Electrodynamics II Exam 1. Part A (152 pts.) Closed Book Relativistic Electrodynamics

Name G. Wysin KSU 2006/04/11

Instructions: Use CGS-Gaussian units. No derivations here, just state your responses clearly, and define your variables in words.

1. (12) Make concise statements of the two Postulates applied by Albert Einstein in his 1905 development of the Special Theory of Relativity:

   1. The Laws of Physics and results of experiments are the same for all inertial reference frames.

   2. The speed of light is the same finite constant for all observers. (It is independent of the motion of the source.)

2. (18) Inertial frame K' moves with velocity $\vec{\beta} = \beta\hat{x}$ with respect to inertial frame K. The xyz coordinate axes of the two frames are parallel. Write out the Lorentz transformation that gives $x'y'z't'$ in terms of $xyzt$.

Assuming the origins coincide at $t = t' = 0$,

$$x_0' \equiv ct' = \gamma(x_0 - \beta x_1) \qquad x_0 \equiv ct.$$
$$x_1' = \gamma(x_1 - \beta x_0)$$
$$x_2' = x_2$$
$$x_3' = x_3$$

3. (12) An object (like a clock) moves in some reference frame K with a variable velocity $\vec{v}(t) = c\vec{\beta}(t)$. Write an expression giving the proper time change $\Delta\tau$ when the time in K evolves from t_1 to t_2.

Proper time interval corresponds to $(d\vec{x}')^2 = 0$

$$ds^2 = c^2dt^2 - d\vec{x}^2 = c^2 d\tau^2, \quad \text{with } d\vec{x} = c\vec{\beta}(t)\,dt$$

$$\Delta\tau = \int_{\tau_1}^{\tau_2} d\tau = \int_{t_1}^{t_2} dt\sqrt{1-\beta^2(t)}$$

4. (6) A particle of rest mass m has a 4-momentum $p = (E/c, \vec{\mathbf{p}})$. With correct factors of c, what is the squared "length" $p \cdot p$ of this 4-vector?

$$p\cdot p = E^2/c^2 - \vec{p}\cdot\vec{p} = m^2c^2$$

5. (6) Explain in your own words why a space-time interval $s_{12}^2 = c^2(t_1 - t_2)^2 - |\vec{x}_1 - \vec{x}_2|^2$ should be an invariant, the same in all inertial reference frames.

If events 1 and 2 are the emission and absorption of light, then $S_{12}^2 = S_{12}'^2$ will identically preserve the speed of light in all reference frames — at $S_{12}^2 = S_{12}'^2 = 0$.

6. (12) Match the types of space-time intervals with their definitions.

1. $s_{12}^2 < 0$	c	a) light-like interval
2. $s_{12}^2 = 0$	a	b) time-like interval
3. $s_{12}^2 > 0$	b	c) space-like interval

7. (20) An electron ($mc^2 = 0.511$ MeV) with total energy of 10.0 MeV is made to collide with another electron initially at rest (both measured in the lab).

a) How large is the total energy W (in MeV) in the center of momentum frame? $[\vec{p}_1' + \vec{p}_2' = 0]$

$$W^2 = (p_1' + p_2')^2 = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 = m^2 + m^2 + 2(E_1 m - \vec{p}_1 \cdot \vec{0})$$

$$W = \sqrt{2m(m+E_1)} = \sqrt{2(0.511)(0.511+10)} = \underline{3.28 \text{ MeV}}.$$

b) How fast is the center of momentum frame moving with respect to the lab?

$$p_{1x}' = \gamma(p_{1x} - \beta E_1), \qquad p_x' = \gamma(p_{1x} - \beta(E_1 + m)) = 0$$

$$p_{2x}' = \gamma(p_{2x} - \beta E_2) = -\gamma\beta m \qquad \beta = \frac{p_{1x}}{E_1 + m} = \frac{\sqrt{E_1^2 - m^2}}{E_1 + m} = \sqrt{\frac{E_1 - m}{E_1 + m}} = \sqrt{\frac{10 - 0.511}{10 + 0.511}} = \underline{0.950}$$

8. (18) A certain unstable elementary particle of rest energy $mc^2 = 140$ Mev has a lifetime of 2.56×10^{-8}s when at rest. If it is given a total energy of 140 GeV in the lab, what are its

a) relativistic factor γ?

$$\gamma = \frac{E}{mc^2} = \frac{140 \text{ GeV}}{140 \text{ MeV}} = 1000$$

b) lifetime in the lab?

$$\Delta t = \gamma \Delta\tau = 1000 \times 2.56 \times 10^{-8} s = 2.56 \times 10^{-5} s.$$

c) mean decay path in the lab?

travels at nearly the speed of light.

$$\Delta x = c\Delta t = \left(3 \times 10^8 \frac{m}{s}\right)\left(2.56 \times 10^{-5} s\right) = 7680 \text{ m}$$

9. (12) Give the definition of the electromagnetic field tensor $F^{\alpha\beta}$. How do you get the electric field components from it?

