Lecture Notes for Nonlinear and Quantum Optics

PHYS 953

Fall 2007

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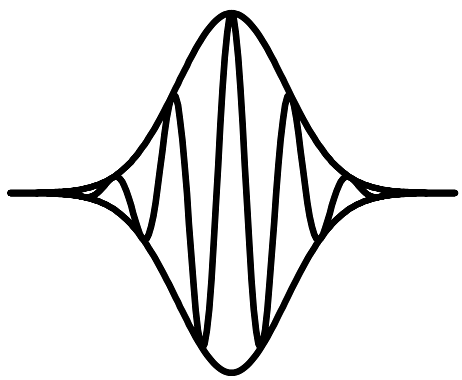
Kansas State University



PHYS 953 Nonlinear and Quantum Optics, Fall 2007
Lectures and Projects

PHYS 953 – Adv. Topics/Non-linear and Quantum Optics - Fall 2007
Lecture: M/W/F, 12:30-1:30 a.m. Willard 25

Textbooks: *Nonlinear Optics,* Boyd; *Introductory Quantum Optics*, Gerry and Knight;

Suggested References: *Introduction to Quantum Optics, From Light Quanta to Quantum Teleportation*, Paul; *The Quantum Challenge*, Greenstein and Zajonc; *Quantum Optics*, Walls and Milburn; *Coherence and Quantum Optics*, Mandel and Wolf; *Nonlinear Optics*, Shen; *Nonlinear Fiber Optics*, Agrawal; *Handbook of Nonlinear Optics*, Sutherland; *Handbook of Nonlinear Optical Crystals,* Dmitriev, Gurzadyan, and Nikogosyan; *Electromagnetic Noise and Quantum Optical Measurements*, Haus;

Instructor: Dr. Brian R. Washburn, CW 36B, (785) 532-2263, washburn@phys.ksu.edu. Office hours: M/W/F 9:30-10:30 PM or by appt.

Prerequisites: A solid foundation in undergraduate-level quantum mechanics, electromagnetism, and optics.

Course Objective: The purpose of this course is to provide an introduction to the field of nonlinear optics, exploring the physical mechanisms, applications, and experimental techniques. Furthermore the fundamentals of quantum optics will be taught in the second half in this course. Connections between quantum and nonlinear optics will be highlighted throughout the semester. My goal is for students to end up with a working knowledge of nonlinear optics and a conceptual understanding of the foundations of quantum optics.

Grading:

Exam 1	150 pts	300 pts
Exam 2	150 pts	
Mini-Projects		500 pts
Final Project		200 pts
Total possible		**1000 pts**

Exams: There will be two exams during the semester. The format will be a take-home exam to be completed over 24 hours.

Mini-Projects: Problems in nonlinear and quantum optics are quite involved, so traditional homework assignments will not properly teach the material. So, the homework for this course will be in the form of mini-projects. The mini-projects will be a detailed solution of interconnected problems related to lecture topics. The problems will need to be solved using resources beyond the textbook and class notes. The purpose of the mini-projects is to mimic problem-solving scenarios found in a research environment.

There will be between 5-7 mini-projects, each given with two or more weeks for completion. Working on the mini-projects in groups is strongly encouraged, but you will need to write up the assignment on your own.

Final Project: There will be a final project for the class but no final exam. The final project will be an investigation of a topic or problem in the areas of nonlinear and quantum optics, that will involve a literature search and some original work. The final project will consist of three parts:

Part 1: Abstract and bibliography
Part 2: 6 page paper plus references
Part 3: 15 minute presentation

Late Projects: No project will be accepted after its due date unless prior arrangements have been made. Sorry! Please inform me with possible conflicts before the due date, and other arrangements will be made (if you ask really nicely).

Class Material: Extra class materials are posted on K-state Online, including papers and tutorials.

Disabilities: If you have any condition such as a physical or learning disability, which will make it difficult for you to carry out the work as I have outlined it or which will require academic accommodations, please notify me and contact the Disabled Students Office (Holton 202), in the first two weeks of the course.

Plagiarism: Plagiarism and cheating are serious offenses and may be punished by failure on the exam, paper or project; failure in the course; and/or expulsion from the University. For more information refer to the "Academic Dishonesty" policy in K-State Undergraduate Catalog and the Undergraduate Honor System Policy on the Provost's web page: http://www.ksu.edu/honor/.

Tentative Course Schedule, Nonlinear and Quantum Optics, PHYS 953, Fall 2007

Date	Topic	Chapters	Projects
Aug. 20 (M)	Introduction to nonlinear optics —Class overview, review of linear optics and the semi-classical treatment of light	B1	
Aug. 22 (W)	—Review of material dispersion: stuff you should know already	B1	
Aug. 24 (F)	—The nonlinear susceptibility: formal definitions	B1	
Aug. 27 (M)	—The nonlinear susceptibility: analogy to anharmonic motion	B1	
Aug. 29 (W)	—The nonlinear susceptibility: properties of materials	B1	
Aug. 31 (F)	—Symmetry and nonlinear optical properties	B1	
Sept. 3 (M)	No Class		
Sept. 5 (W)	—The Maxwell's wave equation in a nonlinear medium		
Sept. 7 (F)	Second order nonlinear effects —Second harmonic generation	B2	MP1 Due
Sept. 10 (M)	—Phase matching in second harmonic crystals	B2	
Sept 12 (W)	—Second harmonic generation with ultrashort pulses	B2	
Sept. 14 (F)	—Difference and sum frequency generation	B2	
Sept. 17 (M)	—Parametric amplification in crystals, optical parametric oscillators*	B2	
Sept. 19 (W)	—Quasi-phasematching in periodically poled materials	B2	
Sept. 21 (F)	Applications for second harmonic generation —Ultrashort pulse measurement: intensity and interferometric autocorrelators		
Sept. 24 (M)	—Ultrashort pulse measurement: FROGs, SPIDERs, and TADPOLEs		
Sept. 26 (W)	Carrier-envelope phase measurement: the *f*-to-2*f* interferometer		
Sept. 28 (F)	Third order nonlinear effects —Intensity dependent refractive index; four-wave mixing	B4	MP2 Due
Oct. 1 (M)	No Class		
Oct. 3 (W)	—Pulse propagation in a third order nonlinear medium: nonlinear fiber optics	Exam 1	
Oct 4 (U)	Exam 1 Due		
Oct. 5 (F)	—Nonlinear fiber optics: solitons and similaritons	B4, B13	
Oct. 8 (M)	—Spatial third order effects: self focusing and light bullets*	B4, B13	
Oct. 10 (W)	Applications of third order effects and high-intensity lasers —Short pulse generation using nonlinear effects	B13	
Oct. 12 (F)	—Nonlinear pulse compression in gases	B13	MP3 Due
Oct. 15 (M)	Spontaneous and stimulated Raman scattering* —Spontaneous Raman scattering	B9	
Oct. 17 (W)	—Stimulated Raman scattering in third order media, CARS spectroscopy*	B9	
Oct. 19 (F)	Introduction to quantum optics —What is a photon? The Hanbury-Brown and Twiss experiment	G1	
Oct. 22 (M)	—What is a photon? The Aspect experiments	G1	
Oct. 24 (W)	Field quantization and coherent states —Quantization of a single mode field	G2	
Oct. 26 (F)	—Vacuum fluctuations and the zero-point energy	G2	MP4 Due
Oct. 29 (M)	—The quantum phase	G3	
Oct. 31 (W)	—Coherent states: light waves as harmonic oscillators	G3	
Nov. 2 (F)	—Properties of coherent states, phase-space pictures	G3	
Nov. 5 (M)	—Review of the density operator, phase-space probability functions	G3	
Nov. 7 (W)	Emission and absorption of radiation by atoms —Atom-field interactions: classical and quantized fields	G4, B6	
Nov. 9 (F)	—Optical Bloch equations, the Rabi model	G4, B6	MP6 Due
Nov. 12 (M)	—Ramsey fringes, the Jaynes-Cumming model*	Exam 2	
Nov. 13 (T)	Exam 2 Due	Final Project Part 1Due	
Nov. 14 (W)	Nonclassical light* —Squeezed states, applications of squeezing in gravity wave detection	G7	
Nov. 16 (F)	—Squeezing and nonlinear fiber optics		
Nov. 19 (M)	Bell's theorem and quantum entanglement —EPR Paradox and Bell's Theorem	G9	
Nov. 21 (W)	No Class		
Nov. 23 (F)	No Class		
Nov. 26 (M)	—Bell's Theorem and the Aspect experiment	G9	MP5 Due
Nov. 28 (W)	—Violation of Bell's theorem using an optical parametric amplifier	G9	
Nov. 30 (F)	Optical tests of quantum mechanics —The Hong-Ou-Mandel interferometers	G9	
Dec. 3 (M)	—Quantum beats, quantum demolition measurements	Final Project Part 2 Due	
Dec. 5 (W)	—The Franson experiment	G9	
Dec 7 (F)	Final Project Presentation	Final Project Part 3 Due	
Dec 10 (M)	Final Project Presentation, final exam period 4:10 p.m. - 6:00 p.m.	Final Project Part 3 Due	

Books: B= Boyd, *Nonlinear Optics*, G= Gerry and Knight, *Introductory Quantum Optics*; * denotes a topic that may be replaced with something much more interesting

BRW 8/16/2007

Nonlinear + Quantum Optics

Lecture Outlines

- Lecture 1 Introduction + Review (50 minutes)
  - Class overview: class syllabus, topics
  - Nonlinear effects: generation of new spectral components
    SC generation in MSF
  - Overlap between nonlinear + ultrafast optics
    High Peak power ⇒ induce nonlinear effects
  - Quantum optics ⇒ small # of photons / Semi classical
  - Review: Notation for electric field
    Real + Complex
  - Fourier Transform ⇒ Important for understanding concepts in class
  - Linear optical properties

$$\vec{P}(\omega) = \epsilon_0 \chi \vec{E}(\omega) \quad \leftarrow \text{not really}$$

real instantaneous field

$$\vec{P}(\omega) = \epsilon_0 \chi \cdot \vec{E} \Rightarrow P_i(\omega) = \epsilon_0 \sum_j \chi_{ij} E_j$$

tensor notation

  - Nonlinear properties in Frequency domain

dyadic notation

$$\vec{P} = \epsilon_0 \chi^{(1)} \cdot \vec{E} + \epsilon_0 \chi^{(2)} : \vec{E}\vec{E} + \epsilon_0 \chi^{(3)} \vdots \vec{E}\vec{E}\vec{E}$$

OR

$$P_i = \epsilon_0 \sum_j \chi^{(1)}_{ij} E_j + \epsilon_0 \sum_{jk} \chi^{(2)}_{ijk} E_j E_k + \epsilon_0 \sum_{jkl} \chi_{ijkl} E_j E_k E_l$$

$$\vec{P} = \sum_i P_i \hat{x}_i \quad \text{Frequency domain}$$

- Time Domain

$$\vec{P}^{(2)} = \varepsilon_0 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi^{(2)}(t-t', t-t'') : \bar{E}(\bar{r},t)\,\bar{E}(\bar{r},t'')\,dt'dt''$$

