Electrodynamics I — Midterm - Part A - Closed Book — KSU 2005/10/17

Name Electro Dynamic

1. (3) Write Gauss' Law in differential form.

$$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \rho(\vec{r})/\epsilon_0, \qquad \text{or} \qquad \vec{\nabla}\cdot\vec{D} = \rho,$$

$\vec{E}$= electric field, ρ=volume charge density, ϵ_0=permittivity of vacuum, $\vec{D}$=electric displacement.

2. (3) Write Poisson's equation for the potential $\Phi(\vec{r})$ produced by a charge density $\rho(\vec{r})$.

$$\nabla^2\Phi(\vec{r}) = -\rho(\vec{r})/\epsilon_0.$$

3. (3) Write a differential equation that a Green function $G(\vec{r},\vec{r}\,')$ must solve.

$$\nabla^2 G(\vec{r},\vec{r}\,') = -4\pi\delta(\vec{r}-\vec{r}\,').$$

$\vec{r}'$ is the source point, $\vec{r}$ is the observation point or point where a response is measured.

4. (3) A problem has no boundaries. How can you write the solution $\Phi(\vec{r})$ to your above Poisson equation using a Green function?

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V d^3r'\; G(\vec{r},\vec{r}\,')\rho(\vec{r}\,').$$

where $\rho(\vec{r})$ is the given charge density within volume V, producing the field.

5. (4) A problem has boundaries with Dirichlet boundary conditions. Now how can you write the solution to the Poisson equation using a Green function?

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V d^3r'\; G(\vec{r},\vec{r}\,')\rho(\vec{r}\,') - \frac{1}{4\pi}\int_S da\,'\frac{\partial G}{\partial n\,'}\,\Phi(\vec{r}\,').$$

where the second integral is over the surface S bounding the system, and $\vec{n}\,'$ is a normal vector pointing out of the system.

6. (3) What is the formula for the free-field Green function in three dimensions?

$$G(\vec{r},\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|}.$$

7. (3) Once you know the solution for potential $\Phi(\vec{r})$, how can you obtain the surface charge density σ on a conductor boundary?

$$\sigma = \epsilon_0 E_s = -\epsilon_0\vec{E}\cdot\hat{n} = \epsilon_0\frac{\partial\Phi}{\partial n},$$

where E_s is the component pointing* out *of the conductor, while $\hat{n}$ is the normal pointing out of volume V,* into *the conductor.

8. (3) How can you use $\Phi(\vec{r})$ to obtain the energy density in an electric field in a linear dielectric medium?

$$w = \frac{1}{2}\vec{E}\cdot\vec{D} = \frac{1}{2}(-\nabla\Phi)\cdot(-\epsilon\nabla\Phi) = \frac{1}{2}\epsilon|\vec{\nabla}\Phi|^2.$$

9. (3) What is the capacitance of a conducting sphere of radius R?

$$C = \frac{Q}{V} = \frac{Q}{Q/(4\pi\epsilon_0 R)} = 4\pi\epsilon_0 R.$$

10. (9) Use delta-functions to express the charge density $\rho(\vec{r})$ for the following charge distributions, in the indicated coordinate systems:

a) A charge Q distributed uniformly over a spherical shell of radius a, in spherical coordinates (r, ϕ, θ).

$$\rho(\vec{r}) = \frac{Q}{4\pi r^2}\, \delta(r - a).$$

b) A point charge q on the z-axis at $z = z_0$, in cylindrical coordinate (ρ, ϕ, z).

$$\rho(\vec{r}) = \frac{q}{2\pi\rho}\, \delta(\rho)\ \delta(z - z_0).$$

c) The same charge, in spherical coordinates (r, ϕ, θ).

$$\rho(\vec{r}) = \frac{q}{2\pi r^2}\, \delta(r - z_0)\ \delta(cos\theta - 1).$$

11. (9) Give the orthogonality relations for

a) Legendre polynomials

$$\int_{-1}^{1} dx\ P_l(x) P_{l'}(x) = \frac{2}{2l+1}\delta_{l,l'}.$$

a) spherical harmonics

$$\int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta) Y^*_{l,m}(\theta,\phi) Y_{l',m'}(\theta,\phi) = \delta_{l,l'}\delta_{m,m'}.$$

a) Bessel functions $J_m(k\rho)$ on $0 \le \rho \le a$

$$\int_0^a \rho\ d\rho\ J_m(\frac{x_{m,n}}{a}\rho) J_m(\frac{x_{m,n'}}{a}\rho) = \frac{a^2}{2}[J_{m+1}(x_{m,n})]^2 \delta_{n,n'},$$

where $x_{m,n}$ and $x_{m,n'}$ are any zeroes of the $J_m(x)$ Bessel functions.