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \qquad E_i = -F^{0i} = F^{i0}$$

10. (8) Write out the usual elementary relativistic Lagrangian for a particle of mass m, charge e, interacting with a given electromagnetic field.

$$L = -mc^2\sqrt{1-\frac{u^2}{c^2}} - \frac{e}{\gamma c} U_\alpha A^\alpha = -mc^2\sqrt{1-\frac{u^2}{c^2}} - e\Phi + \frac{e}{c}\vec{u}\cdot\vec{A}$$

11. (12) A particle with charge e and mass m enters a region with uniform crossed electric and magnetic fields $\vec{E}$ and $\vec{B}$.

a) What situation leads to a net drift of the particle superimposed with a spiral motion? What is the average speed of this drifting?

weaker $\vec{E}$-field: $\vec{E}^2 < \vec{B}^2 \Rightarrow$ some frame with no $\vec{E}$-field.

$\Rightarrow$ general cyclotron motion, superimposed on translation, at velocity

$$\vec{u} = c\,\frac{\vec{E}\times\vec{B}}{B^2}$$

b) What situation leads to a continuous acceleration of the particle in the direction of $\vec{E}$?

weaker $\vec{B}$-field: $\vec{E}^2 > \vec{B}^2 \Rightarrow$ some frame with no $\vec{B}$-field.

$\Rightarrow$ continual acceleration along direction of $\vec{E}$.

12. (16) Based on the electromagnetic field 4-potential A^α and the 4-current J^α,

a) Write out a Lagrangian density $\mathcal{L}$ for the electromagnetic field.

$$\mathcal{L} = -\frac{1}{16\pi} F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{c} J_\alpha A^\alpha$$

$$= -\frac{1}{16\pi}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial^\alpha A^\beta - \partial^\beta A^\alpha) - \frac{1}{c} J_\alpha A^\alpha, \quad \text{use } A_\beta = g_{\beta\nu}A^\nu, \text{ etc.}$$

b) What are the associated inhomogeneous Maxwell's equations in their covariant form?

$$\partial^\beta F_{\beta\alpha} = \frac{4\pi}{c} J_\alpha$$

13. (Bonus=18) Frame K' is moving relative to frame K as in Question 2. Write three equations for how the components of the electric field $\vec{E}'$ are obtained from the EM field components in K.

$E_1' = E_1$ (no change along boost direction)

$$\left.\begin{aligned} E_2' &= \gamma(E_2 - \beta B_3) \\ E_3' &= \gamma(E_3 + \beta B_2) \end{aligned}\right\} \text{ summarized as } \vec{E}_\perp' = \gamma(\vec{E}_\perp + \vec{\beta}\times\vec{B}_\perp)$$

is like the Lorentz force law.

Electrodynamics II Exam 2. Part B (148 pts.) Open Book Relativistic Electrodynamics

Name G. Wysin KSU 2006/04/11

Instructions: Please show the details of your derivations here. Explain your reasoning for full credit. Open-book and 1-page note summary allowed.

1. An electron starts from rest at time $t = 0$ in a region of uniform electric field $\vec{E} = E_0\hat{z}$. There is no magnetic field.

   a) (16) Write the equations of motion for all the components of the 4-velocity, considered as functions of the proper time τ.

   b) (20) Solve these equations for the given initial conditions, evaluating all the constants of integration.

   c) (20) The electron accelerates until it reaches an energy $E = 10mc^2$. How long did this take, measured in the lab frame?

   d) (20) How far did the electron travel to attain the energy $E = 10mc^2$.

   e) (Bonus=20) For extra credit, give numerical answers to c) and d) when the electric field strength is $E_0 = 1.00$ kV/m.

a) $\frac{dU^\alpha}{d\tau} = \frac{e}{mc} F^{\alpha\beta} U_\beta$ is the general covariant form of dynamics.

[Or maybe it is easier to just write out the time & space parts separately.]

