Electrodynamics II Exam 2. Part A (130 pts.) Closed Book Relativistic Electrodynamics

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Instructions: Use CGS-Gaussian units. No derivations here, just state your responses clearly, and define your variables in words. Write on other side if needed.

1. (8) Make a concise statement of the *principle of relativity* (the first of two postulates used by Albert Einstein in 1905):

The Laws of Physics and results of all experiments are the same for all inertial reference frames.

2. (10) Inertial frame K' moves with velocity $\boldsymbol{\beta} = \beta\hat{z}$ with respect to inertial frame K. The coordinate axes of the two frames are parallel. Write out the Lorentz transformation that gives $t'x'y'z'$ in terms of $txyz$.

$$x_0' = ct' = \gamma(x_0 - \beta x_3) \qquad \text{where } x_0 = ct \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$
$$x_1' = x_1 \qquad x_1 = x$$
$$x_2' = x_2 \qquad x_2 = y$$
$$x_3' = \gamma(x_3 - \beta x_0) \qquad x_3 = z$$

3. (8) A rod of length L_0 along the $\hat{z}$ direction is at rest in frame K' of the previous question. Show how to use that transformation to get its length as measured in frame K.

The measurement in K is done at the same time in frame K, locating the ends of the rod and getting its length as $L = \Delta x_3$. Using the last eqn. in #2, with $\Delta x_0 = c\Delta t = 0$,

$$\Delta x_3' = \gamma\,\Delta x_3 \quad \text{or} \quad L_0 = \gamma L \Rightarrow L = \frac{L_0}{\gamma},$$ Lorentz contraction.

4. (8) Besides any space-time point $x = (ct, \mathbf{x})$, give two other examples of quantities that transform as 4-vectors. Show their time and space components (symbols) and state their meaning also in words.

4-momentum $p = (E/c, \vec{p})$, energy and momentum of a particle

4-velocity $U = \gamma(c, \vec{u})$, same as $U = \frac{dx}{d\tau}$, τ = proper time.

4-potential $A = (\Phi, \vec{A})$, scalar and vector potentials of EM field.

5. (8) An object moves in some reference frame K with a variable velocity $\mathbf{v}(t) = c\boldsymbol{\beta}(t)$. Write an expression giving the proper time change $\Delta\tau$ when the time in K evolves from t_1 to t_2.

From $ds^2 = c^2dt^2 - d\vec{x}^2 = c^2 d\tau^2$, and $d\vec{x} = c\vec{\beta}(t)\,dt$

$$\Delta\tau = \int_{\tau_1}^{\tau_2} d\tau = \int_{t_1}^{t_2} dt\sqrt{1-\left(\frac{d\vec{x}}{c\,dt}\right)^2} = \int_{t_1}^{t_2} dt\sqrt{1-\beta^2(t)}$$

6. (6) A particle of rest mass m has a 4-momentum $p = (E/c, \mathbf{p})$. With correct factors of c, what is the squared length $p \cdot p$ of this 4-vector?

$$p \cdot p = (E/c)^2 - \vec{p}^{\,2} = m^2 c^2, \qquad m = \text{rest mass}.$$

7. (8) A particle has energy E and 3-momentum $\mathbf{p} = p_z \hat{z}$ as measured in frame K. Another inertial frame K' moves with respect to K at relative velocity $\boldsymbol{\beta} = \beta\hat{z}$. What is the particle's energy in frame K'?

The 4-momentum $p = (E/c, \vec{p})$ transforms as does x in #2.
Since E/c is the time component, we have

$$E'/c = \gamma\,(E/c - \beta p_3) \qquad \text{where } p_3 = p_z.$$

8. (8) An arbitrary linear transformation between inertial frames K and K' can be written for 4-vectors in the form $x' = \mathsf{A}x$, where A is a 4×4 matrix.

a) (4) For arbitrary Lorentz transformations, what condition must A satisfy?

From $x' = Ax$ and $\tilde{x}' g x' = \tilde{x} g x \Rightarrow \tilde{A} g A = g$

or $(\det A)^2 = 1$ or $\det(A) = \pm 1$.

b) (4) What is the definition of a "proper Lorentz transformation"?

proper Lorentz transforms are continuous with the identity transformation, meaning they can be built from a sequence of infinitesimal changes.

