Electrodynamics II. Exam 3. Part A (150 pts.)
Relativistic Particles, Fields and Radiation Closed Book
Name G. WYSIN KSU 2015/05/11 14:00 - 15:50

Instructions: Some small derivations here, state your responses clearly, define your variables in words if they are not common usage.

1. A particle with charge e, mass m is moving with coordinate $\mathbf{r}(t)$ and velocity $\mathbf{u}(t)$ in some EM field defined in a given inertial frame of reference by scalar and vector potentials $\Phi, \mathbf{A}$.

   a. (10) When the particle is moving nonrelativistically, what is the simplest Lagrangian that describes its dynamics?

$$L = T - V = \frac{1}{2}mu^2 - e\Phi$$

or $$L = -mc^2\sqrt{1-\frac{u^2}{c^2}} - e\Phi$$

   b. (10) Now consider the particle with relativistic motion. Write out the full relativistic Lagrangian needed in the the given inertial frame.

Generalize $L_{int} = -e\Phi = -e\frac{1}{\gamma c}U_0 A^0$, where $U_0 = \gamma c$, $U_i = \gamma u_i$

$$\rightarrow L = -mc^2\sqrt{1-\frac{u^2}{c^2}} - \frac{e}{\gamma c}U_\alpha A^\alpha = -\frac{mc^2}{\gamma} - e\Phi + \frac{e}{c}\vec{u}\cdot\vec{A}$$

   c. (10) For the particle with relativistic motion, use your previous answer to get the canonical momentum $\mathbf{P}$, in terms of the mechanical momentum $\mathbf{p} = \gamma m\mathbf{u}$.

$$P_i = \frac{\partial L}{\partial u_i} = -mc^2\frac{\partial}{\partial u_i}\sqrt{1-\frac{u^2}{c^2}} + \frac{e}{c}A_i =$$

$$P_i = \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}}\frac{u_i}{c^2} + \frac{e}{c}A_i, \quad \text{or} \quad \vec{P} = \gamma m\vec{u} + \frac{e}{c}\vec{A}$$

2. (8) In classical mechanics for periodic motion there are action integrals of the form $J_i = \oint p_i\, dq_i$ that are considered adiabatic quantities (also, they become quantized in amounts nh). What adiabatic invariant for charged particle motion results from this?

$$J = \oint \vec{P}\cdot d\vec{l} = \oint \gamma m\vec{u}\cdot d\vec{l} + \oint \frac{e}{c}\vec{A}\cdot d\vec{l} = \frac{e}{c}B\pi r^2$$

which gives something proportional to the magnetic flux.

3. A particle (mass m charge e) moves in a uniform constant magnetic field $\mathbf{B} = B\hat{z}$.

a. (8) Calculate the angular frequency ω of its cylclotron motion.

$$\frac{d\vec{p}}{dt} = \frac{e}{c}\vec{u}\times\vec{B} \quad \text{at constant } |\vec{u}| \text{ and so constant } \gamma, \quad E = mc^2\gamma$$

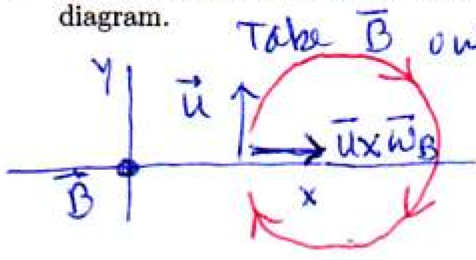
$$|\vec{p}| = m|\vec{u}|\gamma$$

$$m\gamma\frac{d\vec{u}}{dt} = \frac{e}{c}\vec{u}\times\vec{B} \qquad \text{Hence } \vec{\omega} = \frac{e\vec{B}}{mc\gamma}$$

$$\frac{d\vec{u}}{dt} = \vec{u}\times\frac{e\vec{B}}{mc\gamma} \qquad \boxed{\omega = \frac{eB}{mc\gamma}.}$$

b. (8) For a positive charge e, determine the sense of rotation around lines of $\mathbf{B}$ on a diagram.

Take $\vec{B}$ out of page.



The charge moves on the left hand sense around $\vec{B}$.

