1. **Soliton Propagation in a Single-Mode Optical Fiber**

An optical soliton forms due to the interplay of anomalous group velocity dispersion (GVD) and self-phase modulation (SPM) in an optical fiber. For an ultrashort pulse injected into the fiber, GVD causes the pulse temporal envelope to broaden while SPM causes the spectral width to increase. A soliton forms when the two effects are balanced, which happens when the total amount of dispersion and nonlinearity is just right. We can define the nonlinear length \( L_{NL} \) and dispersion length \( L_D \) in the fiber in terms of the peak power \( P_0 \), the pulse duration \( \Delta t \), group velocity dispersion \( \beta_2 \), and the effective nonlinearity \( \gamma \) by

\[
L_{NL} = \frac{1}{\gamma P_0} \quad \text{and} \quad L_D = \frac{T_0^2}{|\beta|} \quad \text{where} \quad T_0 = \frac{\Delta t}{2 \ln(1 + \sqrt{2})} \quad \text{and} \quad \gamma = \frac{n_i \omega}{c \pi \ell^2}. \]

A first order soliton occurs when \( L_{NL} / L_D = 1 \).

A hyperbolic secant pulse with center wavelength \( \lambda_0=1550 \) nm and pulse duration \( \Delta t=100 \) fs FWHM propagates through a length \( L_D \) of a single-mode optical fiber. The optical fiber has a core radius of \( r=4.1 \) µm and an index difference \( \Delta n=0.008 \) between the core and cladding index or refraction. The value for the nonlinear index of refraction is \( n_2=3 \times 10^{-20} \) m²/W. The fiber core consists of germanium-doped fused silica whose index of refraction is given by the three term Sellmeier equation (valid for wavelength in µm):

\[
n^2(\lambda) = 1 + \sum_{i=1}^{3} \frac{B_i \lambda^2}{\lambda^2 - C_i} \quad \text{where} \quad B_1 = 0.711040, B_2 = 0.451885, B_3 = 0.704048 \quad \text{and} \quad C_1 = 0.064270, C_2 = 0.129408, C_3 = 9.45478. \quad (1)
\]

(The fiber cladding consists of fused silica, which has a smaller index of refraction than germanium-doped fused silica. We will not need to use its Sellmeier equation for the problem.) The wave guiding due to the fiber geometry changes the total dispersion that the pulse experiences. The propagation constant \( \beta(\omega) \) for the fiber, which represents the \( z \) component of the wavevector \( k(\omega) \), is given by

\[
\beta(\omega) = n(\omega) \sqrt{1 + 2 \Delta n b(\omega)}.
\]

The propagation constant is expressed where \( \Delta n \) is the index difference between core and cladding, \( r \) is the core radius, and \( b(\omega) \) is the normalized mode propagation constant due to the fiber geometry given in terms of the normalized frequency \( V(\omega) \). An approximate form for \( b(\omega) \) is given by

\[
b(\omega) = 1 - \left( \frac{1 + \sqrt{2}}{1 + \sqrt{4 + V(\omega)}} \right)^2 \quad \text{where} \quad V(\omega) = \frac{r \omega}{c \ell} n(\omega) \sqrt{2 \Delta n}.
\]

1. Show that the value of the second order propagation constant \( \beta_2 \) (i.e. group velocity dispersion) at \( \lambda_0=1550 \) nm is -0.0000180 fs²/nm. \( \beta_2 \) can be determined from

\[
\beta_2(\omega_0) = \frac{d^2 \beta(\omega)}{d\omega^2} \bigg|_{\omega=\omega_0}
\]

2. What is \( L_D \)? Determine the peak power \( P_0 \) for where \( L_{NL} / L_D = 1 \).

3. Consider the pulse propagating through \( L_D \) of fiber experiencing only group velocity dispersion (no nonlinear effects). Plot the temporal chirp \( \omega_{GVD}(t) = \omega_b - \tilde{\omega}_b \varphi_{GVD}(t) \) of the pulse due only to GVD after \( L_D \).

4. Consider the pulse propagating through \( L_D \) of fiber experiencing only self-phase modulation (no dispersion). Plot the temporal chirp \( \omega_{SPM}(t) = \omega_b - \tilde{\omega}_b \varphi_{SPM}(t) \) of the pulse due only SPM after \( L_D \).

5. By comparing \( \omega_{GVD}(t) \) and \( \omega_{SPM}(t) \), explain how the interaction of SPM and GVD leads to soliton formation.
2. **Partially Degenerate Four Wave Mixing in a Single-Mode Optical Fiber**

We wish to determine the pump, signal, and idler frequencies for partially degenerate four-wave mixing (FWM) in an optical fiber. Partially degenerate FWM is described by

\[ 2\omega_p - \omega_s - \omega_i = 0 \]

where we use the terms pump (p), signal (s), and idler (i) as for difference frequency generation. Here we define \( \omega_i > \omega_s \).

A strong continuous wave laser serves as the pump at \( \omega_p \) of power \( P_0 = 0.5 \text{ MW} \). The pump is injected into an optical fiber with a germanium-doped fused silica core. The fiber has a core radius 4.1 µm and index difference \( \Delta n = 0.008 \) between the core and cladding indices (as in Problem 1).

1. Determine the signal and idler wavelengths produced through partial degenerate four wave mixing for pump wavelengths from \( \lambda_p = 900 \) to 2000 nm. To determine this for a given pump frequency \( \omega_p \), you will need to find the signal \( \omega_s \) and idler \( \omega_i \) frequencies that satisfies both energy conservation and phase matching:

\[
2\omega_p - \omega_s - \omega_i = 0 \\
\Delta k = \Delta k_{m} + \Delta k_{w} + \Delta k_{NL} = 0 \\
\text{where} \\
\Delta k_{m} = c^{-1} \left( n(\omega_i)\omega_s + n(\omega_s)\omega_i - 2n(\omega_p)\omega_p \right) \\
\Delta k_{w} = \Delta n c^{-1} \left( b(\omega_i)\omega_s + b(\omega_s)\omega_i - 2b(\omega_p)\omega_p \right) \\
\Delta k_{NL} = 2\gamma P_0
\]

The phase mismatch \( \Delta k \) has contributions due to material dispersion (\( \Delta k_{m} \)), waveguide dispersion (\( \Delta k_{w} \)), and the fiber nonlinearity (\( \Delta k_{NL} \)). To determine the phase mismatch, you will need to use the Sellmeier equation and \( b(\omega) \) from the previous problem.

2. Plot \( \lambda_s \) and \( \lambda_i \) versus \( \lambda_p \).

The zero group velocity dispersion wavelength \( \lambda_{GVD} \) is ~ 1345 nm for this fiber, which is determined using \( \beta(\omega) \). Notice that the behavior of \( \lambda_s \) versus \( \lambda_p \) and \( \lambda_i \) versus \( \lambda_p \) is different on the long and short wavelength sides of \( \lambda_{GVD} \).