Instructions: Use SI units. Where appropriate, define all variables or symbols you use, in words. Try to tell about the physics involved, more than the mathematics, if possible.

1. (8) A charge density $\rho(x)$ is invariant when the system is rotated through any angle around the $z$-axis. How can you write the general solution of Poisson’s equation for the potential $\Phi(x)$ in spherical coordinates in this situation?

2. (8) Write out the orthogonality condition for the Legendre polynomials $P_l(\cos \theta)$, when defined so that $P_l(0) = 1$ for all $l$.

3. (8) An electric dipole $p$ at position $x_0$ is exposed to an external electric field $E(x)$. Write an expression for the interaction energy of the dipole with this field.

4. (12) The electric potential of an electric dipole $p$ at the origin is $\Phi(x) = k \frac{p \cdot x}{r^3}$. From this show how to write an expression for the electric field $E(x)$ at point $x$ due to an electric dipole at some position $x_0$. 
5. (8) Give a definition of electric polarization $\mathbf{P}$ of a medium.

6. (8) How is electric displacement $\mathbf{D}$ of a medium defined?

7. (8) What is a formula that gives the bound charge density $\rho_b$ inside a dielectric medium?

8. (8) A dielectric medium has an electric susceptibility tensor with Cartesian components $\chi_{ij}$, where $i, j$ correspond to $x, y, z$. How is $\chi_{ij}$ used or what does it mean?

9. (8) A current density all over space is given by $\mathbf{J}(\mathbf{x})$. Write an integral expression that gives the magnetic induction $\mathbf{B}(\mathbf{x})$ due to that current density.

10. (8) Write an integral expression for the vector potential $\mathbf{A}(\mathbf{x})$ due to a current density $\mathbf{J}(\mathbf{x})$. 
11. (16) A localized current density $\mathbf{J}(x)$ produces a magnetic dipole moment $\mathbf{m}$.

(a) (8) Write an integral expression for $\mathbf{m}$ in terms of $\mathbf{J}$.

(b) (8) Write an expression for the vector potential produced by $\mathbf{m}$.

12. (8) Give a definition of the magnetization $\mathbf{M}(x)$ of a magnetic material.

13. (8) How is an effective bound current density derived from some magnetization $\mathbf{M}(x)$?

14. (8) How is the magnetic field $\mathbf{H}$ defined?

Part A Score = _________/124
Instructions: Use SI units. Please show the details of your derivations here. Explain your reasoning for full credit. Open-book only, no notes.

1. (36) Inside a sphere of radius $a$ the electric permittivity is a uniform value $\epsilon_1$. Outside there is a another medium with uniform electric permittivity $\epsilon_2$. A point charge $q$ is placed at the center of the sphere (the origin).

(a) (12) How large is the electric field at $r = 2a$?
(b) (12) How large is the free surface charge density at $r = a$?
(c) (12) How large is the bound surface charge density at $r = a$?
2. Consider the problem of finding an electrostatic potential $\Phi(\rho, \phi, z)$ in the region $z > 0$, above an infinite plane, on which the potential given in cylindrical coordinates is $V(\rho, \phi)$.

(a) Using cylindrical coordinates, write an integral expression for the potential $\Phi(\rho, \phi, z)$ above the plane in terms of Bessel functions $J_m(x)$. It may contain some unknown expansion coefficients that depend on $m$.

(b) Using appropriate orthogonality conditions, determine an integral expression for those expansion coefficients.

(c) Now consider that the potential given on the plane is circularly symmetric,

$$V(\rho, \phi) = \begin{cases} V_0, & \rho < a \\ 0, & \rho > a \end{cases}$$

If possible, evaluate the expansion coefficients.

Hint: The recursion relations for the Bessel functions will be helpful.

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x).$$

$$J_{n-1}(x) - J_{n+1}(x) = 2 \frac{d}{dx} J_n(x).$$
3. (48) A magnetic field is created by a localized distribution of permanent magnetization \( \mathbf{M} \), without free currents, \( \mathbf{J}=0 \). Thus, assume \( \mathbf{M} \) can be derived from a magnetic scalar potential by \( \mathbf{H} = -\nabla \Phi_M \). The total magnetic energy to assemble such a system is to be found.

(a) (16) Use the appropriate Maxwell’s equations and get the differential equation that \( \Phi_M \) obeys.

(b) (16) Show that the field energy integral over all space is zero:

\[
W_0 = \frac{1}{2} \int d^3x \; \mathbf{B} \cdot \mathbf{H} = 0.
\]

Integration by parts or vector integral calculus identities might be helpful.

(c) (16) Combine the integral in (b) with the interaction of the dipoles in a field, \( W_{\text{int}} \), as would be based on a single-dipole interaction energy,

\[
u = -\mathbf{m} \cdot \mathbf{B}.
\]

Hint: Turn this into an integral \( W_{\text{int}} \) over the magnetization \( \mathbf{M} \). Consider the process of starting with \( \mathbf{M}=0 \) everywhere and then bringing it to its final values, understanding that \( \mathbf{H} \) and \( \mathbf{B} \) are being generated by \( \mathbf{M} \).

Show that the total system energy is

\[
W = W_0 + W_{\text{int}} = \frac{1}{2} \mu_0 \int d^3x \; (\mathbf{H}^2 - \mathbf{M}^2)
\]

Part B Score = _________/132