12. (3) Give an expression defining the electric dipole moment of a charge distribution:

$$\vec{p} = \int_V d^3x\ \rho(\vec{r})\ \vec{r}.$$

13. (3) Give an expression defining the electric polarization in a medium:

$$\vec{P}(\vec{r}) = \sum_i n_i(\vec{r})\ \langle \vec{p}_i \rangle,$$

where $n_i(\vec{r})$ is the number of i^{th} type of electric dipoles per unit volume at $\vec{r}$, with their average value indicated by the brackets.

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Instructions: Use SI units. Please show the details of your derivations here. Explain your reasoning for full credit. Open-book only, no notes.

14. Within a sphere of radius R there is a non-zero charge density,

$$\rho(\vec{r}) = \rho(r,\theta,\phi) = \rho_0 \frac{R}{r} \sin^2\theta. \tag{1}$$

It is surrounded by an infinite vacuum.

a) (6) Determine the total charge.

$$\begin{aligned} q = \int_V d^3r \rho(\vec{r}) &= \int_0^R dr\ r^2 \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta)\ \rho_0 \frac{R}{r}(1-\cos^2\theta) \\ &= \rho_0 R[\int_0^R dr\ r]\ [2\pi]\ [\int_{-1}^{1} dx\ (1-x^2)] \\ &= \rho_0 R[R^2/2]\ [2\pi]\ [2(1-1/3)] \\ &= \frac{4\pi R^3}{3}\rho_0. \end{aligned}$$

b) (6) Express $\rho(\vec{r})$ in terms of Legendre polynomials. Why this is a good thing to do?

As $\sin^2\theta = 1-\cos^2\theta$, and $P_0(x)=1, P_2(x) = (3x^2-1)/2$, do the replacement:

$$\cos^2\theta = x^2 = (2P_2+1)/3, \qquad \text{then}$$

$$\rho(\vec{r}) = \rho_0 \frac{R}{r}(1-x^2) = \rho_0 \frac{R}{r}[1-(2P_2+1)/3]$$

$$\rho(\vec{r}) = \rho_0 \frac{R}{r}\cdot \frac{2}{3}[P_0(x) - P_2(x)].$$

It shows the symmetry of the charge distribution, which will lead to a similar symmetry in the resulting potential field.

c) (8) Determine the potential $\Phi(r,\theta,\phi)$ for points outside the sphere. Think about different ways to do this, perhaps, before proceeding.

In this region $\rho = 0$, so we are solving Laplace's equation with azimuthal symmetry and $r > r'$, then we can use the general form of solution,

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}[A_l r^l + B_l r^{-(l+1)}]P_l(\cos\theta),$$

The potential must be finite at $r\to\infty$, so all $A_l = 0$. The B_l coefficients can be found if we know the potential on the z-axis. This can be accomplished taking observation point $\vec{r}$ on the z-axis, from

$$\Phi(z) = \frac{1}{4\pi\epsilon_0}\int_V d^3r' \frac{1}{|\vec{r}-\vec{r}\,'|}\ \rho(\vec{r}\,'),$$

$$\text{with} \qquad \frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{l=0}^{\infty} \frac{(r')^l}{z^{(l+1)}} P_l(\cos\theta')$$

Only the $l = 0, 2$ terms will survive from the integration $\int_{-1}^{1} d\cos\theta' P_l P_l' = 2/(2l+1)\delta_{l,l'}$. Then what remains is

$$\Phi(z) = \frac{1}{4\pi\epsilon_0}\int_0^R dr\,'2\pi(r\,')^2\rho_0\frac{R}{r\,'}\frac{2}{3}\left[\frac{2}{z} - \frac{2}{5}\frac{(r\,')^2}{z^3}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{4\pi}{3}\rho_0 R\left[\frac{2R^2}{2z} - \frac{2}{5}\frac{R^4}{4z^3}\right]$$

$$= \frac{\rho_0 R^2}{3\epsilon_0}\left[\frac{R}{z} - \frac{R^3}{10z^3}\right]$$

Then we can read off the coefficients,

$$B_0 = \frac{\rho_0 R^3}{3\epsilon_0}, \qquad B_2 = -\frac{\rho_0 R^5}{30\epsilon_0}$$

the potential <u>outside</u> the sphere of charge is

$$\Phi(r,\theta) = \frac{\rho_0 R^2}{3\epsilon_0}\left[\frac{R}{r}P_0(\cos\theta) - \frac{R^3}{10r^3}P_2(\cos\theta)\right].$$

d) (12 Bonus pts) Determine the potential $\Phi(r,\theta,\phi)$ for points *inside* the sphere. Be more careful here, what equation are you solving?

