Electrodynamics II   Exam 1. Part A (130 pts.) Closed Book   Waveguides, Radiation

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Instructions: Use SI units. No derivations here, just state your responses clearly, and define your variables in words. For the waveguide questions, assume it to be filled with a medium with parameters ϵ, μ, and the axis is the z-axis.

1. (10) For a waveguide made from a very good conductor, which components ($\perp$ or $\|$) of **E** and **H** are nearly zero at the surfaces of the conductor?

Tangential $\vec{E}$ is continuous across boundary $\Rightarrow$ $\vec{E}_{\|} \approx 0$

Normal $\vec{B}$ is continuous across boundary $\Rightarrow$ $\vec{H}_{\perp} \approx 0$

2. (10) Write out the differential equation that any component of the fields should satisfy in a waveguide, for a mode propagating down the axis as $\exp(ikz)$ with wavevector k.

$$\left[\nabla_t^2 + \omega^2 \epsilon\mu - k^2\right]\psi = 0, \quad \nabla_t^2 = \partial_x^2 + \partial_y^2, \quad \psi = \vec{E}, \vec{H}$$

any component.

3. (10) For a transverse magnetic mode (TM) in a waveguide, which component of **E** or **H** acts as ψ that can be used to determine all the other field components?

$\psi = E_z$ with $B_z = 0$.

4. (10) For a TM waveguide mode, what is the boundary condition on the component ψ that you answered in the previous question?

$E_z\big|_S = 0$ Dirichlet B.C.

5. (10) If a certain waveguide mode has some cutoff frequency ω_λ, what does that mean?

The mode will not propagate in the waveguide unless $\omega > \omega_\lambda$, where ω is the frequency of the wave source.

6. (10) A waveguide mode has a cutoff frequency ω_λ. If waves at frequency ω are sent into the waveguide, with what propagation vector k do they propagate?

$$\gamma_\lambda^2 = \omega_\lambda^2 \epsilon\mu = \omega^2\epsilon\mu - k^2$$

$$k = \sqrt{\epsilon\mu}\sqrt{\omega^2 - \omega_\lambda^2}$$ which shows you need $\omega > \omega_\lambda$.

7. (10) A cylindrical resonant cavity with ends at $z = 0$, $z = d$ is oscillating in a TE mode. Which EM field component determines all the other components?

H_z, with $E_z = 0$.

8. (10) For a cylindrical resonant cavity oscillating in a TE mode, what are the boundary conditions on the field component of the previous question a) on the walls b) on the ends at $z = 0$ and $z = d$?

a) $\left.\frac{\partial H_z}{\partial n}\right|_S = 0$ Neumann B.C. b) $H_z = H_\perp = 0$ at ends.

9. (10) Give a definition of the Q of a resonant cavity. Explain what the Q means for practical purposes.

$$Q = 2\pi \frac{\text{stored energy}}{\text{energy loss in a period}}.$$

Q gives an indication of the frequency sharpness of the resonance.

10. (10) The vector potential of an oscillating electric dipole is $\vec{A} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-i\omega\vec{p})$. From that, what is an expression for the magnetic field?

$$\vec{H} = \frac{1}{\mu_0}\vec{B} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A} = \frac{1}{\mu_0}\hat{n}\frac{\partial}{\partial r}\times\vec{A} = \frac{-i\omega}{4\pi}(\hat{n}\times\vec{p})\frac{e^{ikr}}{r}\left(ik - \frac{1}{r}\right),$$ use $\omega = ck$

or $$\vec{H} = \frac{ck^2}{4\pi}(\hat{n}\times\vec{p})\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right).$$

11. (10) Very far from an oscillating electric dipole, use your previous answer to show the polarization direction of the electric field vector.

$$\vec{E} \approx Z_0\vec{H}\times\hat{n} = Z_0\frac{ck^2}{4\pi}\underbrace{\left[(\hat{n}\times\vec{p})\times\hat{n}\right]}_{\text{gives direction of } \vec{E} \text{ in radiation zone.}}\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right).$$