4. Consider the Lagrangian for EM fields.

a. (8) Write a Lagrangian density $\mathcal{L}$ for electromagnetic fields generated by the 4-vector current density J_α.

$$\mathcal{L} = -\frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{c}J_\alpha A^\alpha$$

$$\text{where} \quad F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha.$$

b. (8) What are the inhomogeneous Maxwell's equations (in covariant form) that result from the Lagrangian density in (a)?

$$\text{From} \quad \partial^\beta\left(\frac{\partial\mathcal{L}}{\partial(\partial^\beta A^\alpha)}\right) = \frac{\partial\mathcal{L}}{\partial A^\alpha} \quad \text{Euler-Lagrange eqns.}$$

$$\text{Get} \quad \frac{1}{4\pi}\partial^\beta F_{\beta\alpha} = \frac{1}{c}J_\alpha$$

c. (8) If photons were to have a rest mass proportional to some parameter μ, how should the Lagrangian from (a) be modified?

Add a term proportional to $|A|^2$.

$$\mathcal{L}_{extra} = \frac{\mu^2}{8\pi}A_\alpha A^\alpha$$

d. (8) If photons have rest mass, what is the mathematical form of the electrostatic potential of an isolated point charge q?

The potential gets a screening effect,

$$\Phi = q \frac{e^{-\mu r}}{r}.$$

5. The symmetrical stress tensor is $\Theta^{\alpha\beta} = \frac{1}{4\pi}\left(g^{\alpha\mu}F_{\mu\lambda}F^{\lambda\beta} + \frac{1}{4}g^{\alpha\beta}F_{\mu\lambda}F^{\mu\lambda}\right)$.

a. (8) What is the physical interpretation of the equation, $\partial_\alpha \Theta^{\alpha\beta} = 0$, for $\beta = 0$?

One has $\Theta^{00} = \frac{1}{8\pi}(E^2 + B^2)$ (energy density)

and $\Theta^{i0} = \frac{1}{4\pi}(\vec{E}\times\vec{B})_i$ (poynting vector)

$\partial_\alpha \Theta^{\alpha 0} = 0$ becomes $\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = 0$, conservation of EM energy.

b. (8) What is the physical interpretation of the equation, $\partial_\alpha \Theta^{\alpha\beta} + \frac{1}{c}F^{\beta\lambda}J_\lambda = 0$, for $\beta = i$?

Now the equation represents the conservation of EM momentum combined with momentum in the material particles. (J_λ).

6. A current density produces EM fields satisfying the covariant equations, $\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c}J^\beta$. Suppose you solve it in the Lorenz gauge, where $\partial_\alpha A^\alpha = 0$.

a. (8) Changing the RHS to $\delta^{(4)}(x - x')$, what is the 4D retarded Green's function $G_r(x - x')$ for this equation?

$$G_r(x-x') = \frac{1}{4\pi R}\Theta(x_0 - x_0')\,\delta(x_0 - x_0' - R), \quad R = |\vec{x} - \vec{x}'|.$$

or $$G_r(x-x') = \frac{1}{2\pi}\Theta(x_0 - x_0')\,\delta[(x-x')^2].$$

b. (8) How do write the most general solution for $A^\alpha(x)$ using the retarded Green's function?

$$A^\alpha(x) = A^\alpha_{in}(x) + \int d^4x'\, G_r(x-x')\, A^\alpha(x') \cdot \frac{4\pi}{c}.$$

boundary condition at ∞ past time.

7. (8) Describe the physical meaning of the formula below and the definitions of $U(\tau)$ and τ_0.

$$A^\alpha(x) = \left[\frac{eU^\alpha(\tau)}{U \cdot (x - r(\tau))}\right]_{\tau=\tau_0}$$

$U^\alpha(\tau) = \gamma(c, \vec{u})$ is the path of a charge e, as parameterized in proper time.

The formula gives the radiation potential of the moving charge.

τ_0 is the retarded time on the past light cone.

8. (8) Write a formula that is dimensionally correct (up to a numerical constant) for the instanteous power radiated by an accelerated charge in *nonrelativistic motion.*

$$P = \frac{2}{3}\frac{e^2}{c^3}(\dot{\vec{u}})^2$$

Larmor result.

For an accelerated charge.

9. (8) Generalize your result of the previous question to a covariant formula for an arbitrary relativistic case.