9. (12) A proper Lorentz transformation written as $x' = \mathsf{A}x$ has a matrix expressed as $\mathsf{A} = e^{\mathsf{L}}$, where L is a traceless 4×4 real matrix. For the case of a boost at rapidity ζ along the $\hat{z}$ axis,

a) (4) Write out the matrix A.

b) (4) Write out the matrix L.

$$A = \begin{pmatrix} \cosh\zeta & 0 & 0 & -\sinh\zeta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\zeta & 0 & 0 & \cosh\zeta \end{pmatrix}, \quad \begin{matrix} \sinh\zeta = \beta\gamma \\ \cosh\zeta = \gamma \\ \text{from \#2.} \end{matrix} \qquad L = -\zeta K_3 = \begin{pmatrix} 0 & 0 & 0 & -\zeta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\zeta & 0 & 0 & 0 \end{pmatrix}$$

c) (4) Write out the matrix that is the generator of infinitesimal boosts along the $\hat{z}$ axis.

$$K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \text{Then } A = e^{L} = e^{-\zeta K_3}.$$

10. (10) Give the definition of the electromagnetic field tensor $F^{\alpha\beta}$. How do you get the electric field components from it?

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \qquad E_i = -F^{0i} = +F^{i0}, \quad i = 1,2,3$$

where $A^\alpha = (\Phi, \vec{A})$ i.e. the time-space components.

11. (8) In terms of the EM field tensor $F^{\alpha\beta}$, how do you write the covariant form of equation of motion for the 4-velocity U^α of a charged particle exposed to that field?

4-momentum is $p^\alpha = mU^\alpha$, as from $\vec{p} = \gamma m\vec{u}$, $U^\alpha = (E/c, \gamma\vec{u})$

Then $\frac{dp^\alpha}{d\tau} = m\frac{dU^\alpha}{d\tau} = \frac{e}{c}F^{\alpha\beta}U_\beta$ or $\frac{dU^\alpha}{d\tau} = \frac{e}{mc}F^{\alpha\beta}U_\beta$.

12. (8) Write out a relativistic Lagrangian L for a particle of mass m, charge e, interacting with a given electromagnetic field described by 4-potential A^α.

$$L = -mc^2\sqrt{1-\frac{u^2}{c^2}} - \frac{e}{\gamma c}U_\alpha A^\alpha = -mc^2\sqrt{1-\frac{u^2}{c^2}} - e\Phi + \frac{e}{c}\vec{u}\cdot\vec{A}$$

13. (6) If a particle of mass m, charge e has mechanical (or kinetic) momentum $\mathbf{p}$, how do you express its canonical momentum $\mathbf{P}$ when exposed to EM fields?

$$\vec{P} = \frac{\partial L}{\partial \vec{u}} = m\gamma\vec{u} + \frac{e}{c}\vec{A} = \vec{p} + \frac{e}{c}\vec{A}$$

by the derivative of L in #12.

14. (12) Based on the electromagnetic field tensor $F^{\alpha\beta}$ and the 4-current J^α,

a) (6) Write out a Lagrangian density $\mathcal{L}$ for electromagnetic fields produced by arbitrary J^α.

$$\mathcal{L} = \frac{-1}{16\pi}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{c}J_\alpha A^\alpha$$
$$= \frac{-1}{16\pi}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial^\alpha A^\beta - \partial^\beta A^\alpha) - \frac{1}{c}J_\alpha A^\alpha,$$ and use $A_\beta = g_{\beta\gamma}A^\gamma$, etc

b) (6) What are the associated inhomogeneous Maxwell's equations in their covariant form?

$\partial^\beta F_{\beta\alpha} = \frac{4\pi}{c}J_\alpha$, or other equiv. form such as $\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c}J^\beta$

↳ contraction on adjacent indices, a divergence.

15. (10) The EM field tensor $F^{\alpha\beta}$ and its dual $\mathcal{F}^{\alpha\beta}$ can be used to make two quantities that are invariant under proper Lorentz transformations. What are they? Also express them in terms of electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$.

$F_{\alpha\beta}F^{\alpha\beta} = 2(\vec{B}^2 - \vec{E}^2)$ (is a true scalar)

$F_{\alpha\beta}\mathcal{F}^{\alpha\beta} = -4\vec{E}\cdot\vec{B}$ (is a pseudoscalar)

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Instructions: Please show the details of your derivations here. Explain your reasoning for full credit. Open-book only. Do both problems.

