1}{4}\beta \cos\theta_B s - r_N \frac{\beta^2}{32} \cos^2\theta_B \, s^2 \quad . \tag{B-7}$$

This is the Ansatz as written in the original coordinate system. Note that $\hat{\sigma}_B$ can be obtained from $\hat{\sigma}_A$ by changing $r_N \to -r_N$ and $\theta_A \to \theta_B$.

The Hamiltonian, equation (6-26c), is the sum of exchange, anisotropy and applied field terms,

$$E_{exchange} = \int_{-\infty}^{\infty} dz(\hat{\sigma}_A \cdot \hat{\sigma}_B - \frac{1}{2}\frac{\partial\hat{\sigma}_A}{\partial z}\cdot\frac{\partial\hat{\sigma}_B}{\partial z})$$

$$E_{anisotropy} = \int_{-\infty}^{\infty} dz\ \frac{1}{2}\alpha(\sigma_{Az}^2 + \sigma_{Bz}^2)$$

$$E_{field} = -\int_{-\infty}^{\infty} dz\ \frac{1}{2}\beta(\sigma_{Ax} + \sigma_{Bx}) \quad . \qquad (B-8)$$

Evaluating these using (B-6) is straightforward; the two contributions to the exchange energy are

$$\int dz\ \hat{\sigma}_A \cdot \hat{\sigma}_B = w\{\frac{1}{2}\pi\beta r_N(\cos\theta_A - \cos\theta_B) - \frac{\beta^2}{16}(3\cos^2\theta_A - 2\cos\theta_A\cos\theta_B + 3\cos^2\theta_B)$$

$$+ [1 - \cos(\theta_A - \theta_B)][2r_N^2 - \frac{\pi}{8}\beta r_N(\cos\theta_A - \cos\theta_B) + \frac{\beta^2}{24}(\cos\theta_A - \cos\theta_B)^2]\} \quad (B-9)$$

$$-\int dz\ \frac{1}{2}\frac{\partial\hat{\sigma}_A}{\partial_z}\cdot\frac{\partial\hat{\sigma}_B}{\partial_z} =$$

$$= \frac{1}{w}\{-\frac{\beta^2}{48}\cos\theta_A\cos\theta_B + [r_N^2 - \frac{\pi}{16}\beta r_N(\cos\theta_A - \cos\theta_B) + \frac{\beta^2}{48}(\cos\theta_A - \cos\theta_B)^2]\cos(\theta_A - \theta_B)$$

$$+ [1 - \cos(\theta_A - \theta_B)][\frac{2}{3}r_N^2 - \frac{\pi}{32}\beta r_N(\cos\theta_A - \cos\theta_B) + \frac{\beta^2}{120}(\cos^2\theta_A - \cos\theta_A\cos\theta_B + \cos^2\theta_B]\}$$

(B-10)

The anistropy contribution is much simpler,

$$E_{anisotropy} = w\{\frac{1}{4}\alpha\ \sin^2\theta_A(2r_N^2 - \frac{\pi}{4}\beta r_N\cos\theta_A + \frac{\beta^2}{6}\cos^2\theta_A)$$

$$+ \frac{1}{4}\alpha\ \sin^2\theta_B(2r_N^2 + \frac{\pi}{4}\beta r_N\cos\theta_B + \frac{\beta^2}{6}\cos^2\theta_B)\} \quad . \qquad (B-11)$$

Finally the magnetic field energy is the simplest

$$E_{field} = w\cdot\frac{1}{2}\beta[\frac{1}{2}\beta(\cos^2\theta_A + \cos^2\theta_B) - \pi r_N(\cos\theta_A - \cos\theta_B)] \quad . \qquad \text{(B-12)}$$

Note that the last term in E_{field} cancels the first term in (B-9). The first term in E_{field} cancels the first term in (B-9). The first term in E_{field} can be combined with the second term in (B-9) to simplify things somewhat. Also note that terms cubic in β have been dropped. The functions F and G in equation (6-28) are found from these expresssions by putting $\theta_B = \theta_A + \Delta$, retaining terms up to quadratic order in Δ. The coefficients of the powers of Δ were then approximated to leading order in β and $\sqrt{\alpha}$.