Convolution & response function

- Lecture 2 Dispersion (50 minutes)
  - Wavelength dependent index of refraction $n(\lambda)$

$$\vec{P}(\omega) = \epsilon_0 \chi \vec{E}(\omega) \quad \left(\text{linear isotropic medium}\right)$$

$$P_i(\omega) = \sum_j \epsilon_0 \chi_{ij} E_j(\omega) \quad (\text{Birefringent})$$

- Effects of dispersion : phase velocity
- Effects of dispersion on pulse : group velocity
- Compute pulse Broadening

Offer class using Mathematica

Lecture 3 Nonlinear susceptibility

- Physical oscillator model of linear susceptibility
  Index of Refraction / Absorption

- Section 1.4 Anharmonic model for nonlinear susceptibility
  Review of anharmonic oscillators
  Non centrosymmetric medium / Centrosymmetric media

$\Rightarrow$ Derive $\chi^{(2)}$ + $\chi^{(3)}$ classically

Lecture 4 Complexity of Nonlinear terms
$\chi^{(2)}$ processes
Formal Definitions

Symmetries on χ

Lecture 7: 2nd Harmonic process + the wave equation

Nonlinear wave equation

Coupled differential Equations from wave eq

Solve for non-depleted pump

Lecture 8: Phase matching

Type I + Type II / ooe + oeo

Crystal Orientation

Phase mismatch

deff

Lecture 9: SHG + SFG

Review non depleted pump

Depleted pump case

Energy transfer from fundamental to SHG

Lecture 18 Intro to $\chi^{(3)}$ effects. FWM + SPM

Lecture 19 (W) Self phase modulation + 3ω generation

Lecture 20 (F) Intro to nonlinear optics in fibers

Intro to fiber optics

Phase matching in fibers

Lecture 21 (M) Nonlinear fiber optics

Pulse propagation

Soliton formation

Lecture (22) (W)

Lecture 23 (F) Applications $\Rightarrow$ Short pulse formation

Lecture 24 (M) Spatial Nonlinear effects

Self focusing

Lecture Outines

Lecture 9: Sum frequency generation + SHG generation

Analytic results and functional forms

Nonlinear Length

Lecture 10: Difference frequency generation

Analytic results

Optical parametric oscillator + amplifiers

Lecture 11: Quasiphase matching

More on OPOS

Periodically Poled materials

Lecture 12: SHG with ultrashort pulses

Lecture 13

Nonlinear and Quantum Optics : Lecture 1

Go over class syllabus
Introduction to Nonlinear optics
Introduction to Quantum Optics

— Linear responce of dielectric medium

$$\vec{P} = \epsilon_0 \chi \vec{\mathcal{E}}$$

(linear isotropic homogenous medium)

more general expression (birefringence)

$$P_i = \epsilon_0 \sum_{j=1}^{3} \chi_{ij}^{(1)} \mathcal{E}_j$$

linear susceptibility

Frequency domain equation

$\vec{P}, \vec{\mathcal{E}}$ ≡ real instaneneous fields

Induced Polarization: $P \equiv$ net dipole moment per unit volume

$$P \sim -Ne\vec{r}$$

displacement

dipoles per volume

Linear Material in the Frequency domain

$$\begin{cases} \vec{P}(\omega) = \epsilon_0 \chi^{(1)} \mathcal{E}(t) & \text{(SI)} \\ \vec{P}(\omega) = \chi^{(1)} \mathcal{E}(\omega) & \text{(Gaussian)} \end{cases}$$

OR $\vec{P}(\omega) = \epsilon_0 \chi^{(1)} \vec{E}(\omega)$

Not true for all materials, only for isotropic materials

$$P_i(\omega) = \epsilon_0 \sum_j \chi^{(1)}_{ij} E_j(\omega)$$

So $\vec{P} = \sum_{ij} \chi^{(1)}_{ij} E_j \hat{x}_i$

OR $\vec{P} = \epsilon_0 \chi^{(1)} \cdot \vec{E}$

~~(tensor notation~~

First order term in Taylor Series in $\vec{E}$! Nonlinear optics just add more terms.

$$\vec{P} = \epsilon_0 \left[\chi^{(1)} \cdot \vec{E} + \chi^{(2)} : \vec{E}\vec{E} + \chi^{(3)} \vdots \vec{E}\vec{E}\vec{E} \cdots \right.$$

OR

$$P_i = \epsilon_0 \Big(\sum_j \chi_{ij} E_j + \sum_{jk} \chi^{(2)}_{ijk} E_j E_k + \sum_{jkl} \chi^{(3)}_{ijkl} E_j E_k E_l + \cdots \Big)$$

$\chi^{(2)} \equiv$ 2nd order

~~$\chi^{(3)} \equiv$ 3rd order~~

Time domain ⇒ Material response

$$\vec{P}^{(1)}(t) = \varepsilon_0 \int \chi^{(1)}(t-t') \cdot E(\vec{r}, t-t')\, dt'$$

$$\vec{P}^{(2)} = \varepsilon_0 \int \chi^{(2)}(t-t', t-t'') \vdots \vec{E}(\vec{r}, t-t')\vec{E}(\vec{r}, t-t'')\, dt'\, dt''$$

Response of material ⇒ fs time scale for solids

$$\chi^{(2)}(t-t', t-t'') = \delta(t-t')\delta(t-t'')$$

- What do nonlinear effects do?

$\boxed{\text{Create "new" spectral components}}$

OR a more technically correct phrase is (remember Energy Cons.)

$\boxed{\text{Nonlinear effects create new spectral components by shifting spectral energy to new frequencies}}$

Example: Supercontinuum generation in a photonic crystal fiber.

An ultrashort pulse of bandwidth 12nm can generate a new spectral bandwidth >1000 nm due to third order nonlinear effects!

Ave Power = 100 mW $\quad f_{rep} = 100$ MHz $\quad \lambda_0 = 800$nm

Peak Power = $\quad \Delta t \approx 100$ fs

z = 1 m fiber

$\gamma \sim 100 \frac{1}{\text{km W}}$ Measure of $\chi^{(3)}$ and Power/Area

$$\gamma = \frac{n_2 \omega_0}{c A_{eff}}$$

Find spectral shift: $\boxed{\phi_{NL} = \gamma P_0 z}$

Non linear effect chirps the pulse

Adds a intensity dependent phase shift

$$E(\omega) = \mathcal{F}\left\{ E_0(t) \exp(i \phi_{NL}) \right\}$$

Lecture #2 The Linear Susceptibility

Review

$$\vec{P}(\omega) = \epsilon_0 \chi^{(1)} \vec{\mathcal{E}} \quad \left(\begin{array}{c}\text{linear isotopic}\\ \text{media}\end{array}\right)$$

$\vec{\mathcal{E}} \equiv$ real instantaneous electric field

$$P_i = \epsilon_0 \sum_j \chi_{ij}^{(1)} \mathcal{E}_j$$

Nonlinear susceptibility

$$\chi = \chi(\mathcal{E})$$

$$P_i = \epsilon_0 \left[\sum_j \chi_{ij}^{(1)} \mathcal{E}_j + \sum_{jk} \chi_{ijk}^{(2)} \mathcal{E}_j \mathcal{E}_k + \sum_{jkl} \chi_{ijkl}^{(3)} \mathcal{E}_j \mathcal{E}_k \mathcal{E}_l + \cdots \right]$$

$\chi^{(n)} \equiv$ n+1 rank tensor with 3^{n+1} terms

$\chi^{(n)} \equiv$ terms can be complex → nonlinear "index" / nonlinear "absorption"

$\chi^{(1)} \Rightarrow$ 9 terms

$\chi^{(2)} \Rightarrow$ 27 terms

$\chi^{(3)} \Rightarrow$ 81 terms

Lorentz Model for a linear dielectric medium

Collection of N electrons per unit volume. We want to find the response of the medium to an applied electric field $\vec{E}$

Medium: nonconducting, isotropic

Induced polarization

$$\vec{P} = -Ne\vec{r} \quad \text{(complex)}$$

at equilibrium $\quad -e\vec{E} = k\vec{r}$

atomic restoring force

$$\vec{P} = \frac{Ne^2}{k}\vec{E}$$

Let $\vec{E}$ vary in time, treat medium as collection of harmonic oscillators

$$m\ddot{\vec{r}} + m\gamma\dot{\vec{r}} + k\vec{r} = -e\vec{E} \quad \Big\} \ \gamma \equiv \text{damping}$$

Assume $\vec{E} \sim e^{-i\omega t}$ so $r(t) \sim e^{-i\omega t}$, subsitute this into differential equations

$$(-m\omega^2 - i\omega m\gamma + k)\vec{r} = -e\vec{E}$$

But $\vec{P} = -Ne\vec{r}$ so $\boxed{\vec{P} = \frac{Ne^2}{(+m\omega^2 + i\omega m\gamma + k)}\vec{E}}$

OR write $\chi = \chi_e' + i\,\chi_e''$

Then $$\boxed{\begin{aligned} \bar{n}^2 + 1 &= \chi_e' + i\,\chi_e'' \\ \left(n + i\,\frac{c}{\omega}\frac{\alpha}{2}\right)^2 + 1 &= \chi_e' + \chi_e'' \end{aligned}}$$ Equate real & Imaginary

$$\boxed{E(z,t) = E_0 \exp(-\alpha z/2)\exp(i(kz - \omega t))}$$

$k = \omega n / c$

Plot Frequency response (near resonance it is lorentzian) $\cancel{\dfrac{(\omega_j - \omega_0)}{((\omega - \omega_0)^2 + (\omega\gamma)^2)}}$

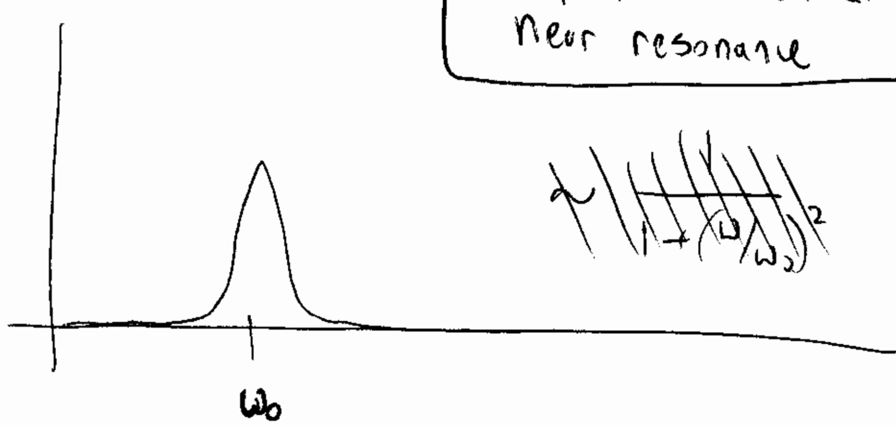
$\omega \simeq \omega_0$

Re χ
(n)