1. (50) A moving electron (rest mass energy $mc^2 = 0.511$ MeV) collides with a positron initially at rest in the lab. The collision converts the electron-positron pair into a muon-antimuon ($\mu\bar{\mu}$) pair, with rest mass energies of 140 MeV each. Ignore any neutrinos that may be required to satisfy all conservation laws of particle physics.

   a) (10) How large is the threshold total energy required for this process, as measured in the center of momentum frame?

   b) (20) Calculate the energy E_1 (in MeV) of the incident electron at threshold, in the lab frame.

   c) (20) At threshold, how fast is the center of momentum frame moving with respect to the lab?

a) $e + \bar{e} \rightarrow \mu + \bar{\mu}$ $\quad M = 140\ \text{MeV}/c^2$.

In the center of momentum frame, at threshold, the $\mu\bar{\mu}$ pair is created at rest. Then the total energy is the total rest energy,

$W' = 2 \times 140\ \text{MeV} = 280\ \text{MeV}.$

b) $\overset{\circ}{e} \longrightarrow E_1, \vec{p}_1 \quad \overset{\circ}{\bar{e}},\ E_2 = mc^2,\ \vec{p}_2 = 0$

Lab frame

$\underset{\vec{p}_1'}{\xrightarrow{E_1'}} \quad \underset{\vec{p}_2'}{\xleftarrow{E_2'}}$ Before collision.

Center of momentum frame, with $\vec{p}_1' + \vec{p}_2' = 0$

$E_1'' = E_2'' = Mc^2$.

$\vec{p}_1'' = \vec{p}_2'' = 0$ After collision.

c.o.m. frame, $\mu\bar{\mu}$ pair

or $p_1'' = (Mc, 0)$

$p_2'' = (Mc, 0)$

(no 3-momentum)

For calculations, the squared 4-momentum is conserved, and also is an invariant between different frames.

$$(p_1 + p_2)^2 = (p_1'' + p_2'')^2$$

$$p_1^2 + p_2^2 + 2p_1 \cdot p_2 = (2Mc)^2$$

zero.

$$m^2c^2 + m^2c^2 + 2(E_1E_2/c^2 - \vec{p}_1 \cdot \vec{p}_2) = 4M^2c^2$$

$$2m^2c^2 + 2E_1 m = 4M^2c^2$$

b) continued

$$E_1 = \frac{2M^2c^2 - m^2c^2}{m} = \frac{2M^2c^4 - m^2c^4}{mc^2} = \frac{2(140)^2 - (0.511)^2}{0.511} = 76700 \text{ MeV}$$

A very large energy is needed because of the large mass change.

c) Transform the electron and positron momenta to c.o.m. frame, as 4-vectors.

$$\left.\begin{aligned} p'_{1x} &= \gamma(p_{1x} - \beta E_1/c) \\ p'_{2x} &= \gamma(\underbrace{p_{2x}}_{\text{zero}} - \beta \underbrace{E_2}_{mc^2}/c) \end{aligned}\right\} \text{add} \Rightarrow p'_x = p'_{1x} + p'_{2x} = \gamma[p_{1x} - \beta(E_1 + mc^2)/c] = 0$$

positron at rest

Total x-momentum in c.o.m. is zero

Then $p_{1x}c - \beta(E_1 + mc^2) = 0$ use $\frac{E_1^2}{c^2} - p_{1x}^2 = m^2c^2$

Solving $$\beta = \frac{p_{1x}c}{E_1 + mc^2} = \frac{\sqrt{E_1^2 - m^2c^4}}{E_1 + mc^2} = \frac{\sqrt{(E_1+mc^2)(E_1-mc^2)}}{E_1+mc^2}$$

$$\beta = \sqrt{\frac{E_1 - mc^2}{E_1 + mc^2}}.$$ Numerically this is very close to 1.

Use result of b) to simplify. $E_1 - mc^2 = 2\frac{M^2c^2}{m} - 2mc^2$, $E_1 + mc^2 = 2\frac{M^2c^2}{m}$

$$\beta = \sqrt{\frac{2M^2c^2/m - 2mc^2}{2M^2c^2/m}} = \sqrt{1 - \frac{m^2}{M^2}}.$$

or $\beta^2 = 1 - \frac{m^2}{M^2}$. and $1 - \beta^2 = \frac{m^2}{M^2}$ so $\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{M}{m}$.

