1. A two-dimensional isotropic oscillator has the Hamiltonian

$$H = \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{2}m\omega^2(1 + bxy)(x^2 + y^2) \tag{1}$$

The anharmonic term of strength b represents the deviation of the potential from harmonic form.

a) If $b = 0$, determine the wavefunctions and energies of the three lowest energy states. [Hint: the wavefunctions separate in the form $\Psi(x, y) = f(x)g(y)$.]

b) Now the anharmonic perturbation $b \ll 1$ is turned on. Determine the first order perturbation corrections to the energies of the three lowest states.

c) Evaluate the energy corrections in units of $\hbar\omega$ when $b = 0.1\left(\frac{\hbar}{m\omega}\right)$.

2. Consider a hydrogen atom where the potential is of the Yukawa form,

$$V(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{e^{-\mu r}}{r} \tag{2}$$

where $\mu = m_\gamma c/\hbar$ is produced by a nonzero photon mass m_γ.

Use the Bohr 1s wavefunction as a trial variational wavefunction,

$$\psi(\mathbf{r}) = \sqrt{\frac{Z^3}{\pi}}\, e^{-Zr} \tag{3}$$

where Z is the variational parameter, and determine

a) Expression for $\langle T\rangle$

b) Expression for $\langle V\rangle$

c) Expression for $\langle H\rangle$, with approximation $\mu/Z \ll 1$, to order $(\mu/Z)^2$.

d) Estimate the approximate value of Z that minimizes $\langle H\rangle$ [Hint: It is a *small* correction to the value Z would have when $\mu = 0$.].

3. Consider an ultrarelativistic degenerate gas of N electrons, such that the kinetic energy $\hbar^2k^2/2m$ is replaced by $\hbar ck$. A k-space volume of π^3/V still has one orbital which can hold two electrons, where V is the system volume.

a) Write a formula for the energy dE from the states within a k-space shell of thickness dk at radius k.

b) By integrating over the shells in k-space, determine the total electronic energy for N electrons in a volume V.