$\sim \cancel{\dfrac{\omega}{\left(1 + (\omega/\omega_0)\right)^k}}$

$\boxed{\text{Complex Lorentzian near resonance}}$ $\dfrac{1}{}$

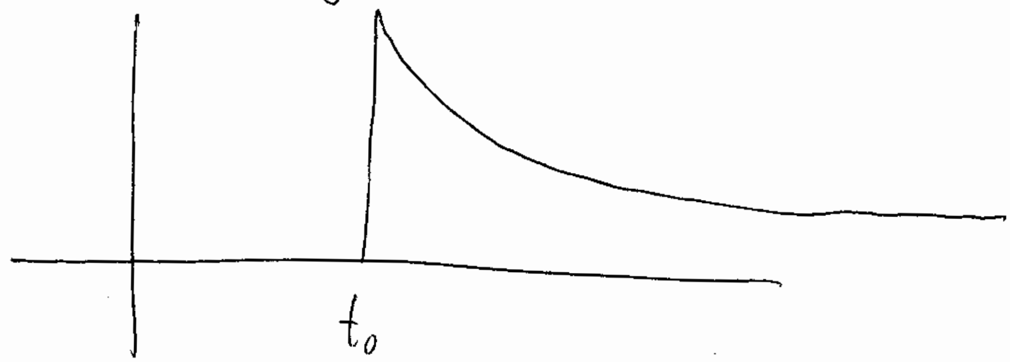
Im χ
(α) OR (K)



$\sim \cancel{\dfrac{1}{\left(1 + (\omega/\omega_0)\right)^2}}$ $\cancel{\dfrac{\omega\gamma}{(\omega_0 - \omega)^2 + (\omega\gamma)^2}}$

Actually we have many resonances $\bar{n} = 1 + \sum_{j=1}^{\infty}\left[\dfrac{Ne^2}{m\epsilon_0}\dfrac{1}{(\omega_j^2 - \omega^2 - i\gamma\omega)}\right.$

Corresponds to time response of thru



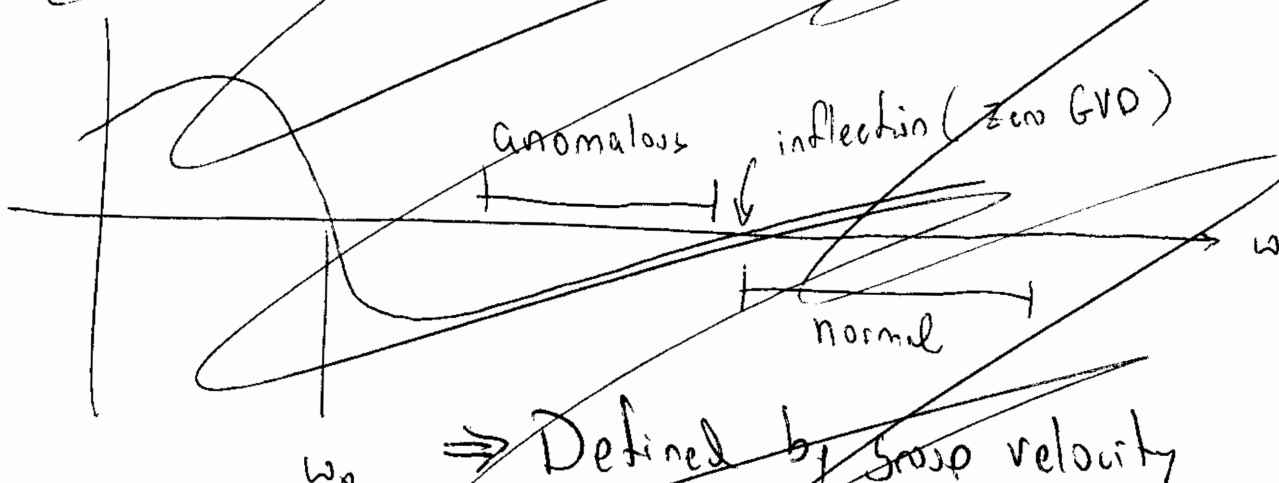
$$f(t) = \begin{cases} 0 & t < t_0 \\ e^{-\alpha t} & t > t_0 \end{cases}$$

Normal + Anomalous dispersion

Work far from ω_0 $\Rightarrow$ good for optical χ

Re χ has a point of inflection

$\Rightarrow$ zero group velocity dispersion



$\Rightarrow$ Defined by group velocity

$\frac{\partial}{\partial \omega}\left(\frac{1}{v_g}\right) > 0$ Normal | < 0 Anomalous

Define $\omega_0 = \sqrt{k/m}$

$$\vec{P} = \left(\frac{Ne^2/m}{\omega_0^2 - \omega^2 - i\omega\gamma} \right) \vec{E} = \epsilon_0 \underbrace{\chi^{(1)}}_{\text{linear susceptibility}} \vec{E}$$

The induced polarization is a source term for the wave eq

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} + \frac{1}{c^2} \partial_t^2 \vec{E} = \mu_0 \partial_t^2 \vec{P}$$

Since we have a source free medium $\vec{\nabla} \cdot \vec{D} = 0$ so $\vec{\nabla} \cdot \vec{E} = 0$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla}^2 \vec{E} = -\nabla^2 E$$

So

$$\boxed{+\nabla^2 \vec{E} - \frac{1}{c^2} \partial_t^2 \vec{E} = \mu_0 \partial_t^2 \vec{P}} \qquad c^2 = \frac{1}{\epsilon_0 \mu_0}$$

Sub in $\vec{P}$

$$\boxed{\nabla^2 E = \frac{1}{c^2} \left(1 + \frac{Ne^2}{m\epsilon_0} \frac{1}{(\omega_0^2 - \omega^2 - i\gamma\omega)} \right) \partial_t^2 E}$$

Define complex index $\bar{n}^2$

$$\boxed{\bar{n}^2 = \left(1 + \frac{Ne^2}{m\epsilon_0} \left(\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} \right) \right)}$$

This term has both the index and absorption

OR

$\bar{n} = n + i\kappa$ OR $\boxed{\bar{n} = n + i \frac{c}{\omega} \frac{\alpha}{2}}$ why 1/2?

Sellemeyer Equations

- Far from resonance

Approx 1) $\chi'' < 1 + \chi_e'$

2) Far from resonances

$$\chi_e' \sim \frac{Ne^2}{(2\pi c)\,\epsilon_0 m}\left(\frac{\lambda^2}{\lambda^2 - \lambda_0^2}\right)$$

OR

$$n^2(\lambda) = 1 + \sum_i \frac{A_i \lambda^2}{(\lambda^2 - \lambda_{0i}^2)}$$

Get terms A_i λ_i for approximation

Way to think of wave equation

$$\boxed{\nabla^2 \vec{E} - \frac{1}{c^2}\partial_t^2 \vec{E} = \mu_0 \partial_t^2 \vec{P}} \quad (2)$$

and

$$\boxed{\vec{P} = \chi \epsilon_0 \vec{E}} \quad (1)$$

1) $\vec{E}$ generated polarization $\vec{P}$ using (1)

2) $\vec{P}$ is used as source term to generate a new $\vec{E}$ using the wave equation (2)

Lecture 3

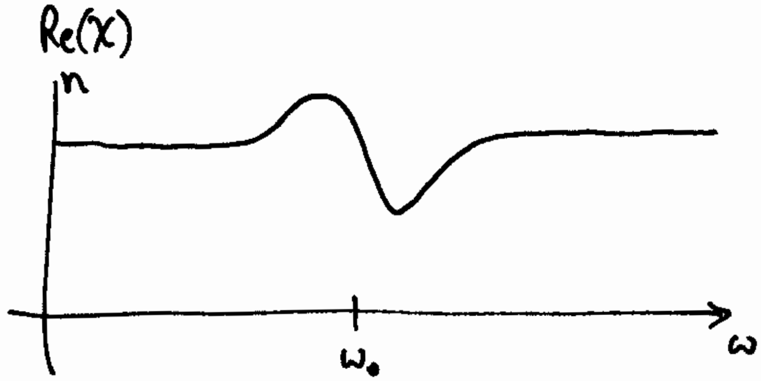
- Review: Derived frequency response of a linear dielectric media

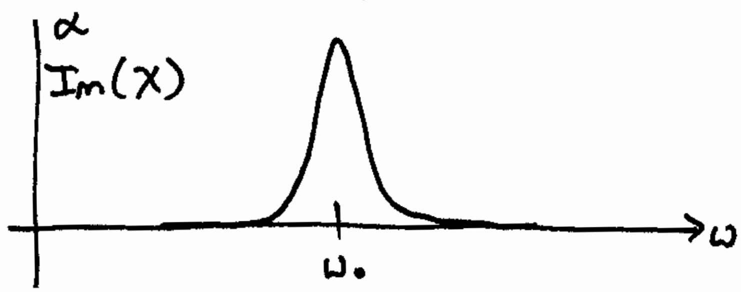
equate real + imaginary parts
$$\begin{cases} \bar{n}^2 = 1 + \dfrac{Ne^2}{m\epsilon_0}\left(\dfrac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}\right) \\[2ex] \bar{n} = n + i\,\dfrac{c}{\omega}\dfrac{\alpha}{2} \end{cases}$$

Real: $$n^2 - \frac{\alpha^2 c}{2\omega^2} = 1 + \frac{Ne^2}{m\epsilon_0}\left(\frac{\omega_0^2 - \omega^2}{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}\right)$$

Imaginary: $$\frac{2n\alpha^2 c}{2\omega^2} = \frac{Ne^2}{m\epsilon_0}\left(\frac{\gamma\omega}{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}\right)$$

Solve for $n(\omega)$ + $\alpha(\omega)$





Optical Pulse : Group + Phase Velocities

Two plane waves of frequency $\omega_1 + \omega_2 \Rightarrow$ Mix to get beat frequency

optical pulse $\Rightarrow$ finite temporal duration

$\Rightarrow$ finite spectral bandwidth

Time + Frequency domains related by Fourier Transform

Phase + Group Velocity

Phase velocity $\Rightarrow$ velocity of one spectral component

$$\boxed{v_p = c/n(\omega)}$$

Remember

$$c = \lambda_0 f_0 = \frac{\lambda_0 \omega_0}{2\pi}$$

so $\lambda_0 = \frac{2\pi c}{\omega_0}$

Group velocity $\Rightarrow$ velocity of spectral packet about the carrier frequency ω_0

$$\boxed{V_g = \frac{d\omega}{d(\beta(\omega))}}$$

$\beta(\omega) \Rightarrow$ propagation along $\hat{z}$

$$\beta(\lambda) = n(\lambda)\frac{2\pi}{\lambda}$$

$$\beta(\omega) = \frac{n(\omega)\omega}{c}$$

Group Delay + Group Index

$$\tau_g = z\frac{1}{v_g} = z\frac{d\beta}{d\omega} = z\frac{d}{d\omega}(n k_0)$$