Use the definitions, $U^\alpha = (\gamma c, \gamma\vec{u})$, and with no $\vec{B}$-field, but $F^{0i} = -E_i$, as well as $U_\beta = (\gamma c, -\gamma\vec{u})$, we'll have simplified eqns. for components,

$$\boxed{\frac{dU^0}{d\tau} = \frac{e}{mc} E_0 U^3}$$

solve simultaneously

$$\frac{dU^1}{d\tau} = 0, \quad \frac{dU^2}{d\tau} = 0, \quad \boxed{\frac{dU^3}{d\tau} = \frac{e}{mc} E_0 U^0}$$

Note that also we needed to apply $F^{i0} = -F^{0i} = +E_i$ for the U^3 eqn.

b) As electron starts from rest, initial condition is, at $\tau = 0$,

$U^0(0) = c, \quad U^1(0) = U^2(0) = U^3(0) = 0.$

$U^1(\tau)$ and $U^2(\tau)$ do not change, they remain zero.

The U^0 and U^3 components evolve in coupled fashion.

Let $\boxed{\omega \equiv \frac{eE_0}{mc}}$ = an inverse time scale.

$$\left.\begin{aligned} \frac{dU^0}{d\tau} &= \omega U^3 \\ \frac{dU^3}{d\tau} &= \omega U^0 \end{aligned}\right\} \Rightarrow \quad \frac{d}{d\tau}\begin{pmatrix} U^0 \\ U^3 \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix}\begin{pmatrix} U^0 \\ U^3 \end{pmatrix}$$ in matrix format.

The matrix $M = \begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix}$ have eigenvalues $\lambda = \pm\omega$, and the unnormalised eigenvectors,

$$\tilde{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \lambda = +\omega, \qquad \tilde{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},\ \lambda = -\omega.$$

The initial condition is $\begin{pmatrix} U^0(0) \\ U^3(0) \end{pmatrix} = \begin{pmatrix} c \\ 0 \end{pmatrix} = \frac{1}{2}c\tilde{v}_1 + \frac{1}{2}c\tilde{v}_2$.

Solution of $\dot{U} = MU$ is $U(\tau) = e^{M\tau}U(0)$, so we get,

$$\begin{aligned} \begin{pmatrix} U^0(\tau) \\ U^3(\tau) \end{pmatrix} &= e^{M\tau}\left(\tfrac{1}{2}c\tilde{v}_1 + \tfrac{1}{2}c\tilde{v}_2\right) = \tfrac{1}{2}c\left(e^{\omega\tau}\tilde{v}_1 + e^{-\omega\tau}\tilde{v}_2\right) \\ &= \tfrac{1}{2}c\left[e^{\omega\tau}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-\omega\tau}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] = c\begin{pmatrix} \frac{1}{2}(e^{\omega\tau} + e^{-\omega\tau}) \\ \frac{1}{2}(e^{\omega\tau} - e^{-\omega\tau}) \end{pmatrix} \\ &= c\begin{pmatrix} \cosh\omega\tau \\ \sinh\omega\tau \end{pmatrix} \rightarrow p = mc\begin{pmatrix} \cosh\omega\tau \\ \sinh\omega\tau \end{pmatrix} \end{aligned}$$ (for correct units.)

Complete solution summarized is

$$\boxed{\begin{aligned} U^0(\tau) &= \gamma c = c\cosh\left(\frac{eE_0}{mc}\tau\right) \\ U^1(\tau) &= \gamma u_x = 0 \\ U^2(\tau) &= \gamma u_y = 0 \\ U^3(\tau) &= \gamma u_z = c\sinh\left(\frac{eE_0}{mc}\tau\right) \end{aligned}}$$

For an electron, then, consider e as a negative constant.

c) If electron accelerates up to energy $E = 10mc^2 = \gamma mc^2$

this occurs when the proper time satisfies $U^0 = 10\,c = c\cosh\left(\frac{eE_0}{mc}\tau\right)$

or at proper time $\frac{eE_0}{mc}\tau = \cosh^{-1}(10)$.

Can convert this to laboratory time via the transformation, $c\,d\tau = \sqrt{c^2dt^2 - d\vec{x}^2}$

$$\Rightarrow \quad d\tau = dt\sqrt{1 - \frac{1}{c^2}\left(\frac{d\vec{x}}{dt}\right)^2} = dt\sqrt{1 - \left(\frac{\vec{u}}{c}\right)^2}\ , \text{ where } \vec{u} = \frac{U^3}{\gamma}\hat{z}.$$

$$dt = \frac{d\tau}{\sqrt{1 - \frac{1}{c^2}u_z^2}} \quad \text{but } \frac{u_z}{c} = \frac{U^3}{U^0} = \frac{\sinh\frac{eE_0}{mc}\tau}{\cosh\frac{eE_0}{mc}\tau} = \tanh\left(\frac{eE_0}{mc}\tau\right)$$

$$dt = \frac{d\tau}{\sqrt{1 - \tanh^2\omega\tau}} = \frac{d\tau}{\operatorname{sech}\omega\tau} = (\cosh\omega\tau)\,d\tau.$$

$$\boxed{t = \int_0^\tau d\tau\,\cosh\omega\tau = \frac{1}{\omega}\sinh\omega\tau.} \quad [@\ \cosh\omega\tau = 10].$$