12. (10) A radiation source produces certain fields $\vec{E}$ and $\vec{H}$ far from the source. How do you use them to obtain the power radiated per unit solid angle, along a direction $\hat{n}$?

$$\frac{dP}{d\Omega} = r^2\,\hat{n}\cdot\underbrace{\frac{1}{2}\mathrm{Re}(\vec{E}\times\vec{H}^*)}_{\text{time-averaged Poynting vector.}}$$

13. (10) What is the definition of a "vector spherical harmonic?" List one orthogonality property that they have.

$$\underbrace{\vec{X}_{lm} = \frac{1}{\sqrt{l(l+1)}}\vec{L}}_{\text{angular momentum operator}}Y_{lm} \qquad \int d\Omega\, \vec{X}^*_{lm}\cdot\vec{X}_{l'm'} = \delta_{ll'}\delta_{mm'}$$

Electrodynamics II Exam 1. Part B (120 pts.) Open Book Waveguides, Radiation

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Instructions: Use SI units. Please show the details of your derivations here. Explain your reasoning for full credit. Open-book.

The starting point for these problems are the fields (9.18) of an oscillating electric dipole $\mathbf{p}$:

$$\mathbf{H} = \frac{ck^2}{4\pi}(\mathbf{n}\times\mathbf{p})\left(1-\frac{1}{ikr}\right)\frac{e^{ikr}}{r} \qquad (1)$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\left\{k^2(\mathbf{n}\times\mathbf{p})\times\mathbf{n}+[3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})-\mathbf{p}]\left(\frac{1}{r^2}-\frac{ik}{r}\right)\right\}\frac{e^{ikr}}{r} \qquad (2)$$

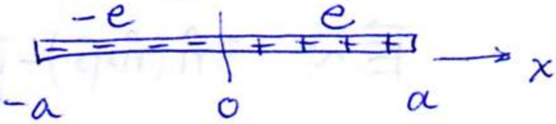
1. A thin rod of length $2a$ has a uniform distribution of positive charge $+e$ on one half and negative charge $-e$ on the other half, forming an electric dipole. It lies in the xy plane and is set rotating at angular frequency ω around the z-axis.

   a) (20) Determine its time-dependent electric dipole moment $\mathbf{p}$. Write it in complex form with an assumed dependence on $\exp(-i\omega t)$.

   b) (20) Find the formula for the time-averaged power radiated per unit solid angle, $dP/d\Omega$, far from the dipole. Give the result as a function of spherical angles (θ, ϕ) that describe some direction in space, outward from the dipole.

   c) (20) Show the calculation of the total power radiated by the dipole.

a) The magnitude of static dipole moment

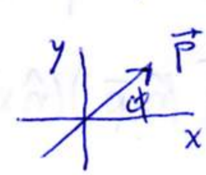


is $P_x = \int \rho\, x\, dx = \int_{-a}^{0} -\frac{e}{a} x\, dx + \int_0^a \frac{e}{a} x\, dx$

$$P_x = -\frac{e}{a}\frac{x^2}{2}\Big|_{-a}^{0} + \frac{e}{a}\frac{x^2}{2}\Big|_0^a = \frac{ea}{2}+\frac{ea}{2} = ea.$$

Now it is set rotating, with angle $\phi = \omega t$, to the $\hat{x}$-axis.

$$\vec{p} = \hat{x}\, ea\cos\omega t + \hat{y}\, ea\sin\omega t$$



$$\vec{p} = \mathrm{Re}\left(ea(\hat{x}+i\hat{y})\,e^{-i\omega t}\right)$$

or just $\boxed{\vec{p} = p(\hat{x}+i\hat{y})}$, $p = ea$.

and real part and $e^{-i\omega t}$ are implied.