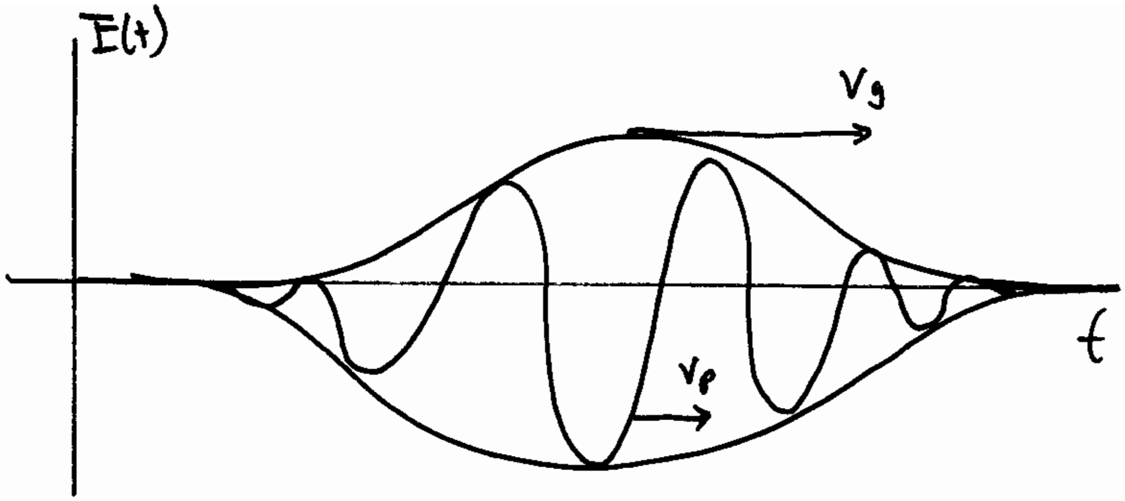
$k_0 = \frac{2\pi}{\lambda_0}$

$$= z\frac{d\lambda}{d\omega}\frac{d}{d\lambda}\left(n(\lambda)\frac{2\pi}{\lambda_0}\right)$$

$$= \frac{z}{c}\left(n - \lambda\, dn/d\lambda\right)$$

Where

So $v_g = \frac{z}{\tau_g} = \frac{z}{z}\frac{c}{(n - \lambda\, dn/d\lambda)} \equiv \boxed{\frac{c}{N}}$ $\boxed{N = n - \lambda\frac{dn}{d\lambda}}$ Group Index



Anomalous + Normal Dispersion

Jackson ~~Anomalous~~ Normal n increases with increasing ω

Anomalous n decrease with increasing ω

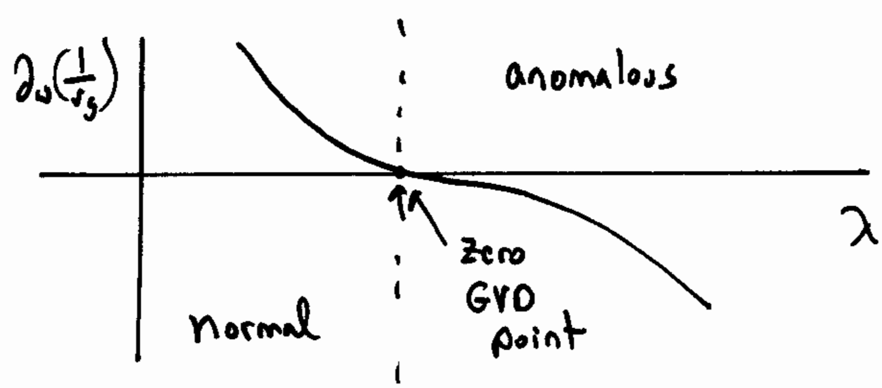
Define using group velocity

$$\partial_\omega \left(1/v_g \right) > 0 \quad \text{normal}$$

$$\partial_\omega \left(1/v_g \right) < 0 \quad \text{anomalous}$$

$$c \frac{\partial}{\partial \omega} \left(\frac{1}{n - \lambda \, dn/d\lambda} \right) = c \frac{\partial \lambda}{\partial \omega} \frac{\partial}{\partial \lambda} \left(\frac{1}{n - \lambda \, dn/d\lambda} \right)$$

$$= \frac{\lambda^2}{2\pi} \frac{\partial}{\partial \lambda} \left(\frac{1}{n(\lambda) - \lambda \, dn/d\lambda} \right)$$

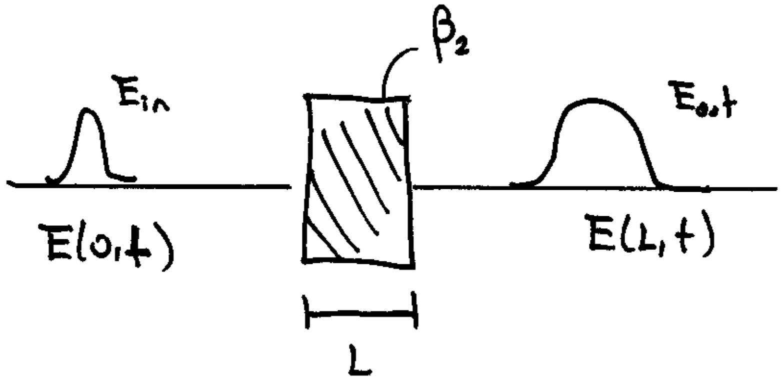


Pulse propagation

Dispersion induces a phase distortion on pulse $\phi(\omega)$

$$\beta(\omega) = n(\omega)\,\omega/c = \beta_0 + \beta_1(\omega-\omega_0)^2 + \tfrac{1}{2}\beta_2(\omega-\omega_0)^2 + \cdots$$

$$\phi(\omega) = \beta(\omega) L$$



$$E(0,t) = A(t)\exp(i\omega_0 t)$$

$\left\{ \mathcal{F} \equiv \text{Fourier Transform} \right.$

$$E(L,t) = \mathcal{F}^{-1}\left\{ \mathcal{F}\{E(0,t)\}\exp(i\phi(\omega))\right\}$$

Each spectral component propagated over L, and "sees" dispersion $\beta(\omega)$
Each spectral component "sees" a delay with respect to the carrier frequency ω_0

Causes Frequency Chirp

Positive chirp : "redir" components travel faster than "bluer"
Negative chirp : "bluer" components travel faster than "reder"

$$\beta_1 = \frac{1}{v_g} = \frac{1}{c}\left(n + \omega\frac{dn}{d\omega}\right) \qquad \beta_3 =$$

$$\beta_2 = \frac{1}{c}\left(2\left(\frac{dn}{d\omega}\right) + \omega\frac{d^2n}{d\omega^2}\right) \simeq \frac{\lambda^3}{2\pi c^2}\frac{d^2n}{d\lambda^2}$$

$\boxed{\begin{array}{l}\beta_2 > 0 \quad \text{Normal} \\ \beta_2 < 0 \quad \text{Anamolous}\end{array}}$

Material Dispersion for Fused Silica

This notebook determines the wavelength index of refraction, group index, group velocity dispersion, quadratic and cubic dispersion coefficients for bulk fused silica.

▪ Intial Definitions

Use c as the speed of light (in nm/fs).

```
c = 299.792458 ;
```

▪ Determine Sellmeier equations and the material dispersion for bulk fused silica

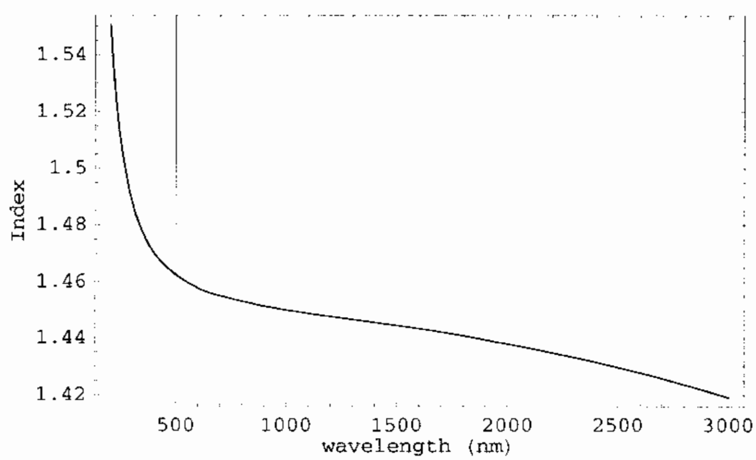
Define the Sellmeier equation and coefficients for fused silica, values taken from "Fundamentals of Optical Fibers", J.A. Buck., pg 127. The equation is good for wavelengths in nanometers.

```
B1 = 0.6961663; B2 = 0.4076426; B3 = 0.8974794;
C1 = 0.0684043; C2 = 0.1162412; C3 = 9.896161;
```

$$n_o[\lambda\_] = \sqrt{\frac{B1\ (\lambda/1000)^2}{(\lambda/1000)^2 - C1^2} + \frac{B2\ (\lambda/1000)^2}{(\lambda/1000)^2 - C2^2} + \frac{B3\ (\lambda/1000)^2}{(\lambda/1000)^2 - C3^2} + 1};$$

Plot the index as a function of wavelength

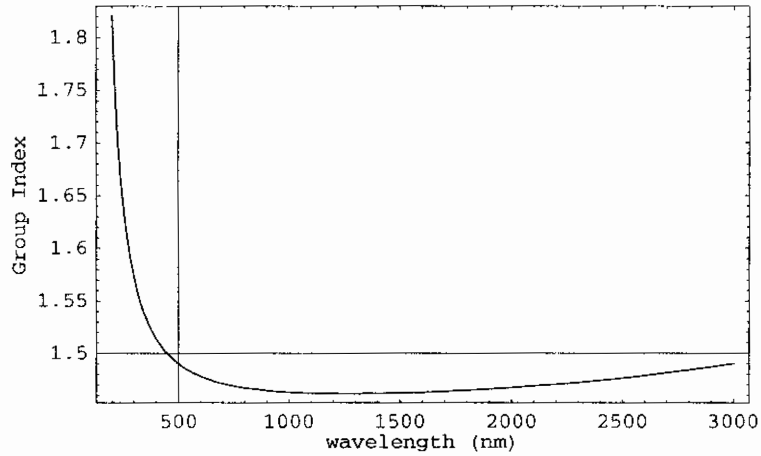
```
Plot[nₒ[λ], {λ, 200, 3000}, PlotRange -> {All, All}, Frame -> True,
    FrameLabel -> {"wavelength (nm)", "Index"}];
```



Deterimine the group index N_g using the expresion we derived in class.

$$N_g[\lambda\_] = n_o[\lambda] - \lambda\,\partial_\lambda\ (n_o[\lambda]);$$

```
Plot[Ng[λ], {λ, 200, 3000}, PlotRange -> {All, All}, Frame -> True,
    FrameLabel -> {"wavelength (nm)", "Group Index"}];
```