Next, consider the kinetic integral K, in particular the contribution from the A sublattice first,

$$K_A = \frac{1}{2} \int dz\ \sigma_{Az} \frac{d}{dt} \tan^{-1} \frac{\sigma_{Ay}}{\sigma_{Ax}} = -\frac{1}{2}v \int dx\ \sigma_{Az} \frac{d}{dx} \tan^{-1} \frac{\sigma_{Ay}}{\sigma_{Ax}} \quad , \qquad \text{(B-13)}$$

and consider a slightly different integral, by changing $\sigma_{Az} \rightarrow \sigma_{Az} - 1$,

$$K_A' = -\frac{1}{2}v \int dx(\sigma_{Az}-1)\frac{d}{dx} \tan^{-1} \frac{\sigma_{Ay}}{\sigma_{Ax}} = K_A + \frac{1}{2}v \tan^{-1} \frac{\sigma_{Ay}}{\sigma_{Ax}} \Bigg|_{x=-\infty}^{x=\infty} \qquad \text{(B-14)}$$

$$= K_A + v \cos^{-1}(\frac{1}{4}\beta) \quad . \qquad \text{(B-14)}$$

K_A' therefore differs from K_A by an additive constant which is independent of the variational parameters, and therefore irrelevant. Thus we compute K_A' instead, since it can be written

$$K' = \frac{1}{2}v \int dx \ \frac{\sigma_x\dot{\sigma}_y - \sigma_y\dot{\sigma}_x}{1 + \sigma_z} = vP' \quad . \qquad \text{(B-15)}$$

The A subscripts have been dropped for simplicity. Since other terms in the Lagrangian have been evaluated to quadratic order in β, the same should be done for P'. This is done by systematically expanding $(1 + \sigma_z)^{-1}$.

First, the numerator of the integrand is

$$\sigma_x\dot{\sigma}_y - \sigma_y\dot{\sigma}_x = \cos\theta\cdot s - \tfrac{1}{4}\beta\ell\,\cos 2\theta\cdot s^2 - \tfrac{1}{8}\beta^2\cos\theta\,\sin^2\theta\cdot s^3 \quad , \qquad \text{(B-16a)}$$

while the denominator is

$$1 + \sigma_z = 1 + r_N\sin\theta\cdot s - \tfrac{1}{4}\beta\sin\theta\,\cos\theta\cdot s^2 + r_N\,\frac{\beta^2}{32}\,\sin\theta\,\cos^2\theta\cdot s^3$$

$$= 1 + as + bs^2 + c(s) \quad , \qquad \text{(B-16b)}$$

where

$$a = \ell\,\sin\theta$$

$$b = -\tfrac{1}{4}\beta\,\sin\theta\,\cos\theta$$

$$c(s) = \ell\,\frac{\beta^2}{32}\,\sin\theta\,(s^3\,\cos^2\theta - s) \qquad \text{(B-16c)}$$

Here $\ell = +1$ for the A sublattice and $\ell = -1$ for the B sublattice. Then these expressions apply to both sublattices, with $\theta = \theta_A$ or $\theta = \theta_B$. Now expand $(1 + \sigma_z)^{-1}$ as

$$(1 + \sigma_z)^{-1} = \frac{1}{1+as}\;\; 1 - \frac{bs^2}{1+as} - \frac{c(s)}{1+as} + \;\frac{bs^2}{1+as}^{\;2} + \ldots \quad . \qquad \text{(B-17)}$$

The leading order term in P' is independent of β,

$$P'_0 = \frac{1}{2} \int dx \, \frac{\cos\theta \cdot s}{1+as} = \frac{1}{2} \cos\theta \, \frac{1}{\sqrt{1-\sin^2\theta}} \quad \pi - 2 \tan^{-1} \frac{\ell \sin\theta}{\sqrt{1-\sin^2\theta}}$$

$$\equiv \frac{1}{2} f(\theta) \quad . \tag{B-18}$$

We consider this result separately on the two sublattices. For the A sublattice, $\ell = +1$, $\theta = \theta_A$, and we get