1b) Far from the dipole, use $\vec{E} \simeq \frac{k^2}{4\pi\epsilon_0} (\hat{n}\times\vec{p})\times\hat{n} \; \frac{e^{ikr}}{r}$

$$\vec{S} = \frac{1}{2} Re(\vec{E}\times\vec{H}^*) \qquad \vec{H} \simeq \frac{ck^2}{4\pi}(\hat{n}\times\vec{p}) \; \frac{e^{ikr}}{r}$$

$$\vec{S} = \frac{1}{2}\frac{k^2}{4\pi\epsilon_0}\cdot\frac{ck^2}{4\pi}\left[(\hat{n}\times\vec{p})\times\hat{n}\right]\times(\hat{n}\times\vec{p}^*)\,\frac{1}{r^2} \qquad \text{(real part implied)}$$

$$r^2\vec{S} = \frac{ck^4}{32\pi^2\epsilon_0}\,\hat{n}\,(\hat{n}\times\vec{p})\cdot(\hat{n}\times\vec{p}^*)$$

$$\frac{dP}{d\Omega} = r^2\,\hat{n}\cdot\vec{S} = \frac{ck^4}{32\pi^2\epsilon_0}\,|\hat{n}\times\vec{p}|^2$$

Let $\hat{n} = \frac{1}{r}(x\hat{x} + y\hat{y} + z\hat{z}) = \frac{\vec{r}}{r}$. $\quad \vec{p} = p(\hat{x} + i\hat{y})$

$$\hat{n}\times\vec{p} = \frac{p}{r}\left(xi\hat{z} - y\hat{z} + z\hat{y} - zi\hat{x}\right)$$

$$\hat{n}\times\vec{p} = \frac{p}{r}\left(-iz\,\hat{x} + z\,\hat{y} + (ix-y)\,\hat{z}\right)$$

$$|\hat{n}\times\vec{p}|^2 = \frac{p^2}{r^2}\left(z^2 + z^2 + x^2 + y^2\right) = \frac{p^2}{r^2}\left(r^2\cos^2\theta + r^2\right)$$

$$= p^2(1+\cos^2\theta)$$

$$\boxed{\frac{dP}{d\Omega} = \frac{ck^4p^2}{32\pi^2\epsilon_0}(1+\cos^2\theta)} \quad \text{where } p = ea.$$

1c) Total power radiated $\Rightarrow$ integrate over all solid angle.

$$\int d\Omega\,(1+\cos^2\theta) = \int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\theta)\,(1+\cos^2\theta) = 2\pi\left(\cos\theta + \frac{\cos^3\theta}{3}\right)_{-1}^{1} = 2\pi\cdot 2\cdot\frac{4}{3}$$

$$P = \int \frac{dP}{d\Omega}\,d\Omega = \frac{ck^4p^2}{32\pi^2\epsilon_0}\,2\pi\cdot\frac{8}{3} = \boxed{\frac{ck^4p^2}{6\pi\epsilon_0}}$$

Note that this is the total Power of two dipoles — one along $\hat{x}$ and one along $\hat{y}$ with a 90° phase difference.

2. Consider an oscillating electric dipole $\mathbf{p}$ with fields given by (9.18) in Jackson's text.

a) (40) Show that the dipole radiates electromagnetic angular momentum at a rate given by

$$\frac{d\mathbf{L}}{dt} = \frac{k^3}{12\pi\epsilon_0}\mathrm{Im}\left\{\mathbf{p}^* \times \mathbf{p}\right\}. \tag{3}$$

Hint: The angular momentum comes from more than the transverse (radiation zone) components of the fields.

b) (20) For the rotating dipole in Problem 1, use this result to find the rate at which it radiates angular momentum. What Cartesian components are present? Discuss whether your result makes sense.

a) The flux of angular momentum density is $\frac{1}{2c}\mathrm{Re}\left(\vec{r}\times(\vec{E}\times\vec{H}^*)\right)$

But $\vec{r} = r\hat{n}$. We note that the far zone terms in $(\vec{E}\times\vec{H}^*)$ produce 0 in this expression! $\sim \hat{n}\times\hat{n}\,(\hat{n}\times\vec{p})\cdot(\hat{n}\times\vec{p}^*) = 0$

One must use the second term in $\vec{E}$ to get a nonzero result.