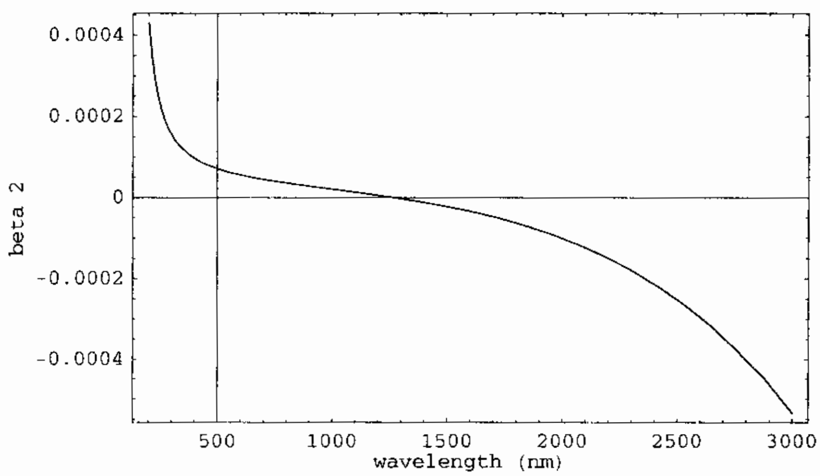
The wavevector can be written in terms of a Taylor series as a function of ω. Define the dispersion coefficients as β_2, β_3, and β_1. The units of the dispersion terms are nm/fs ,fs^2 / nm , and fs^3 / nm respectively.

$$\beta1[\lambda\_] = \frac{c}{N_g[\lambda]};$$

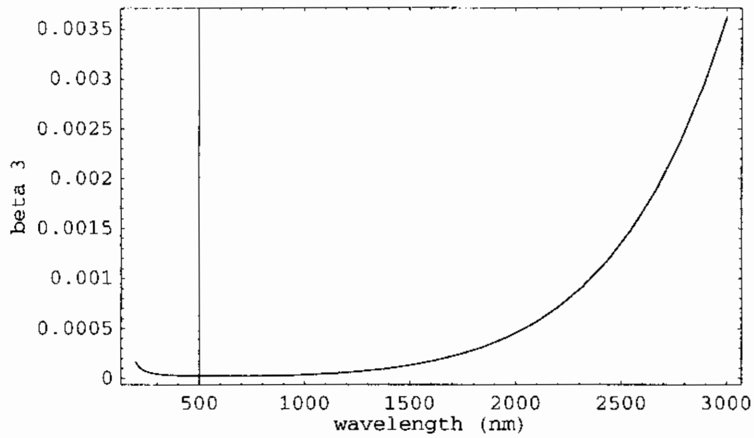
$$\beta2[\lambda\_] = \frac{\lambda^3}{2\,\pi\,c^2}\,\partial_{\lambda\lambda} n_o[\lambda];$$

$$\beta3[\lambda\_] = \frac{-\lambda^4}{4\,\pi^2\,c^3}\,(3\,(\partial_{\lambda\lambda} n_o[\lambda]) + \lambda\,(\partial_{\lambda\lambda\lambda} n_o[\lambda]));$$

```
Plot[β2[λ], {λ, 200, 3000}, PlotRange -> {All, All}, Frame -> True,
    FrameLabel -> {"wavelength (nm)", "beta 2"}];
```



```
Plot[β3[λ], {λ, 200, 3000}, PlotRange -> {All, All}, Frame -> True,
    FrameLabel -> {"wavelength (nm)", "beta 3"}];
```



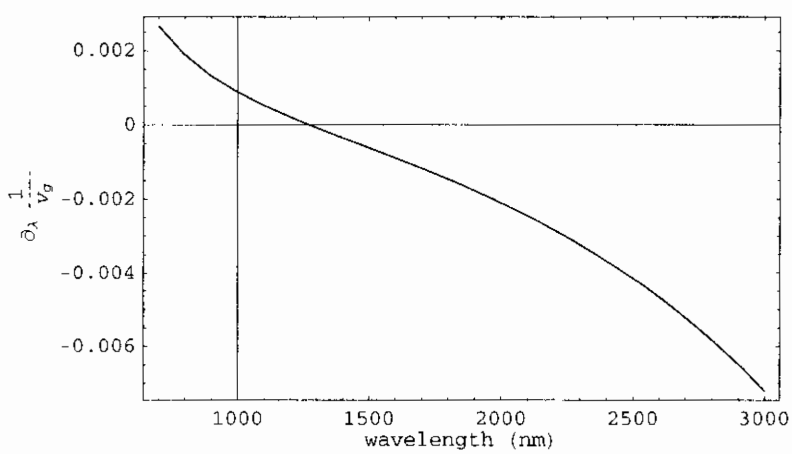
■ Regions of normal and anomalous dispersion in fused silica

To find the region of normal and anomalous dispersion, we need to find the derivative of $1/v_g$, which is related to the group velocity dispersion.

The normal dispersion region is where $\partial_\lambda \frac{1}{v_g} > 0$ The anamolous dispersion region is where $\partial_\lambda \frac{1}{v_g} < 0$

$$\text{dvgd}\lambda[\lambda\_] = \frac{\lambda}{2\pi}\,\partial_\lambda \frac{1}{N_g[\lambda]};$$

```
Plot[dvgdλ[λ], {λ, 700, 3000}, PlotRange -> {All, All}, Frame -> True,
    FrameLabel -> {"wavelength (nm)", "∂λ 1/vg"}];
```



From the graph we find that the zero group velocity dispersion wavelength is 1272 nm.

Dispersion + Pulse Broadening

Use Sellmeier Equation to describe $n(\lambda)$

$$n^2 - 1 = \sum_j \left(\frac{A_j \lambda^2}{\lambda^2 - \lambda_j^2} \right)$$

Pulse Broadening in Bulk material

Mode propagation constant $\beta(\omega)$

component of wavevector along propagation direction

Describe $\beta(\omega)$ as a Taylor Series

$$\beta(\omega) = n(\omega)\,\omega/c = \beta_0 + \beta_1(\omega - \omega_0) + \frac{1}{2}\beta_2(\omega - \omega_0)^2 + \ldots$$

$\omega_0 \equiv$ Carrier frequency

$$\beta_1 = 1/v_g \quad \text{group velocity}$$

$$\beta_n = \left. \frac{d^n \beta}{d\omega^n} \right|_{\omega_0}$$

$$v_g = \frac{c}{N}$$

Group Index $\boxed{N = n - \lambda\, dn/d\lambda}$

Need to determine β_n from $n(\lambda)$

$$\beta_1 = \frac{1}{c}\left(n + \omega \frac{dn}{d\omega}\right)$$

$$\beta_2 = \frac{1}{c}\left(2\frac{dn}{d\omega} + \omega \frac{d^2 n}{d\omega}\right) \simeq \frac{\lambda^2}{2\pi c^2}\frac{d^2 n}{d\lambda^2}$$

$$\beta_3 \simeq -\frac{\lambda^4}{2\pi^2 c^2}\left(3\frac{d^2 n}{d\lambda^2} + \lambda \frac{d^3 n}{d\lambda^3}\right)$$

Sign of Group velocity dispersion $\Rightarrow$ Sign of β_2

$\beta_2 > 0$	Positive,	Normal
$\beta_2 < 0$	Negative,	Anomalous

Lecture 4

Review: Dispersion

Pulse Broadening

$$\boxed{E(z,t) = \mathcal{F}^{-1}\left\{ \mathcal{F}\{E(0,t)\}\, e^{i\phi(\omega)} \right\}}$$

where $\phi(\omega) = \beta(\omega) z = z\left[\sum_n \frac{\beta_n}{n!} (\omega-\omega_0)^n \right]$

$\beta_2 \equiv$ GVD $\quad (s^2/m)$

$\phi_2 \equiv$ 2nd order phase distortion coefficient $\quad (s^2)$

Today: Properties of $\chi^{(n)}$ $\quad (n > 1)$

The nonlinear susceptibility: Generate new spectral components

$$\cancel{P_i = \epsilon_0 \sum_{jk} \sum_{(nm)} \chi_{ijk}^{(2)} E_j E_k}$$

$$\left.\begin{aligned} E(t) &= \sum_n E(\omega_n)\, e^{-i\omega_n t} \\ P(t) &= \sum_n P(\omega_n)\, e^{-i\omega_n t} \end{aligned}\right\} \text{sum over } \cancel{\text{generated}}/\text{input spectral components}$$

$$P_i(\omega_n + \omega_m) = \sum_{jk} \sum_{(nm)} \chi_{ijk}^{(2)}(\omega_m+\omega_n; \omega_n, \omega_m)\, E_j(\omega_n) E_k(\omega_m)$$

~~Lecture 13~~ Formal Definitions

Nonlinear susceptibility of a lossless, dispersionless material.

$\Rightarrow$ ignore nonlinear dispersion

$$\vec{\mathcal{E}} = \mathcal{E}_x \hat{x} + \mathcal{E}_y \hat{y} + \mathcal{E}_z \hat{z}$$

$\mathcal{E} \Rightarrow$ Real electric field

For each component we need to sum over all frequency components

$$\mathcal{E}_x = \sum_{n,m} \frac{1}{2} E^{nm} \exp(-i\omega_m t)\exp(i\vec{k}_n \cdot \vec{r}) + \text{c.c.}$$

Need c.c. since $\mathcal{E}_x$ must be real

Cannot ignore c.c. terms since the susceptibilities lead to products of terms $\Rightarrow$ new frequency components

Then Get polarization

$$\boxed{\mathcal{P}_i = \epsilon_0 \chi^{(2)}_{ijk} \mathcal{E}_j(\omega_n) \mathcal{E}_k(\omega_m)}$$

This polarization is used as a source in the wave eq

$$\boxed{\nabla^2 \mathcal{E} - \mu_0 \epsilon_0 \epsilon_r \partial_t^2 \mathcal{E} = \mu_0 \partial_t^2 \mathcal{P}}$$

The susceptibility and P are in the frequency domain

To get the time domain, must do a convolution

$$P_i^{(2)} = \epsilon_0 \int\int \chi_{ijk}^{(2)}(t-t', t-t'') : \mathcal{E}_j(\vec{r}, t') \, \mathcal{E}_k(\vec{r}, t'') \, dt' dt''$$

For most cases $\quad \chi_{ijk}(t-t') \Rightarrow \delta(t-t')$

Another notation

$$P_i(\omega_n + \omega_m) = \sum_{jk} \sum_{jk} \chi_{ijk}(\omega_n + \omega_m; \omega_n, \omega_m) \, \mathcal{E}_j(\omega_n) \, \mathcal{E}_k(\omega_m)$$

$$\chi_{ijk}(\underbrace{\omega_n + \omega_m}_{\text{result}}; \underbrace{\omega_n, \omega_m}_{\text{input}})$$

Nonlinear susceptibility : anharmonic oscillator

Restoring force ⇒ Beyond Hooke's Law

Two types of media

Noncentrosymmetric ⇒ Lacks inversion symmetry

Centrosymmetric ⇒ Inversion center as symmetry

Lacks Inversion symmetry ⇒ Special properties

Inversion symmetry / center ⇒ reflection about point brings compound back it itself

Crystal KDP ⇒ Noncentrosymmetric (lacks inversion symmetry)
Fused silica SiO_2 ⇒ Centrosymmetric since it is a symmetric molecule.