$$P'_{0A} = \frac{1}{2} f(\theta_A) \quad . \tag{B-19}$$

For the B sublattice, $\ell = -1$, $\theta = \theta_B$; we get

$$P'_{0B} = \frac{1}{2} f(-\theta_B) \quad . \tag{B-20}$$

Careful examination shows that $f(\theta)$ has a discontinuity at $\theta = -\frac{1}{2}\pi$. Generally, one has

$$f(\theta) = \pi - 2\theta \quad \text{for} \quad -\frac{\pi}{2} < \theta < \pi$$

$$f(\theta) = -3\pi - 2\theta \quad \text{for} \quad -\pi < \theta < -\frac{\pi}{2} \quad . \tag{B-21}$$

Provided we always assume $0 < \theta_A < \pi$, there is no problem with P'_{0A}. However, P'_{0B} has a discontinuity at $\theta_B = \frac{1}{2}\pi$, and generally since we expect that $\theta_B \simeq \theta_A$, this may lead to problems. The simplest way to avoid the problem entirely is to move the discontinuity to $\theta_B = -\frac{1}{2}\pi$ by re-defining K'_B, and always assuming $0 < \theta_B < \pi$, as was assumed for θ_A. The new definition for K'_B is

$$K'_B = -\frac{1}{2} v \int dx (\sigma_{Bz}+1) \frac{d}{dx} \tan^{-1} \frac{\sigma_{By}}{\sigma_{Bx}} = K_B - \frac{1}{2} v \tan^{-1} \frac{\sigma_{By}}{\sigma_{Bx}} \Bigg|_{x=-\infty}^{x=\infty} \tag{B-22}$$

$$= K_B + v[\cos^{-1}(\tfrac{1}{4}\beta) - \pi] \quad , \tag{B-22}$$

or

$$K'_B = -\frac{1}{2}v \int dx \; \frac{\sigma_{Bx}\dot{\sigma}_{By} - \sigma_{By}\dot{\sigma}_{Bx}}{1 - \sigma_{Bz}} = vP'_B \quad . \tag{B-23}$$

Using this new definition, the leading term in P'_B is

$$P'_{0B} = -\frac{1}{2}f(\theta_B) \quad . \tag{B-24}$$

Therefore, the term in the total momentum P'_0 is

$$P'_0 = P'_{0A} + P'_{0B} = \frac{1}{2}f(\theta_A) - \frac{1}{2}f(\theta_B) = \theta_B - \theta_A \quad . \tag{B-25}$$

One continues systematically evaluating the order β and order β^2 contributions to K'_A and K'_B, using definitions (B-15) and (B-23). After some more tedious calculation of integrals, the order β terms are found to be

$$P'_{1A} = \frac{\pi}{8}\beta \sin\theta_A$$

$$P'_{1B} = \frac{\pi}{8}\beta \sin\theta_B \quad . \tag{B-26}$$

And finally, after <u>still</u> more tedious integration, one finds that the order β^2 terms sum to zero![1] So the final result for the kinetic integral is

$$K_{tot} = K'_A + K'_B = v[(\theta_B - \theta_A) + \frac{\pi}{8}\beta(\sin\theta_A + \sin\theta_B)] \quad . \tag{B-27}$$

Expanding θ_B around θ_A, $\theta_B = \theta_A + \Delta$, leads then to equation (6-28a).

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[1]This is assuming $\theta_A = \theta_B$, i.e., the leading order β^2 term is zero, while there may be nonzero terms proportional to $\beta^2\Delta$, $\beta^2\Delta^2$, etc., but these can be dropped. In any case this term would not involve Δ and would therefore be an irrelevant constant.