In this case you have a region filled with charge, so you are solving the *Poisson* ***equation. The solution must be expressed as***

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V d^3r\,'\frac{1}{|\vec{r}-\vec{r}\,'|}\;\rho(\vec{r}\,'),$$

and the best way to go now is to use

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{l,m}^*(\theta',\phi')Y_{l,m}(\theta,\phi).$$

It is convenient to express the charge density in terms of Y_{lm}, via

$$P_l(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}}Y_{l,0}(\theta,\phi).$$

$$\rho(\vec{r}) = \rho_0\frac{R}{r}\cdot\frac{2}{3}\sqrt{4\pi}\left[Y_{0,0}(\theta,\phi) - \frac{1}{\sqrt{5}}Y_{2,0}(\theta,\phi)\right].$$

Putting these together, doing the angular integrations, and using the orthogonality of the spherical harmonics, then going back to the Legendre polynomial description, there remains some ugly radial integration:

$$\Phi(r,\theta) = \frac{1}{4\pi\epsilon_0}\int_0^R dr'(r')^2\rho_0\frac{R}{r\,'}\frac{2}{3}\sqrt{4\pi}4\pi\left[\frac{1}{r_>}Y_{0,0}(\theta,\phi) - \frac{1}{5}\frac{r_<^2}{r_>^3}\frac{1}{\sqrt{5}}Y_{2,0}(\theta,\phi)\right]$$

$$\Phi(r,\theta) = \frac{1}{\epsilon_0}\int_0^R dr\,'(r\,')^2\rho_0\frac{R}{r\,'}\frac{2}{3}\left[\frac{1}{r_>}P_0(\cos\theta) - \frac{1}{5}\frac{r_<^2}{r_>^3}P_2(\cos\theta)\right]$$

The integrations need to be split into two parts:
1) $r\,'$ *goes from* 0 *to* r*, with* $r_< = r\,', r_> = r$*.*
2) $r\,'$ *goes from* r *to* R*, with* $r_< = r, r_> = r\,'$*.*

For P_0 *coefficient:*

$$\int_0^R dr\,'\frac{r\,'}{r_>} = \int_0^r dr\,'\frac{r\,'}{r} + \int_r^R dr\,'\frac{r\,'}{r\,'}$$

$$= \frac{r}{2} + (R - r) = R - \frac{r}{2}.$$

For P_2 *coefficient:*

$$\int_0^R dr\,'r\,'\frac{r_<^2}{r_>^3} = \int_0^r dr\,'\frac{r\,'^3}{r^3} + \int_r^R dr\,'\frac{r^2}{r\,'^2}$$

$$= \frac{r}{4} - r^2(\frac{1}{R} - \frac{1}{r}) = \frac{5r}{4} - \frac{r^2}{R}.$$

Putting these results all together, the potential *__inside__* *the sphere of charge is:*

$$\Phi(r,\theta) = \frac{\rho_0 R^2}{3\epsilon_0}\left[(2 - \frac{r}{R})P_0(\cos\theta) - \frac{1}{5}(\frac{5r}{2R} - \frac{2r^2}{R^2})P_2(\cos\theta)\right]$$

Check the result at $r = R$*, where the inside and outside solutions should match.*

$$\Phi(R,\theta) = \frac{\rho_0 R^2}{3\epsilon_0}\left[P_0(\cos\theta) - \frac{1}{10}P_2(\cos\theta)\right]$$

so in fact these match correctly! But note carefully that the internal potential does not take the simple form with single powers of r *multiplying Legendre Polynomials, because it is a solution of Poisson's equation, not Laplace's equation.*

15. A very long right circular cylinder of uniform permittivity ϵ, radius a, is placed into a vacuum containing a previously uniform electric field $\vec{E}_0$ oriented perpendicular to the axis of the cylinder.

a) (4) Ignoring end effects, write general expressions for the potential inside and outside the cylinder.

Since the cylinder is very long, this is a two-dimensional electrostatics problem, with a potential $\Phi(\rho, \phi)$. Then the expected standard form of the potential applies.
Inside the cylinder, which inlcudes $\rho \to 0$, there can only be positive powers of ρ, so we assume:

$$\Phi_{\text{in}} = \sum_{\nu=1}^{\infty} [A_\nu \cos(\nu\phi) + B_\nu \sin(\nu\phi)]\rho^\nu.$$

Outside the cylinder, there can only be decaying powers of ρ, except for a term to give a uniform asymptotic field, so we assume:

$$\Phi_{\text{out}} = -E_0 \rho \cos\phi + \sum_{\nu=1}^{\infty} [C_\nu \cos(\nu\phi) + D_\nu \sin(\nu\phi)]\rho^{-\nu}.$$

The first term produces an electric field strength E_0 in the x-direction.

b) (8) State and apply the appropriate boundary conditions at the surface of the cylinder.