With $\cosh^2x - \sinh^2x = 1$, when $\cosh x = 10$, $\sinh x = \sqrt{10^2-1} = \sqrt{99}$.

$$\boxed{t = \frac{1}{\omega}\sinh\omega\tau = \frac{1}{\omega}\sqrt{\cosh^2\omega\tau - 1} = \frac{1}{\omega}\sqrt{99} = \sqrt{99}\,\frac{mc}{eE_0}}$$

d) Distance travelled? Could calculate distance along z, if you like.

$$u_z(\tau) = c\cdot\frac{U^3}{U^0} = c\tanh(\omega\tau)$$

but $dt = \cosh\omega\tau\ d\tau$

$$z = \int u_z\,dt = c\int\frac{\sinh\omega\tau\ dt}{\cosh\omega\tau} = \frac{c}{\omega}\int^{\cosh\omega\tau}\frac{d(\cosh\omega\tau)}{\cosh\omega\tau} = \frac{c}{\omega}\cdot(\cosh\omega\tau)\Big|_{\tau=0}^{\tau}$$

$$z = \frac{c}{\omega}\times[\cosh\omega\tau - 1] \quad \text{Final point when } \cosh\omega\tau = 10.$$

$$\boxed{z = \frac{c}{\omega}\times 9 = \frac{mc^2}{eE_0}\times 9}$$

change in KE.

Same as if you looked at work $eE_0 z = 9\,mc^2$ ✓ ok!

e) Bonus. Numerical answers for $E_0 = 1.00\ \mathrm{kV/m}$.

This is an SI unit, so perhaps be careful in application of formulas.

For the time to get to $E = 10\,mc^2$, formula is OK as is

$$t = \sqrt{99}\cdot\frac{(9.11\times10^{-31}\,\mathrm{kg})(2.9979\times10^{8}\,\mathrm{m/s})}{(1.602\times10^{-19}\,\mathrm{C})(1000\ \mathrm{V/m})} = 1.70\times10^{-5}\,\mathrm{s} = \underline{17.0\,\mu\mathrm{s}.}$$

For the distance, also the formula will give correct units,

$$z = 9\,\frac{mc^2}{eE_0} = 9\,\frac{(9.11\times10^{-31}\,\mathrm{kg})(2.9979\times10^{8}\,\mathrm{m/s})^2}{(1.602\times10^{-19}\,\mathrm{C})(1000\ \mathrm{V/m})} = \underline{4600\,\mathrm{m}}$$

2. Consider a positron ($mc^2 = 0.511$ MeV, $q = +e$) moving at constant velocity (along $\hat{x}_1$-axis) as it passes close to a neutral atom at impact parameter $b = 1.0\ \mu$m. Jackson gives the formula (11.152) for the fields seen in the lab frame (the atom is at rest in the lab frame)

$$E_1 = E_1' = -\frac{q\gamma vt}{(b^2+\gamma^2v^2t^2)^{3/2}} \tag{1}$$

$$E_2 = \gamma E_2' = \frac{q\gamma b}{(b^2+\gamma^2v^2t^2)^{3/2}} \tag{2}$$

$$B_3 = \gamma\beta E_2' = \beta E_2 \tag{3}$$

a) (16) Do the "conversion" of these CGS formulas to SI units, using the basic relations, $\vec{E}_{\rm SI} = \frac{1}{4\pi\epsilon_0}\vec{E}_{\rm CGS}$, $\vec{B}_{\rm SI} = \frac{\mu_0 c}{4\pi}\vec{B}_{\rm CGS}$.

b) (20) If the positron is travelling at a low speed $v = 0.01c$, what are the peak electric and magnetic field strengths felt by the atom? Give numbers in V/m and tesla.

c) (20) If the positron instead has total energy of 511 Mev, what are the peak electric and magnetic field strengths felt by the atom? Give numbers in V/m and tesla.

d) (16) For a 511 MeV positron, over what time interval (in seconds) is the magnetic field strength appreciable?

e) (Bonus=20) For extra credit, what positron energy would be necessary to produce a peak magnetic field strength of 1.0 tesla?