Appendix C

Matrix Elements for Two Spin-1 Models

#1. For the spin-1 operator $\hat{V}_{n,n+1}$ given by

$$\hat{V}_{n,n+1} = -J_x \hat{S}^x_n \hat{S}^x_{n+1} - J_y \hat{S}^y_n \hat{S}^y_{n+1} \quad , \tag{C-1}$$

matrix elements of $\exp(-\frac{\beta}{m}\hat{V}_{n,n+1})$ are desired, and can be obtained from equations (9-21) and (9-22). This can be done by using the matrix representations of the spin-1 operators, starting from the well-known $\hat{S}^x$ and $\hat{S}^y$ matrices,

$$\hat{S}^x = \frac{1}{2}(\hat{S}^+ + \hat{S}^-) \rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{C-2}$$

$$\hat{S}^y = \frac{1}{2i}(S^+ - S^-) \rightarrow \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

The operator products in Chapter 9 are obtained from these fundamental matrices by direct multiplication. For instance,

$$\hat{S}^x \hat{S}^x = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad . \tag{C-4}$$

We make the following definitions,

$$\hat{M} = -\frac{\beta}{m}\hat{V}_{n,n+1} \quad , \quad \hat{A} = e^{\hat{M}}$$

$$K_{x,y} = \frac{\beta}{m} J_{x,y} \quad , \quad K^2_{xy} = K^2_x + K^2_y \quad . \tag{C-5}$$

For arbitrary J_x and J_y, and using $\langle S_{n,r} S_{n+1,r} | \hat{A} | S_{n,r+1} S_{n+1,r+1} \rangle$ notation, the 41 nonzero matrix elements are as follows:

<u>25 from the even powers of $\hat{M}$</u>

$$\langle 00|\hat{A}|00\rangle = \cosh K_{xy}$$

$$\langle 01|\hat{A}|01\rangle = \langle 0\text{-}1|\hat{A}|0\text{-}1\rangle = \langle 10|\hat{A}|10\rangle = \langle \text{-}10|\hat{A}|\text{-}10\rangle$$

$$= \frac{1}{2}(\cosh K_x + \cosh K_y)$$

$$\langle 01|\hat{A}|0\text{-}1\rangle = \langle 0\text{-}1|\hat{A}|01\rangle = \langle 10|\hat{A}|\text{-}10\rangle = \langle \text{-}10|\hat{A}|10\rangle$$

$$= \frac{1}{2}(\cosh K_x - \cosh K_y)$$

$$\langle 11|\hat{A}|11\rangle = \langle \text{-}1\text{-}1|\hat{A}|\text{-}1\text{-}1\rangle = 1 + \frac{(K_x - K_y)^2}{4K_{xy}^2}(\cosh K_{xy} - 1)$$

$$\langle 1\text{-}1|\hat{A}|1\text{-}1\rangle = \langle \text{-}11|\hat{A}|\text{-}11\rangle = 1 + \frac{(K_x + K_y)^2}{4K_{xy}^2}(\cosh K_{xy} - 1)$$

$$\langle 11|\hat{A}|\text{-}1\text{-}1\rangle = \langle \text{-}1\text{-}1|\hat{A}|11\rangle = \frac{(K_x - K_y)^2}{4K_{xy}^2}(\cosh K_{xy} - 1)$$

$$\langle 1\text{-}1|\hat{A}|\text{-}11\rangle = \langle \text{-}11|\hat{A}|1\text{-}1\rangle = \frac{(K_x + K_y)^2}{4K_{xy}^2}(\cosh K_{xy} - 1)$$

$\langle 11|\hat{A}|1\text{-}1\rangle$ = four others with a single -1 =

$\langle \text{-}1\text{-}1|\hat{A}|\text{-}11\rangle$ = four others with a single +1

$$= \frac{1}{4K_{xy}^2}\left(K_y^2 \cosh K_x - K_x^2 \cosh K_y + (K_x^2 - K_y^2)\cosh K_{xy}\right)$$