* { Media that lack inversion symmetry have nonzero $\chi^{(2n)}$ (even orders) } *

Noncentrosymmetric materials

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x + a x^2 = -eE(t)/m$$

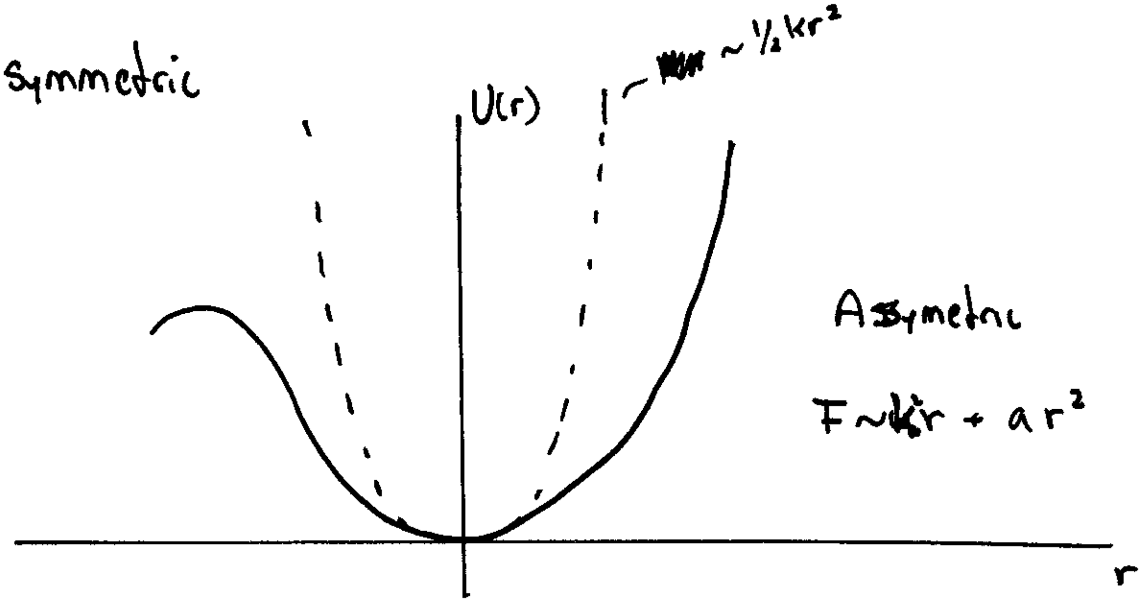
Restoring force $\vec{F} = -m\omega_0^2 x - m a x^2$

How to solve this eq ⇒ method of successive approximations

Perturbative series

Non centrosymmetric

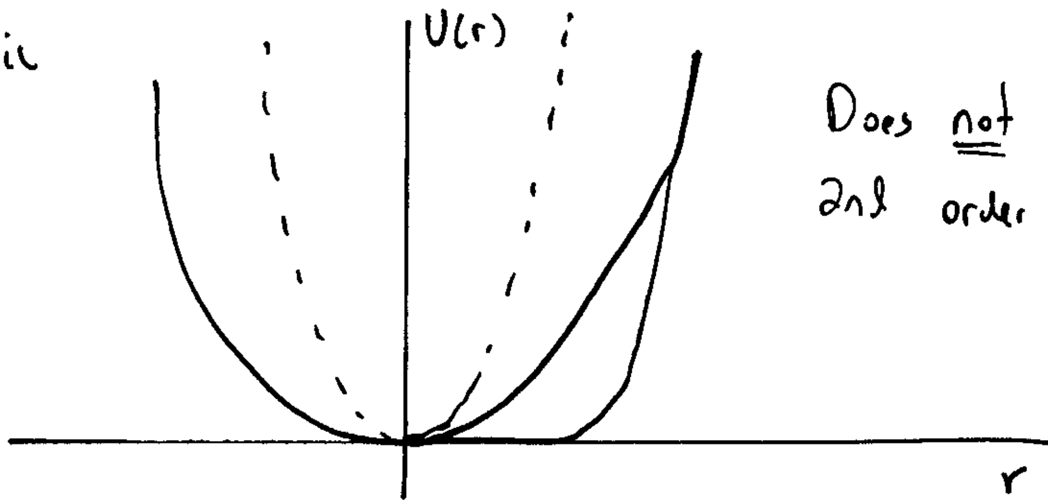
Has 2nd order terms



Assymetric

$F \sim kr + ar^2$

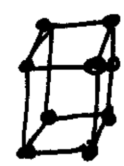
Centrosymmetric



Does <u>not</u> have 2nd order terms

Symmetric

KDP Non centrosymmetric { tetragonal crystal group



fused silica Centrosymmetric { Amorphous no crystal symmetric Symmetric

Solve for case without driving force

Example $\boxed{\ddot{x} + \omega_0^2 x^2 = -\alpha x^2 - \beta x^3}$

Solution $x = x_0 + \cancel{x_0} + \cancel{x_2} + \cdots$

$\omega = \omega_0 + \omega_1 + \omega_2 - -$

$$x_0 = a\cos(\omega t)$$

Sub in DE $x(t) = x_1 + x_0 = x_1 + a\cos(\omega t)$

$$= x_1 + a\cos((\omega_0 + \omega_1)t)$$

$$\dot{x}(t) = \dot{x}_1 - a\sin((\omega_0+\omega_1)t)(\omega_0+\omega_1)$$

$$\ddot{x}(t) = \ddot{x}_1 - a(\omega_0+\omega_1)^2\cos((\omega_0+\omega_1)t)$$

$$\ddot{x}_1 + \ddot{x}_0 + \omega_0^2(x_1+x_0) = -\alpha(x_1+x_0)^2 - \beta(x_0+x_1)^2$$

$$\left(\ddot{x}_1 - a\omega^2\cos(\omega t) + \omega_0^2 x_1\right) + \omega_0^2 a\cos\omega t = -\alpha(x_1+x_0)^2 - \beta(\quad)^3$$

$$\ddot{x}_1 - a(\omega_1+\omega_0)^2\cos(\omega t) + \omega_0^2 x_1 + \omega_0^2 a\cos(\omega t) = \text{''} \qquad \text{''}$$

$$\ddot{x}_1 - a\omega_1^2 \overset{\to\text{small}}{\cos(\omega t)} - a\omega_0^2\cancel{\cos(\omega t)} - 2\omega_0\omega_1\cos(\omega t) + \omega_0^2 x_1$$

$$+ \cancel{\omega_0^2 a\cos\omega t} = \text{''} \qquad \text{''}$$

$$\ddot{x}_1 + \omega_0 x_1 \simeq 2a\omega_1\omega_0\cos\omega t - \alpha a^2\cos^2(\omega t)$$

$\therefore$ $\omega_1 \simeq 0$ no resonant term with ω

~~Then $x = x_0 + x_1 + x_2$ Solve for x_2~~

~~$\omega_0 = \omega_0 + 0 + \omega_2$ ω_2~~

Solve for ω_1 + x_1

$$\boxed{\begin{aligned} \omega_1 &= 0 \\ x_1 &= \frac{\alpha a^2}{2\omega_0^2} + \frac{\alpha a^2}{6\omega_0^2}\cos(2\omega t) \end{aligned}}$$

Next Solution $\quad x = x_0 + x_1 + x_2$

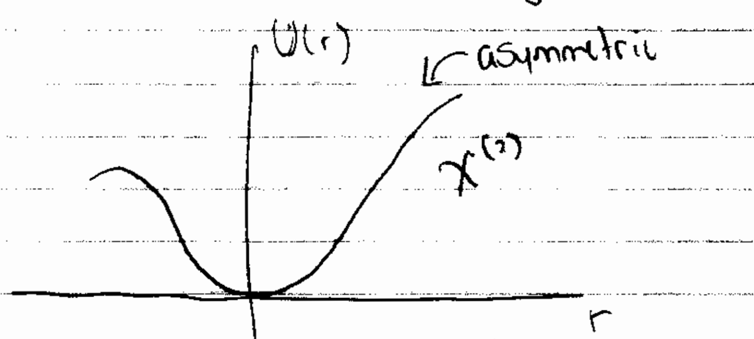
$$\omega = \omega_0 + \omega_1 + \omega_2$$

$$\boxed{\begin{aligned} \omega_2 &= \frac{3\gamma}{8\omega_0} - \frac{5\alpha^2}{12\omega_0^2}a^2 \\ x_2 &= \frac{a^3}{16\omega_0}\left(\frac{\alpha^2}{3\omega_0^2} - \frac{1}{2}\beta\right)\cos(3\omega t) \end{aligned}}$$

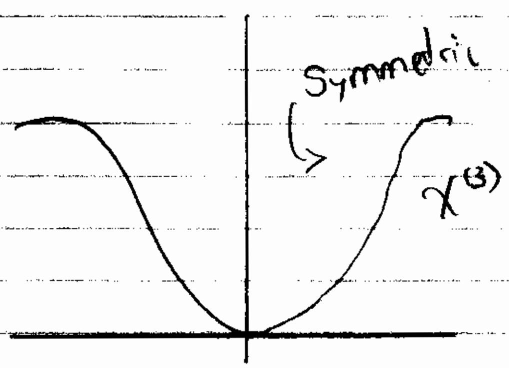
$\Rightarrow$ Notice if $\alpha = 0$ then no 2ω term!

$\alpha = 0 \Rightarrow$ Centrosymmetric medium

Potentials $\quad U(r) = -\int \vec{F} \cdot d\vec{r}$



Noncentrosymmetric



Centrosymmetric

Look at solving (with a driving force)

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x + a x^2 = -eE(t)/m$$

Form of Electric field

$$E(t) = \left(E_1 e^{-i\omega_1 t} + E_2 e^{-i\omega_2 t}\right) + \text{c.c.}$$

Use perturbative solution

Replace $E(t) \rightarrow \lambda E(t)$

Solution in power series expansion

$$x(t) = \lambda x^{(1)} + \lambda^2 x^{(2)} + \lambda^3 x^{(3)}$$

Terms of λ^n satisfy sides of equation

$$\dot{x}(t) = \lambda \dot{x}_1^{(1)} + \lambda^2 \dot{x}^{(2)} + \lambda^3 \dot{x}^{(3)}$$
$$\ddot{x}(t) = \lambda \ddot{x}_1^{(1)} + \lambda^2 \ddot{x}^{(2)} + \lambda^{(3)} \ddot{x}^{(3)}$$

Sub into DE

Lorentz model $\Rightarrow \lambda\left[\ddot{x}^{(1)} + 2\gamma\dot{x}^{(1)} + \omega_0^2 x^{(1)}\right] = -eE(t)/m \;\lambda$ (1)

$$\lambda^2\left[\ddot{x}^{(2)} + 2\gamma\dot{x}^{(2)} + \omega_0^2 x^{(2)} + a\left(x^{(1)}\right)^2\right] = 0\lambda^2 \quad (2)$$

$$\lambda^3\left[\ddot{x}^{(3)} + 2\gamma x^{(3)} + \omega_0^2 x^{(3)} + 2a x^{(1)} x^{(2)}\right] = 0\lambda^3 \quad (3)$$

Steady state solution for $x^{(1)}$

$$\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = \frac{-e}{m}\left(E_1 e^{-i\omega_1 t} + E_2 e^{-i\omega_2 t} \right)$$

Steady state solution $\Rightarrow$ ignore transient solution

$$\boxed{x^{(1)}(t) = \frac{-e E_1}{m D(\omega_1)} e^{-i\omega_1 t} + \frac{-e E_2}{m D(\omega_2)} e^{-i\omega_2 t} + \text{c.c.}}$$

$$\text{where} \quad D(\omega_j) = \omega_0^2 - \omega_j^2 - 2i\omega_j\gamma$$

(Eq 1.4.8 wrong?)