Let $P \to \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{d\vec{p}}{dt}\cdot\frac{d\vec{p}}{dt}\right) \to \frac{2}{3}\frac{e^2}{m^2c^3}\frac{dp_\alpha}{d\tau}\frac{dp^\alpha}{d\tau} \times (-1)$.

required for correct sign on space parts

10. (8) The angular distribution of radiated power from an accelerated charge is defined from the formula

$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c}\frac{\left|\mathbf{n} \times \left[(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right]\right|^2}{(1 - \mathbf{n}\cdot\boldsymbol{\beta})^5}.$$

Physically, how does the angular distribution of power radiated change as one goes from the nonrelativistic case to the extreme relativistic limit?

The factor in the denominator becomes small and causes the radiation to be strongest mostly near the forward direction, close to $\vec{\beta}$'s direction.

$\theta_{max} \sim \frac{1}{2\gamma}$ as $\beta \to 1$.

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Part B (100 pts.)
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Instructions: Please show the details of your derivations. Explain your reasoning for full credit. Open-book and 1-page note summary allowed.

1. A point charge e with mass m moves on a trajectory with 4-vector $r(\tau)$ and 4-velocity $U(\tau)$ parametrized by proper time τ.

   a. (10) Write the covariant equation of motion for the 4-velocity of the charge in an electromagnetic field described by EM field tensor $F^{\alpha\beta}$.

   b. (15) Write out separately the equation of motion for the time component, $dU^0/d\tau$, showing how $F^{\alpha\beta}$ becomes the EM fields **E** and **B** in the equation.

   c. (15) Write out separately the equations of motion for space components, $dU^i/d\tau$, showing how $F^{\alpha\beta}$ becomes the EM fields **E** and **B** in the equations.

a. $\frac{dp^\alpha}{d\tau} = \frac{e}{c} F^{\alpha\beta} U_\beta$ where $p^\alpha = mU^\alpha$

or $\boxed{\frac{dU^\alpha}{d\tau} = \frac{e}{mc} F^{\alpha\beta} U_\beta .}$ also where $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$

b. When $\alpha = 0$, with $U^\alpha = (\gamma c, \gamma\vec{u})$, $U^0 = \gamma c$, $U^i = \gamma u_i$

The components of $\vec{E}, \vec{B}$ are $E_i = -F^{0i} = F^{i0}$, $B_i = -\frac{1}{2}\epsilon_{ijk} F^{jk}$

Then $\frac{dU^0}{d\tau} = \frac{e}{mc} F^{0i} U_i = \frac{e}{mc}(-E_i)(-\gamma u_i)$

sign change (normal vector here)

Then $\boxed{\frac{dU^0}{d\tau} = \frac{e}{mc}\gamma \vec{E}\cdot\vec{u}}$ This shows the work done on the charge by the electric field.

Also $\frac{dU^0}{dt} = \frac{e}{mc}\vec{E}\cdot\vec{u}$ because $\gamma = \frac{dt}{d\tau}$.

c. $$\frac{dU^i}{d\tau} = \frac{e}{mc} F^{i\alpha} U_\alpha = \frac{e}{mc}\Big(\underbrace{F^{i0}}_{E_i} \underbrace{U_0}_{c\gamma} + F^{ij} \underbrace{U_j}_{-cu_j} \Big)$$

For the term with F^{ij}, note that $B_i = -\frac{1}{2}\epsilon_{ijk} F^{jk}$.

Then also $$\epsilon_{ilm} B_i = \epsilon_{ilm}\epsilon_{ijk} F^{jk}\left(-\tfrac{1}{2}\right) = -\tfrac{1}{2}\left(\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj}\right) F^{jk}$$

$$\epsilon_{ilm} B_i = -\tfrac{1}{2}\left(F^{lm} - F^{ml} \right) = -F^{lm}.$$

or equivalently, $F^{ij} = -\epsilon_{kij} B_k$.

Then $$F^{ij} U_j = -\epsilon_{kij} B_k U_j = -\epsilon_{ijk} U_j B_k = -(\vec{U}\times\vec{B})_i$$

$$F^{ij} U_j = -(\gamma\vec{u}\times\vec{B})_i$$

This gives

$$\boxed{\frac{dU^i}{d\tau} = \frac{e}{mc}\left(c\gamma E_i + (\gamma\vec{u}\times\vec{B})_i \right)}$$

which then leads to

$$\frac{d\vec{u}}{dt} = \frac{e}{m}\left(\vec{E} + \frac{1}{c}\vec{u}\times\vec{B} \right)$$ as it should.