16 from the odd powers of $\hat{M}$

$$<10|\hat{A}|01> = <-10|\hat{A}|0-1> = <01|\hat{A}|10> = <0-1|\hat{A}|-10>$$

$$= \frac{1}{2}(\sinh K_x + \sinh K_y)$$

$$<10|\hat{A}|0-1> = <-10|\hat{A}|01> = <01|\hat{A}|-10> = <0-1|\hat{A}|10>$$

$$= \frac{1}{2}(\sinh K_x - \sinh K_y)$$

$$<00|\hat{A}|1-1> = <00|\hat{A}|-11> = <1-1|\hat{A}|00> = <-11|\hat{A}|00>$$

$$= \frac{K_x+K_y}{2K_{xy}} \sinh K_{xy}$$

$$<00|\hat{A}|11> = <00|\hat{A}|-1-1> = <11|\hat{A}|00> = <-1-1|\hat{A}|00>$$

$$= \sinh K_x - \sinh K_y - \frac{(K_x-K_y)}{2K_{xy}} \sinh K_{xy} \quad .$$

#2. For the spin-1 operator $\hat{V}_{n,n+1}$ given by

$$\hat{V}_{n,n+1} = -J(\hat{S}^x_n\hat{S}^x_{n+1} + \hat{S}^y_n\hat{S}^y_{n+1} + \hat{S}^z_n\hat{S}^z_{n+1}) \quad , \qquad \text{(C-6)}$$

matrix elements of $\hat{A} = \exp\,(-\frac{\beta}{m}\hat{V}_{n,n+1})$ are desired. Using the method and notation as described in Sections 9.2 and 9.3, the following operator expression can be obtained, where $k = \frac{\beta J}{m}$,

$$\hat{A} = 1 + [x_1x_2 + y_1y_2 + (xy)_1(yx)_2 + (yx)_1(xy)_2]\sinh K$$

$$+ [x_1^2x_2^2 + y_1^2y_2^2 + (xyyx)_1(yxxy)_2 + (yxxy)_1(xyyx)_2](\cosh K - 1)$$

$$+ [(xy)_1(xy)_2 + (yx)_1(yx)_2] \cdot \frac{1}{3}(e^{-2K} - e^K)$$

$$+ (xxyy)_1(xxyy)_2 \ (1 - \tfrac{1}{3}e^{K} - e^{-K} + \tfrac{1}{3}e^{-2K})$$

$$+[(xyyx)_1(xyyx)_2 + (yxxy)_1(yxxy)_2] \ (\tfrac{1}{6}e^{K} - \tfrac{1}{2}e^{-K} + \tfrac{1}{3}e^{-2K})$$

$$-[(xxy)_1(xxy)_2 + (yyx)_1(yyx)_2 + (xyy)_1(xyy)_2 + (yxx)_1(yxx)_2]$$

$$(\tfrac{1}{6}e^{K} - \tfrac{1}{2}e^{-K} + \tfrac{1}{3}e^{-2K}) \quad . \qquad \text{(C-7)}$$

The nonzero matrix elements of $\hat{A}$ are as follows:

$$\langle 00|\hat{A}|00\rangle = \tfrac{1}{3}(2e^{K} + e^{-2K})$$

$$\langle 01|\hat{A}|01\rangle = \langle 0\text{-}1|\hat{A}|0\text{-}1\rangle = \langle 10|\hat{A}|10\rangle = \langle \text{-}10|\hat{A}|\text{-}10\rangle = \cosh K$$

$$\langle 01|\hat{A}|10\rangle = \langle 0\text{-}1|\hat{A}|\text{-}10\rangle = \langle 10|\hat{A}|01\rangle = \langle \text{-}10|\hat{A}|0\text{-}1\rangle = \sinh K$$

$$\langle 11|\hat{A}|11\rangle = \langle \text{-}1\text{-}1|\hat{A}|\text{-}1\text{-}1\rangle = e^{K}$$

$$\langle 1\text{-}1|\hat{A}|1\text{-}1\rangle = \langle \text{-}11|\hat{A}|\text{-}11\rangle = \tfrac{1}{2}[\tfrac{2}{3}(e^{-2K} + e^{-K}) + \tfrac{1}{3}(e^{K} + e^{-K})]$$

$$\langle 1\text{-}1|\hat{A}|\text{-}11\rangle = \langle \text{-}11|\hat{A}|1\text{-}1\rangle = \tfrac{1}{2}[\tfrac{2}{3}(e^{-2K} - e^{-K}) + \tfrac{1}{3}(e^{K} - e^{-K})]$$

$$\langle 00|\hat{A}|1\text{-}1\rangle = \langle 00|\hat{A}|\text{-}11\rangle = \langle 1\text{-}1|\hat{A}|00\rangle = \langle \text{-}11|\hat{A}|00\rangle = \tfrac{1}{3}(e^{K} - e^{-2K}) \quad .$$

The same vertices are nonzero in the isotropic limit of model #1.