Square $x^{(1)}$ and substitute into (2)

The square contains terms

$$\left\{ \begin{array}{l} \pm 2\omega_1 \quad \pm 2\omega_2 \quad \pm(\omega_1+\omega_2) \quad \pm(\omega_1-\omega_2) \\ \text{and} \quad 0 \end{array} \right.$$

$$\left(x^{(1)}(t)\right)^2 = \frac{+e^2}{m^2}\left[\frac{E_1}{(\omega_0^2-\omega_1^2-2i\omega_1\gamma)} e^{-i\omega_2 t} + \frac{E_2}{(\omega_0^2-\omega_2^2-2i\omega_2\gamma)} e^{-i\omega_1 t} \right.$$

$$\left. + \frac{E_1}{(\omega_0^2-\omega_1^2+2i\omega_1\gamma)} e^{+i\omega_2 t} + \frac{E_2}{(\omega_0^2-\omega_2^2+2i\omega_2\gamma)} e^{+i\omega_1 t} \right]^2$$

$$= \frac{e^2}{m^2}\left[\frac{E_1^2}{(\omega_0^2-\omega_1-2i\omega\gamma)^2} e^{-i2\omega_2 t} + \frac{E_2^2}{(\omega_0^2-\omega_2^2-2i\omega_2\gamma)^2} e^{-i2\omega_1 t} \right.$$

$$+ \frac{E_1^2}{(\omega_0^2-\omega_1^2+2i\omega_1\gamma)^2} e^{i2\omega_2 t} + \frac{E_2^2}{(\omega_0^2-\omega_2^2+2i\omega_2\gamma)^2} + \frac{E_1 E_2}{(\quad)(\quad)} e^{-i(\omega_1+\omega_2)}$$

$$+ \frac{E_1^2}{(\quad)(\quad)} e^{0} + \quad E$$

Solve for Linear Case : Lorentz model

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{-e}{m}\mathcal{E}(t)$$

$$\mathcal{E}(t) = E_1 e^{-i\omega_1 t} + E_2 e^{-i\omega_2 t} + E_1^* e^{+i\omega_1 t} + E_2^* e^{i\omega_2 t}$$

Say $x \sim e^{i\omega_1 t} + e^{i\omega_2 t}$

For one frequency ω_1

$$x(t)\left(\omega_1^2 - \omega_0^2 - 2i\omega_1\gamma\right) = -e/m\,\mathcal{E}$$

so $$x(t) = \frac{-e/m}{(\omega_0^2 - \omega_1^2 - 2i\omega_1\gamma)}\, e/m\,\mathcal{E}(t)$$

For two frequencies $\omega_1 + \omega_2$

$$x^{(1)}(t) = x^{(1)}(\omega_1)e^{-i\omega_1 t} + x^{(1)}(\omega_2)e^{-i\omega_2 t} + \text{c.c.}$$

where $$x^{(1)}(\omega_j) = -e/m\, E_j / D(\omega_j)$$

$$D(\omega_j) \equiv \omega_0^2 - \omega_j - 2i\omega_j\gamma$$

Look at $+2\omega_1$ term

$$\ddot{x}^{(2)} + 2\gamma \dot{x}^{(2)} + \omega_0^2 x^{(2)} = \frac{a\, e E_1^2/m^2}{D^2(\omega_1)} e^{-2i\omega_1 t}$$

Look for Solution $\boxed{x^{(2)}(t) = x^{(2)}(2\omega_1) e^{-2i\omega_1 t}}$

Find $$\boxed{x^{(2)}(2\omega_1) = \frac{-a(e/m)^2 E_1^2}{D(2\omega_1) D^2(\omega_1)}}$$

Now Find $\chi^{(2)}$

$$P^{(1)}(\omega_j) = \epsilon_0 \chi^{(1)}(\omega_j) E(\omega_j)$$

From Before $P^{(1)}(\omega_j) = -Ne\, x^{(1)}(\omega_j)$

$$\left\{ x^{(1)} = \frac{P}{-Ne}, \quad \chi = \frac{P}{\epsilon_0 E} \right.$$

So $$\boxed{\chi^{(1)}(\omega_j) = \frac{Ne^2/m\epsilon_0}{D(\omega_j)}}$$ Solve for $\chi^{(1)}$

For $\chi^{(2)}$

$$P^{(2)}(2\omega_1) = \chi^{(2)}(2\omega_1, \omega_1, \omega_1) E^2(\omega_1)$$
$$P^{(2)}(2\omega_1) = -Ne\, x^{(2)}(2\omega_1)$$

So $$\boxed{\chi^{(2)}(2\omega_1; \omega_1, \omega_1) = \frac{N(e^3/m)a}{D(2\omega_1) D^2(\omega_1)}}$$

or in terms of $\chi^{(1)}$

$$\boxed{\chi^{(2)} = \frac{ma}{N^2 e^3} \chi^{(1)}(2\omega_1) \left(\chi^{(1)}(\omega_1) \right)^2}$$

Centro Symmetric materials

$$F = -m\omega_0^2 x + mbx^3$$

Symmetric potential

Find $\chi^{(2)}$ + $\chi^{(3)}$ using same procedure

$$E(t) = E_1 e^{-i\omega_1 t} + E_2 e^{-i\omega_2 t} + E_3 e^{-i\omega_3 t} + c.c.$$

Find $r^{(2)} = 0$ So $\chi^{(2)} = 0$

$$\boxed{\begin{aligned} &\ddot{r}^{(1)} + 2\gamma \dot{r}^{(1)} + \omega_0^2 r^{(1)} = -eE(t)/m \\ &\ddot{r}^{(2)} + 2\gamma \dot{r}^{(1)} + \omega_0^2 r^{(2)} = 0 \Leftarrow \text{no driving term} \\ &\ddot{r}^{(3)} + 2\gamma r^{(3)} + \omega_0^2 r^{(1)} - b(\bar{r}^{(1)} \cdot \bar{r}_1^{(1)})\bar{r}^{(1)} = 0 \end{aligned}}$$

damped eq

$$r^{(3)}(\omega_q) = -\sum_{(mnp)} \frac{be^3 \left(\bar{E}(\omega_m) \cdot \bar{E}(\omega_n)\right) E(\omega_p)}{m^3 D(\omega_q) D(\omega_m) D(\omega_n) D(\omega_p)}$$

Find $\chi^{(3)}$ where

$$P_i^{(3)}(\omega_q) = \sum_{jkl} \sum_{(mnp)} \chi^{(3)}_{ijkl}(\omega_q \omega_m \omega_n \omega_p) E_j(\omega_m) E_k(\omega_n) E_l(\omega_p)$$

So

$$\boxed{\chi_{ijkl}(\omega_q \omega_m \omega_n \omega_p) = \frac{Nb'e^4 \delta_{jk} \delta_{il}}{m^3 D(\omega_q) D(\omega_m) D(\omega_n) D(\omega_p)}}$$

Complications of Nonlinear optics

Full expansion of terms for $\chi^{(2)}$

Wave Equation

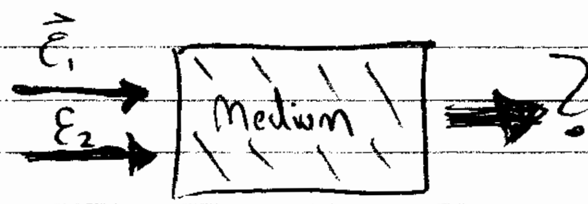
$$\boxed{\nabla^2 \vec{\mathcal{E}} - \mu_0 \epsilon_0 n^2 \partial_t^2 \vec{\mathcal{E}} = \mu_0 \partial_t^2 \vec{\mathcal{P}}_{NL}}$$

Source term

$$\boxed{\vec{\mathcal{P}}_{NL} = \epsilon_0 \chi^{(2)} \vdots \vec{\mathcal{E}} \vec{\mathcal{E}}}$$

- Find new electric field generated by $\chi^{(2)}$ medium

Procedure
1) Find total input $\vec{\mathcal{E}}(t)$
2) Determine nonlinear polarization
$$\vec{\mathcal{P}} = \epsilon_0 \chi^{(2)} \vec{\mathcal{E}}$$
3) Find new electric fields from wave equation



$$\vec{\mathcal{E}}(t) = \sum_n \vec{E}(\omega_n) e^{-i\omega_n t} = \sum_n A(\omega_n) e^{i(\vec{k}_n \cdot \vec{r} - \omega_n t)}$$

complex amplitude

$$\vec{\mathcal{P}}(t) = \sum_n \vec{P}(\omega_n) e^{-i\omega_n t}$$

$$\vec{\mathcal{E}}(t) = E_1 e^{-i\omega_1 t} + E_2 e^{-i\omega_2 t} + E_1^* e^{+i\omega_1 t} + E_2^* e^{+i\omega_2 t}$$

$$= E_1 e^{-i\omega_1 t} + E_2 e^{-i\omega_2 t} + c.c.$$

$$\vec{P}(t) = \chi^{(2)} \mathcal{E}(t)\, \mathcal{E}(t)$$

$$\vec{P}(t) = \chi^{(2)} \left[E_1 e^{-i\omega_1 t} + E_2 e^{-i\omega_2 t} + E_1 e^{i\omega_1 t} + E_2 e^{i\omega_2 t} \right]^*$$

(not exactly true since this will be different

Resultant terms $\Rightarrow$ Eight

$$\vec{P}(t) = \sum \vec{P}(\omega_n)\, e^{-i\omega_n t}$$

$$P(2\omega_1) = \chi^{(2)} E_1^2 \quad (SHG)$$
$$P(2\omega_2) = \chi^{(2)} E_2^2$$

$$P(\omega_1+\omega_2) = 2\chi^{(2)} E_1 E_2 \quad (SFG)$$

$$P(\omega_1-\omega_2) = 2\chi^{(2)} E_1 E_2^* \quad (DFG)$$

$$P(0) = 2\chi^{(2)}(E_1 E_1^* + E_2 E_2^*) \quad (OR)$$

Four different non zero frequency components ($2\omega_1$, $2\omega_2$, $\omega_1+\omega_2$, $\omega_1-\omega_2$)

$$P(-2\omega_1) = \chi^{(2)} E_1^{*2} \qquad P(-\omega_1-\omega_2) = 2\chi^{(2)} E_1^* E_2^*$$
$$P(-2\omega_2) = \chi^{(2)} E_2^{*2}$$

$$P(\omega_2-\omega_1) = 2\chi^{(2)} E_2 E_1^*$$

These Polarizations will induce a new electric field!

