1. A particle of mass m is confined in an infinite 1D square well of width a centered on the origin, that has a delta-function spike in the middle, of strength α:

$$\begin{aligned} V(x) = &\ \alpha\delta(x), \qquad \text{for} \quad |x| < a/2, \\ = &\ \infty, \qquad\quad \text{for} \quad |x| \geq a/2. \end{aligned} \tag{1}$$

We want to look for bound states of this potential.

a) Write down general expressions for the wavefunction $\psi_I(x)$ in the region $x < 0$, and for $\psi_{II}(x)$ in the region $x > 0$.

b) Apply appropriate boundary conditions at $|x| = a/2$ and thereby eliminate some of your undetermined constants.

c) Apply any needed matching conditions on $\psi(x)$ at $x = 0$.

d) IF you have time (look at problems 2 and 3 first!), find an equation that will determine the allowed wavevectors, k. Do not solve it.

2. The usual result for the energy levels of a hydrogen atom is (MKSA)

$$E_n = \frac{-m_e}{2}\left(\frac{e^2}{4\pi\epsilon_0\hbar}\right)^2 \frac{1}{n^2} = \frac{-13.6\text{eV}}{n^2}. \tag{2}$$

Consider the $n = 2$ to $n = 1$ (Lyman-α) transition.

a) Find a *formula* to estimate the energy splitting in this line if the hydrogen is a mixture of normal hydrogen and heavy hydrogen (deuterium). Hint: What are the corrections due to nuclear motion? Proton mass $\approx$ neutron mass $\approx 1.67 \times 10^{-27}$ kg.

b) Estimate numerically this splitting in electron volts.

3. Consider a 1D square well of width a, with $0 < x < a$, with single-particle states,

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right), \tag{3}$$

$$E_n = \left(\frac{\pi^2\hbar^2}{2ma^2}\right) n^2 \equiv E_1 n^2, \qquad n = 1, 2, 3... \tag{4}$$

a) Suppose you put two identical spin-1/2 particles in this well. where the wavefunction is a product over space and spin parts, $\Psi = \psi(x_1, x_2)\chi(\vec{S}_1, \vec{S}_2)$. Give an expression for the ground state wavefunction Ψ, and give its energy in terms of E_1.

b) Repeat a) for the first excited state.

c) Now suppose you put two identical spin-1 particles in the well, again in a product over space and spin parts, $\Psi = \psi(x_1, x_2)\chi(\vec{S}_1, \vec{S}_2)$. If the spin part is a $j = 2$ state

$$\chi(\vec{S}_1, \vec{S}_2) = |j = 2, m_j = 0\rangle = \sqrt{\frac{1}{6}}\Big\{|1\rangle|-1\rangle + 2|0\rangle|0\rangle + |-1\rangle|1\rangle\Big\} \tag{5}$$

(numbers in brakets are m_{s_1} and m_{s_2}, respectively), then what is the associated lowest energy space wavefunction $\psi_0(x_1, x_2)$ and energy.

d) Repeat c) for a $j = 1$ spin state:

$$\chi(\vec{S}_1, \vec{S}_2) = |j = 1, m_j = 0\rangle = \sqrt{\frac{1}{2}}\Big\{|1\rangle|-1\rangle - |-1\rangle|1\rangle\Big\} \tag{6}$$