Do "algebra" just on the vector parts - add other factors later.

$$\vec{E} \sim 3\hat{n}(\hat{n}\cdot\vec{p}) - \vec{p} \qquad \vec{H}^* \sim \hat{n}\times\vec{p}^*$$

$$\vec{E}\times\vec{H}^* \sim \left[3\hat{n}(\hat{n}\cdot\vec{p})-\vec{p}\right]\times(\hat{n}\times\vec{p}^*) = 3(\hat{n}\cdot\vec{p})\left[\hat{n}(\hat{n}\cdot\vec{p}^*)-\vec{p}^*\right] - \hat{n}|\vec{p}|^2 + \vec{p}^*(\hat{n}\cdot\vec{p})$$

use $\vec{r} = r\hat{n}$.

$$\hat{n}\times(\vec{E}\times\vec{H}^*) \sim -3(\hat{n}\cdot\vec{p})(\hat{n}\times\vec{p}^*) + (\hat{n}\times\vec{p}^*)(\hat{n}\cdot\vec{p}) = -2(\hat{n}\cdot\vec{p})(\hat{n}\times\vec{p}^*)$$

As for power radiated, we must integrate this over solid angle.

$$-2\int(\hat{n}\cdot\vec{p})(\hat{n}\times\vec{p}^*)\,d\Omega = -2\int n_\alpha p_\alpha\, \epsilon_{ijk}\,\hat{e}_i\, n_j\, p_k^*\, d\Omega$$

Use identity $\int n_\alpha n_j\, d\Omega = \frac{4\pi}{3}\delta_{\alpha j}$ here.

$$= -2\cdot\frac{4\pi}{3}\,\epsilon_{ijk}\,\hat{e}_i\, p_j\, p_k^* = \frac{-8\pi}{3}\,\vec{p}\times\vec{p}^*$$

2a) continued... $-2\int (\hat{n}\cdot\vec{p})(\hat{n}\times\vec{p}^*)\,d\Omega = -\frac{8\pi}{3}\vec{p}\times\vec{p}^*$.

Total radiated angular momentum rate is then

$$\frac{d\vec{L}}{dt} = \int d\Omega \, \frac{1}{2c} \mathrm{Re}\left(\vec{r}\times(\vec{E}\times\vec{H}^*)\right) r^2$$

area element: $(dA = r^2 d\Omega)$

$$= \mathrm{Re}\, \underbrace{\frac{1}{2c}}_{\text{time-averaging}} \underbrace{\frac{-ik}{4\pi\epsilon_0 r^2}}_{\text{from } \vec{E}} \underbrace{\frac{ck^2}{4\pi r}}_{\text{from } \vec{H}^*} r \cdot \left(-\frac{8\pi}{3}\vec{p}\times\vec{p}^*\right)$$

from $\vec{r} = r\hat{n}$. from $\int d\Omega$ $\hat{n}\times(\vec{E}\times\vec{H}^*)$

$$\boxed{\frac{d\vec{L}}{dt} = \mathrm{Re}\left(\frac{ik^3}{12\pi\epsilon_0}\vec{p}\times\vec{p}^*\right) = \mathrm{Im}\left(\frac{k^3}{12\pi\epsilon_0}\vec{p}^*\times\vec{p}\right)}$$

2b) With $\vec{p} = ea(\hat{x}+i\hat{y}) = p(\hat{x}+i\hat{y})$

$\vec{p}^* = p(\hat{x}-i\hat{y})$

$\vec{p}^*\times\vec{p} = p^2(\hat{x}-i\hat{y})\times(\hat{x}+i\hat{y}) = p^2\cdot(i\hat{z}+i\hat{z}) = 2ip^2\hat{z}$.

$$\frac{d\vec{L}}{dt} = \mathrm{Im}\left(\frac{k^3}{12\pi\epsilon_0}2ip^2\hat{z}\right) = \frac{k^3p^2}{6\pi\epsilon_0}\hat{z}.$$

There is only a $\hat{z}$-component of radiated angular momentum! This seems reasonable because the rotating charged rod only has a component of $\vec{L}$ along the z-axis.

Whatever power source causes it to rotate must also supply a small amount of angular momentum!