2. A particle of charge q, mass m, and initial speed v_0 is incident on a path near a fixed charge Q, at an impact parameter b large enough so that the particle's path remains nearly a straight line as it passes Q.

   a. (20) Assuming nonrelativistic motion, get the instantaneous acceleration and then write an expression for the instantaneous power radiated as a function of the separation r between q and Q.

   b. (10) If the particle is now assumed to move at constant speed on a straight line (it is a contradiction to part (a) but gives a way to complete the calculation), write the relation for $r(t)$.

   c. (20) Combining the above results, write an integration over dr to determine the total energy ΔW radiated as the particle with (q, m) passes the fixed charge Q.

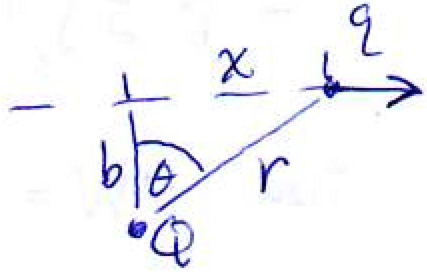
   d. (10) Evaluate the integral from part c to get the net energy radiated.

a. The acceleration is determined by the Coulomb force between $q + Q$.

$\vec{a} = \frac{\vec{F}}{m} = \left(\frac{qQ}{mr^2}\right)$, Then use it in the Larmor formula.

$$P = \frac{dW}{dt} = \frac{2}{3}\frac{q^2}{c^3}\vec{a}^2 = \boxed{\frac{2}{3}\frac{q^2}{c^3}\left(\frac{qQ}{mr^2}\right)^2}.$$

b. q is at $x = vt$ on x-axis. $\longrightarrow$



Although $v(t)$ varies, assume a constant.

$r^2 = b^2 + x^2 = b^2 + v^2t^2$

or $\boxed{r = \sqrt{b^2 + v^2t^2}}$

c. To find the total energy radiated, write $dW = P\,dt$.

and $\Delta W = \int P\,dt = \int P \frac{dt}{dr}\,dr$ where $\frac{dr}{dt} = \frac{v^2 t}{\sqrt{b^2+v^2t^2}} = \frac{v^2 t}{r}$

and use $\frac{dr}{dt} = \frac{v^2 t}{r} = \frac{v}{r}(vt) = \frac{v}{r}\sqrt{r^2 - b^2}$.

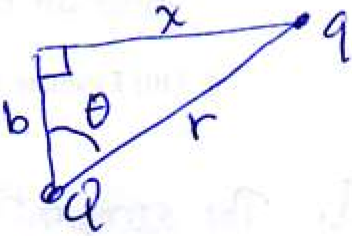
c. continued.

$$\Delta W = 2\int_b^{\infty} P\cdot \frac{r}{v\sqrt{r^2-b^2}}\,dr = 2\cdot\frac{2}{3}\,\frac{q^2}{c^3 v}\left(\frac{qQ}{m}\right)^2 \int_b^{\infty}\frac{1}{r^4}\,\frac{r\,dr}{\sqrt{r^2-b^2}}$$

The factor of 2 is needed because one includes both the approach to r=b and then again moving away.

Clarify answer: $$\boxed{\Delta W = \frac{4}{3}\,\frac{q^4 Q^2}{m^2 c^3 v_0}\int_b^{\infty}\frac{dr}{r^3\sqrt{r^2-b^2}}}$$

d. The integral is transformed to θ.



$$b/r = \cos\theta. \qquad r^2-b^2 = x^2 = b^2\tan^2\theta.$$

$$r = b\sec\theta. \qquad dr = -b\sec^2\theta\,\sin\theta\,.d\theta$$

$$\int\frac{dr}{r^3\sqrt{r^2-b^2}} = \int\frac{b\sec^2\theta\ \sin\theta\ d\theta}{b^3\sec^3\theta\cdot b\tan\theta} = \frac{1}{b^3}\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$= \frac{1}{b^3}\cdot\frac{1}{2}\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta = \frac{1}{b^3}\cdot\frac{\pi}{4}.$$

Then $$\Delta W = \frac{4}{3}\,\frac{q^4Q^2}{m^2c^3v_0}\,\frac{1}{b^3}\cdot\frac{\pi}{4} = \boxed{\frac{\pi q^4 Q^2}{3m^2c^3v_0 b^3}}$$