2. (70) An electron starts from rest (in the lab frame) at time $t = 0$ in a region of uniform electric field $\mathbf{E} = E_0\hat{x}$. There is no magnetic field.

a) (20) Write the equations of motion for the components (U^0, U^1) of the 4-velocity, considered as functions of the proper time τ.

b) (20) Solve these equations for the given initial conditions, evaluating all the constants of integration.

c) (15) The electron accelerates until it reaches an energy $E = 5mc^2$. How long did this take, measured in the lab frame?

d) (15) How far did the electron travel to attain the energy $E = 5mc^2$.

a) The covariant eqn. of motion is $\frac{dU^\alpha}{d\tau} = \frac{e}{mc} F^{\alpha\beta} U_\beta$,

where $U^\alpha = \gamma(c, \vec{u}) = (U^0, \vec{U})$

The nonzero parts of $F^{\alpha\beta}$ are $E_i = -F^{0i} = +F^{i0}$.

We need only choose $\alpha = 0$ and $\alpha = 1$ on $\frac{dU^\alpha}{d\tau}$, which gives

$$\frac{dU^0}{d\tau} = \frac{e}{mc} F^{01} U_1 = \frac{e}{mc}(-E_0)(-U^1) = \frac{eE_0}{mc} U^1$$

$$\frac{dU^1}{d\tau} = \frac{e}{mc} F^{10} U_0 = \frac{e}{mc}(E_0)(+U^0) = \frac{eE_0}{mc} U^0 \, ; \quad \omega \equiv \frac{eE_0}{mc}.$$

b) Both eqns have positive signs on RHS — they are hyperbolic.

(also note that U^2 and U^3 are constants of the motion, equal to zero.)

The fundamental solutions are of form $\cosh(\omega\tau)$ and $\sinh(\omega\tau)$, or you could also use $e^{+\omega\tau}$ and $e^{-\omega\tau}$. Particle starts from rest, so

let. $U^0 = A\cosh\omega\tau$, $\frac{dU^0}{d\tau} = A\omega\sinh\omega\tau = \omega U^1$ (from a))

Then $U^1 = A\sinh\omega\tau$

The 4-velocity has square $U_\alpha U^\alpha = U^{0\,2} - U^{1\,2} = c^2$, which forces $A = c$.

Soln is $U^0(\tau) = c\cosh\omega\tau$

$U^1(\tau) = c\sinh\omega\tau$.

Part B Score = ________/120

c) The U^0 component is $U^0(\tau) = \gamma c$, which gives energy as $E(\tau) = mcU^0$.

The final value is $E(\tau) = 5mc^2$; the initial value is $E(0) = mc^2$.

To transform to lab time, use $\frac{dt}{d\tau} = \gamma$ or $dt = \gamma\, d\tau = \frac{1}{c} U^0\, d\tau$.

$$\Delta t = \int dt = \int \frac{1}{c} U^0\, d\tau = \int_0^\tau \cosh \omega\tau'\, d\tau' = \frac{1}{\omega} \sinh \omega\tau = \frac{U^1(\tau)}{c\omega}$$

At the final time we have $E(\tau) = 5mc^2 = mcU^0 = mc^2 \cosh \omega\tau$

So that $\underline{\cosh \omega\tau = 5}$ then $\sinh \omega\tau = \sqrt{1+\cosh^2\omega\tau} = \sqrt{24}$

The time interval is then $\Delta t = \frac{\sinh \omega\tau}{\omega} = \frac{\sqrt{24}}{\omega} = \sqrt{24}\, \frac{mc}{eE_0}$.

d) The distance is found from the speed in the lab frame, $u_x = \frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{U^1}{\gamma}$

or $\frac{dx}{dt} = \frac{U^1}{U^0/c} = c\, \frac{\sinh \omega\tau}{\cosh \omega\tau} = c \tanh \omega\tau$ $\qquad \gamma = U^0/c = \cosh \omega\tau$

$$\Delta x = \int dx = \int c \tanh \omega\tau \cdot dt = \int c \tanh \omega\tau \cdot \frac{U^0}{c}\, d\tau$$

$$\Delta x = c \int_0^\tau \sinh \omega\tau'\, d\tau' = c\, \frac{1}{\omega}(\cosh \omega\tau - 1) = \frac{c}{\omega}(5-1)$$

$$\Delta x = 4\frac{c}{\omega} = 4\, \frac{c}{eE_0/mc} = 4\, \frac{mc^2}{eE_0}.$$

The distance travelled is proportional to $(\gamma-1)mc^2$, which is the kinetic energy or work done by the applied field!

$$\text{work} = \underbrace{F \cdot \Delta x = \Delta KE}.$$

$(\gamma-1)mc^2 = eE_0 \cdot \Delta x$. Usual conservation of energy!