Nonlinear Processes for the Generation of Quadrature Squeezed Light

This project investigates the use of a nonlinear optical process for the generation of nonclassical light. Do the first three questions for full credit. The other questions will be extra credit.

Consider the superposition state $|\psi\rangle = a|0\rangle + b|1\rangle$ where *a* and *b* are complex and satisfy the relationship $|a|^2 + |b|^2 = 1$.

1. Calculate the variances of the quadrature operators $\hat{X}_1$ and $\hat{X}_2$ (see Eq. 2.52 and Eq. 2.53). The variance of an operator is given by

$$\left\langle (\Delta\hat{X}_i)^2 \right\rangle = \left\langle \hat{X}_i^{\,2} \right\rangle - \left\langle \hat{X}_i \right\rangle^2$$

   Remember that $\hat{X}_1$ is called the in-phase component and $\hat{X}_2$ is the in-quadrature component.
2. Show that there exits values of the parameters *a* and *b* for which either of the quadrature variances become *less* than for a vacuum state. Hint: let $b = \sqrt{1-|a|}e^{i\varphi}$ and $a^2 = |a|^2$ (this is done without the loss of generality). Plot the variance as a function of $|a|^2$ for different φ.
3. For the cases where the quadrature variances become less than for a vacuum state, check to see if the uncertainty principle is violated.
4. Verify that the quantum fluctuations of the field quadrature operators are the same for the vacuum when the field is in coherent state (*i.e.* verify Eq. 3.16).

The above result illustrate a case where the expectation value of the quadrature operator becomes less than a vacuum state, even though the quadrature operators must satisfy the minimum uncertainty relationship. Squeezing is the process when one canonical (conjugate) variable has a variance less than the vacuum state but the other canonical variable will have a larger variation in order to satisfy the uncertainty principle. The quadrature operators $\hat{X}_1$ and $\hat{X}_2$ are canonical variables and do not commute, thus they have an uncertainty relationship given by Eq. 2.56. Quadrature squeezing occurs when

$$\left\langle (\Delta\hat{X}_1)^2 \right\rangle < \frac{1}{4} \text{ or } \left\langle (\Delta\hat{X}_2)^2 \right\rangle < \frac{1}{4}.$$

We can plot a phase space diagram of a normal and squeezed state (see below and on page 154). The area in phase space remains must constant to maintain the minimum uncertainty relationship. However, we can "squeeze" the circle into an ellipse while keeping the area constant (like squeezing the Charmin done in class).

Not Squeezed Squeezed

Quadrature squeezed light can be produced by the second order nonlinear effect known as **degenerate parametric down-conversion**. This process involves two signal (*s*) waves produced by one pump wave (*p*), *i.e.* $\omega_s + \omega_s - \omega_p = 0$. This process is a degenerate form of difference frequency generation with the signal wave equal to the idler wave. The Hamiltonian for this degenerate parametric down-conversion is given by

$$\hat{H} = \frac{\hbar}{2}\left(\chi^{(2)}\hat{a}_s^\dagger \hat{a}_p \hat{a}_s^\dagger + \chi^{*(2)}\hat{a}_s \hat{a}_p^\dagger \hat{a}_s\right)$$

5. Use the Heisenberg equations of motion (Eq. 2.19) to derived two coupled first order differential equation for $\frac{d\hat{a}_s}{dt}$ and $\frac{d\hat{a}_s^\dagger}{dt}$.
6. What are the solutions to these differential equations if we assume a non-depleted pump? Integrate from time 0 to *T*.
7. Show that the quadrature operators have the solution

$$\begin{bmatrix} \hat{X}_1(T) \\ \hat{X}_2(T) \end{bmatrix} = \begin{bmatrix} e^{-\delta T} & 0 \\ 0 & e^{\delta T} \end{bmatrix} \begin{bmatrix} \hat{X}_1(0) \\ \hat{X}_2(0) \end{bmatrix} \text{ where } \delta \equiv i\chi^{(2)}\hat{a}_p$$

8. Consider a coherent state $|\alpha\rangle$ Show the mean square fluctuations (variance) result in

$$\begin{bmatrix} \langle\alpha|(\hat{X}_1(T))^2|\alpha\rangle - \langle\alpha|\hat{X}_1(T)|\alpha\rangle^2 \\ \langle\alpha|(\hat{X}_2(T))^2|\alpha\rangle - \langle\alpha|\hat{X}_2(T)|\alpha\rangle^2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} e^{-2\delta T} \\ e^{+2\delta T} \end{bmatrix}$$

This result states that the mean square fluctuations of the in-phase component $\hat{X}_1$ is exponentially smaller by $e^{-2\delta T}$ and mean square fluctuations of the in-quadrature component $\hat{X}_2$ are exponentially larger by $e^{2\delta T}$. So the above picture depicts the squeezing performed by the nonlinear process. The bizarre thing about this analysis is that it is also true for a **vacuum state**. One can have vacuum and squeezed vacuum.

9. A third order nonlinear can also be used to produce squeezed light instead. Name a third order nonlinear process that will give rise to squeezed light (Hint: we discussed a third order process that "looks" like difference frequency generation. What was that process?).

Squeezed light and squeezed vacuum has many important applications, specifically for light detection at levels below the quantum noise (*i.e.* shot noise) level. See Henry *et al*, Amer. J. Phys. Vol 56 (4) p. 318 (1988) for more information.