At the cylinder surface, we must have continuity of the normal component of $\vec{D}$, and the tangential component of $\vec{E}$.
For normal $\vec{D}$, match the ρ components:

$$-\epsilon \frac{\partial \Phi_{\text{in}}}{\partial \rho}|_{\rho=a} = -\epsilon_0 \frac{\partial \Phi_{\text{out}}}{\partial \rho}|_{\rho=a}.$$

Then matching the coefficients of the sines and cosines, which are linearly indepedent, we get:

$$\epsilon A_1 = \epsilon_0[-E_0 - C_1 a^{-2}], \qquad (\nu = 1)$$

$$\epsilon A_\nu = -\epsilon_0 C_\nu a^{-\nu-1}, \qquad (\nu > 1)$$

$$\epsilon B_\nu = -\epsilon_0 D_\nu a^{-\nu-1}, \qquad (\nu > 1)$$

For tangential $\vec{E}$, match the ϕ components:

$$-\frac{1}{a}\frac{\partial \Phi_{\text{in}}}{\partial \phi}|_{\rho=a} = -\frac{1}{a}\frac{\partial \Phi_{\text{out}}}{\partial \phi}|_{\rho=a}.$$

Again matching coefficients,

$$A_1 = -E_0 + C_1 a^{-2}, \qquad (\nu = 1)$$

$$A_\nu = C_\nu a^{-\nu-1}, \qquad (\nu > 1)$$

$$B_\nu = D_\nu a^{-\nu-1}, \qquad (\nu > 1)$$

c) (8) Determine the potential inside and outside the cylinder.

The homogenous equations for $\nu > 1$ mean those coefficients are all zero. We can easily solve for A_1 and C_1,

$$\frac{\epsilon}{\epsilon_0}(-E_0 + C_1 a^{-2}) = -E_0 - C_1 a^{-2},$$

$$C_1 = \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} E_0 a^2$$

$$A_1 = -\frac{\epsilon_0}{\epsilon}[E_0 + C_1 a^{-2}] = \frac{-2\epsilon_0}{\epsilon + \epsilon_0} E_0.$$

$$\Phi_{\text{in}} = A_1 \rho \cos\phi = \frac{-2\epsilon_0}{\epsilon + \epsilon_0} E_0 \rho \cos\phi = \frac{-2\epsilon_0}{\epsilon + \epsilon_0} E_0 x.$$

$$\Phi_{\text{out}} = -E_0 \rho \cos\phi + C_1 \cos\phi \rho^{-1} = -E_0 \rho \cos\phi + \left(\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\right) \frac{a^2}{\rho^2} E_0 \rho \cos\phi$$

$$\Phi_{\text{out}} = \left[-1 + \left(\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\right) \frac{a^2}{\rho^2}\right] E_0 \rho \cos\phi.$$

d) (8) Determine the electric field inside and outside the cylinder. How does the field strength inside compare to that outside, is it what you expect based on physical arguments?

Inside, the field is along x, with a uniform strength,

$$E_{\text{in, x}} = -\frac{\partial \Phi_{\text{in}}}{\partial x} = \frac{2\epsilon_0}{\epsilon + \epsilon_0} E_0.$$

This is smaller than the field at great distance from the cylinder, as expected due to the polarization induced in the dielectric.
Outside, there is a superposition of the uniform applied field, together with one decaying as ρ^{-2}

$$E_{\text{out}, \rho} = -\frac{\partial \Phi_{\text{out}}}{\partial \rho} = \left[1 + \left(\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\right) \frac{a^2}{\rho^2}\right] E_0 \cos\phi.$$

$$E_{\text{out}, \phi} = -\frac{1}{\rho} \frac{\partial \Phi_{\text{out}}}{\partial \phi} = \left[-1 + \left(\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\right) \frac{a^2}{\rho^2}\right] E_0 \sin\phi.$$

Convert it to Cartesian components, it is:

$$E_{\text{out, x}} = E_{\text{out}, \rho} \cos\phi - E_{\text{out}, \phi} \sin\phi = E_0 + \left(\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\right) \frac{a^2}{\rho^2} E_0 \cos 2\phi$$

$$E_{\text{out, y}} = E_{\text{out}, \rho} \sin\phi + E_{\text{out}, \phi} \cos\phi = \left(\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\right) \frac{a^2}{\rho^2} E_0 \sin 2\phi$$