a) $$E_1 = E_1' = -\frac{q\gamma vt/4\pi\varepsilon_0}{(b^2+\gamma^2v^2t^2)^{3/2}}$$

$$E_2 = \gamma E_2' = \frac{q\gamma b/4\pi\varepsilon_0}{(b^2+\gamma^2v^2t^2)^{3/2}}$$

$$B_3 = \frac{\mu_0 c}{4\pi}\gamma\beta E_2' = \frac{\mu_0 c}{4\pi}\beta E_2$$

old formulas, equation (2).

i.e. The last equation should be written out,

$$B_3 = \frac{\mu_0 c}{4\pi}\beta \cdot \frac{q\gamma b}{(b^2+\gamma^2v^2t^2)^{3/2}}$$ everything here is SI units.

or, $$B_3 = 4\pi\varepsilon_0\frac{\mu_0 c}{4\pi}\beta\cdot\frac{q\gamma b/4\pi\varepsilon_0}{(b^2+\gamma^2v^2t^2)^{3/2}} = \frac{\beta}{c}\left[\frac{q\gamma b/4\pi\varepsilon_0}{(b^2+\gamma^2v^2t^2)^{3/2}}\right]$$

or, $\boxed{B_3 = \frac{\beta}{c}E_2}$ All in SI.

the new SI, E_2.

b) At a nonrelativistic velocity $v = 0.01c$, use $\gamma \simeq 1$, $\beta = 0.01$.
The fields are approximated by

$$E_1 \simeq \frac{qvt/4\pi\varepsilon_0}{(b^2+v^2t^2)^{3/2}}, \quad E_2 \simeq \frac{qb/4\pi\varepsilon_0}{(b^2+v^2t^2)^{3/2}}, \quad B_3 \simeq \frac{\mu_0 c}{4\pi}\beta\frac{qb}{(b^2+v^2t^2)^{3/2}}$$

Their peak values occur at the point of closest approach, @ $t = 0$.

$$E_1 = 0, \quad E_2 = \frac{q/4\pi\varepsilon_0}{b^2}, \quad B_3 = \frac{\mu_0 c\beta}{4\pi}\frac{q}{b^2}$$

$$\text{Peak } E = \frac{1}{4\pi\varepsilon_0}\frac{q}{b^2} = \frac{1.602\times10^{-19}\,C}{4\pi(8.854\times10^{-12}\frac{F}{m})(10^{-6}m)^2} = \underline{1440\ V/m.}$$

$$\text{peak } B = \frac{\mu_0}{4\pi}\frac{qv}{b^2} = \left(10^{-7}\frac{Tm}{A}\right)\frac{(1.602\times10^{-19}C)(3\times10^{6}\,m/s)}{(10^{-6}m)^2} = \underline{4.8\times10^{-8}\ T}$$

c) Positron with $E = 511$ MeV, has $\gamma = \frac{E}{m} = 10^3$, $\beta \simeq 1$
The peak fields again occur at $t=0$, but now keep the γ-factors, in field components E_2 and B_3:

$$\text{peak } E = \frac{1}{4\pi\varepsilon_0}\frac{\gamma q}{b^2} = \frac{10^3(1.602\times10^{-19}C)}{4\pi\varepsilon_0\,(10^{-6}m)^2} = \underline{1.44\ MV/m.}$$

$$\text{peak } B = \frac{\mu_0}{4\pi}\frac{\gamma qv}{b^2} = \left(10^{-7}\frac{Tm}{A}\right)\frac{10^3(1.602\times10^{-19}C)(3\times10^{8}\frac{m}{s})}{(10^{-6}m)^2} = \underline{4.8\times10^{-3}\ T}$$

d) Time interval of large $\vec{B}$ must be when $\gamma vt \simeq \pm b$

$$\text{or} \quad \Delta t \simeq \frac{2b}{\gamma v} \simeq \frac{2b}{\gamma c} = \frac{2(10^{-6}m)}{10^3(3\times10^8\frac{m}{s})} = \underline{6.7\times10^{-18}\ s}$$

e) Bonus. To get $B_{peak} = \frac{\mu_0}{4\pi}\frac{\gamma qv}{b^2} \simeq 1.0$ Tesla, put $v \simeq c$, required γ is

$$\gamma = \frac{(1.0T)(10^{-6}m)^2}{\frac{\mu_0}{4\pi}\,qc} = \frac{(1.0T)(10^{-6}m)^2}{10^{-7}\frac{Tm}{A}(1.602\times10^{-19}C)(3\times10^8\frac{m}{s})} = 208000.$$

$$E = \gamma mc^2 = (208000)(0.511\,MeV) = 106000\ MeV = \underline{106\ GeV.}$$