- Similar process for $\chi^{(3)} \Rightarrow 44$ different frequency components!!

Formal Definitions & Example in $\chi^{(2)}$

Being a bit sloppy here, should write the polarization for $\chi^{(2)}$

$$P_i(\omega_n+\omega_m) = \epsilon_0 \sum_{jk} \sum_{(nm)} \chi_{ijk}^{(2)}(\omega_n+\omega_m; \omega_n \omega_m) E_j(\omega_n) E_k(\omega_m)$$

Example

Sum Frequency generation

$$P_i(\omega_3) = \epsilon_0 \sum_{jk} \Big[\chi_{ijk}^{(2)}(\omega_3; \omega_1, \omega_2) E_j(\omega_1) E_k(\omega_2)$$

$$+ \chi_{ijk}^{(2)}(\omega_3; \omega_2, \omega_1) E_j(\omega_2) E_k(\omega_1) \Big]$$

Symmetries require that

$$\chi_{ijk}^{(2)}(\omega_m+\omega_n, \omega_m, \omega_n) = \chi_{ijk}^{(2)}(\omega_m+\omega_n; \omega_n \omega_m)$$

So

$$P_i(\omega_3) = 2\epsilon_0 \sum_{jk} \chi_{ijk}^{(2)}(\omega_3; \omega_1 \omega_2) E_j(\omega_1) E_k(\omega_2)$$

If inputs are polarized along $\hat{x}$

$$P_i(\omega_3) = \epsilon_0 2 \chi_{ixx}(\omega_3; \omega_1, \omega_2) E_x(\omega_1) E_x(\omega_2)$$

Lecture 5 Properties of the nonlinear susceptibility

Cover : 2nd order process
Symmetries to reduce complexity of $\chi^{(2)}$
Voit notation

Consider a 2nd order process

$$P_i(\omega_n+\omega_m; \omega_n, \omega_m) = \epsilon_0 \sum_{jk} \sum_{(nm)} \chi^{(2)}_{ijk}(\omega_n+\omega_m; \omega_n, \omega_m) E_j(\omega_n) E_k(\omega_m)$$

mutual interaction of three waves at ω_1, ω_2 & ω_3

For different combinations we will need to know tensors

$$\chi_{ijk}(\omega_1; \omega_3, -\omega_2) \qquad \chi_{ijk}(\omega_1, -\omega_2, \omega_3)$$

etc.

$$\chi_{ijk}(\omega_2; \omega_3, -\omega_1) \qquad \chi_{ijk}(\omega_2; -\omega_1, \omega_3)$$

$$\chi_{ijk}(\omega_3; \omega_1, \omega_2) \qquad \chi_{ijk}(\omega_3; \omega_2, \omega_1)$$

Another Six

$$\chi_{ijk}(-\omega_1; -\omega_3, \omega_2) \quad \text{etc} \ldots$$

Picture of Nonlinear process

$$\chi^{(2)}(\omega_n + \omega_m ; \omega_n , \omega_n)$$

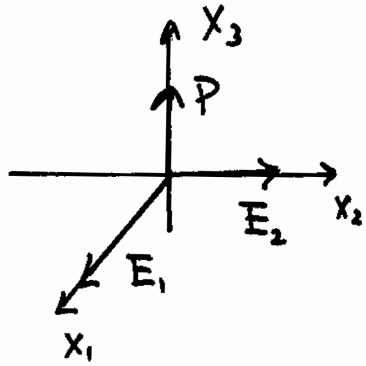
Specifically $\omega_1 , \omega_2 \Rightarrow \omega_3 = \omega_1 + \omega_2$

$\omega_1 = \omega_3 - \omega_2$

$\omega_2 = \omega_3 - \omega_1$

$$P_i(\omega_n + \omega_m) = \sum_{jk}\sum_{(nm)} \chi^{(2)}_{ijk}(\omega_n + \omega_m ; \omega_n, \omega_m) E_j(\omega_n) E_k(\omega_m)$$

One case, might look like:



$E_1 \sim e^{-i\omega_1 t}$

$E_2 \sim e^{-i\omega_2 t}$

of course there are many more possible situations

The number of tensors to describe all interactions

$3! \times 2 = 12$ tensors

$\times\ 27$ component

324 complex numbers!

$\cancel{\chi_{ijk}(\pm\omega_1, \pm\omega_2, \pm\omega_3)}$

$\cancel{\chi_{ijk}(\pm\omega_2, \pm\omega_1, \pm\omega_3)}$

$\chi_{ijk}(\pm\omega_3, \pm\omega_2 \pm\cancel{\omega_1})$

$\vdots$

$\chi^{(3)} \Rightarrow$ Four fields $\quad \chi^{(3)}(\omega_4 ; \omega_1\ \omega_2\ \omega_3)$

$4! * 2 = 24$

81

1944 Complex numbers

Review of symmetries

1) Reality of Fields $\quad \bar{P} = P_i(\omega_n+\omega_m)e^{-i(\omega_n+\omega_m)t} + P_i(-\omega_n-\omega_m)\,e^{+i(\omega_n+\omega_m)t}$

$$\chi^{(2)}_{ijk}(-\omega_n-\omega_m; -\omega_n, -\omega_m) = \left(\chi^{(2)}_{ijk}(\omega_n+\omega_m; \omega_n, \omega_m)\right)^*$$

since negative + positive frequency components of P

$$P_i(-\omega_n-\omega_m) = P_i(\omega_n+\omega_m)^*$$

2) Intrinsic permutation symmetry (Matter of convienence "forced" symmetry)

Require the susceptibility to be unchanged by simultaneous interchange of last two frequency arguments.

$$\chi^{(2)}_{ijk}(\omega_n+\omega_m; \omega_n, \omega_m) = \chi^{(2)}_{ikj}(\omega_n+\omega_m; \omega_m, \omega_n)$$

3) Loss less Media

Far from resonance $\quad \chi^{(2)}_{ijk}(\omega_n+\omega_m; \omega_n, \omega_m)$ is <u>REAL</u>!

4) Full Permutation Symmetry (Lossless media)

All frequency components of nonlinear susceptibility can be changed as long as corresponding Cartesian indices are changed simultaneously.

$$\chi^{(2)}_{ijk}(\omega_3 = \omega_1+\omega_2) = \chi^{(2)}_{jki}(-\omega_1 = \omega_2-\omega_3) \quad \left\{ \begin{array}{l} i \to \omega_3 \\ j \to \omega_1 \\ k \to \omega_2 \end{array} \right.$$

Due reality of fields

$$\cancel{\chi^{(2)}_{ijk}(\omega_3 = \omega_1+\omega_2) = }$$

$$\chi_{jki}^{(2)}(-\omega_1 = \omega_2 - \omega_3) = \chi_{jki}^{(2)}(\omega_1 = -\omega_2 + \omega_3)^*$$

which due to the reality of $\chi^{(2)}$ is equal to $\chi_{jki}^{(2)}(\omega_1 = -\omega_2 + \omega_3)$

$$\text{So} \Rightarrow \quad \chi_{ijk}^{(2)}(\omega_3 = \omega_1 + \omega_2) = \chi_{jki}^{(2)}(\omega_1 = -\omega_2 + \omega_3)$$

5) Kleinman Symmetry

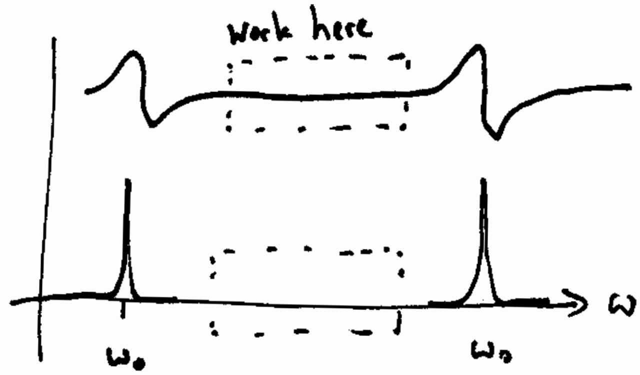
Assume: 1) frequencies are smaller than resonant

$\omega < \omega_0$

$\Rightarrow$ $\chi^{(2)}$ is frequency independent

Just like for χ'

- Index monotonically increases
- absorption is small



$\Rightarrow$ Response of $\chi^{(2)}$ is nearly "instantaneous"

$$\vec{P}(t) = \epsilon_0 \chi^{(2)} \mathcal{E}^2(t)$$

instead of

$$\vec{P}(t) = \epsilon_0 \iint \chi^{(2)}(t-t', t-t'')\, \mathcal{E}(t-t')\, \mathcal{E}(t-t'')\, dt\, dt$$

More on Kleinman Symmetry

Material is lossless $\Rightarrow$ Full permutation symmetry

Implies indices can be permuted as long as frequencies are permuted.

Example $\quad \chi^{(2)}_{ijk}(\omega_3 = \omega_1 + \omega_2) = \chi^{(2)}_{jki}(\omega_1 = -\omega_2 + \omega_3)$

$$= \chi^{(2)}_{kij}(\omega_2 = \omega_3 - \omega_1) \qquad \begin{cases} i \to \omega_3 \\ j \to \omega_1 \\ k \to \omega_2 \end{cases}$$

Assume $\chi^{(2)}$ does not depend on frequency

$\Rightarrow$ permute indices <u>without</u> permuting frequencies

So $$\chi^{(2)}_{ijk}(\omega_3 = \omega_1 + \omega_2) = \chi_{jki}(\omega_3 = \omega_1 + \omega_2)$$
$$= \chi_{kij}(\omega_3 = \omega_1 + \omega_2)$$
$$\vdots$$

Ignore nonlinear dispersion

Contracted Notation : Voit notation

$$\boxed{d_{ijk} \equiv \frac{1}{2} \chi^{(2)}_{ijk}}$$

useful for Kleinman symmetry ⇒ • Symmetric in last two indices
• symmetric where $\omega_n = \omega_m$ (SHG)

jk :	11	22	33	23, 32	31, 13	12, 21
l :	1	2	3	4	5	6

Susceptibility is a 3×6 matrix

$$d_{il}$$

Kleinman symmetry any indice d_{ijk} can be freely permuted

$$d_{12} = d_{122} = d_{212} = d_{26}$$

$$d_{14} = d_{123} = d_{213} = d_{25}$$

Relate back to nonlinear polarization

$$\begin{pmatrix} P_x(2\omega) \\ P_y(2\omega) \\ P_z(2\omega) \end{pmatrix} = 2 \left[\quad d_{il} \quad \right] \begin{bmatrix} E_x^2(\omega) \\ E_y^2(\omega) \\ E_z^2(\omega) \\ 2E_y E_z \\ 2E_x E_z \\ 2E_x E_y \end{bmatrix}$$

d_{eff}

Fixed geometry → polarization
→ propagation direction

SHG $$P(2\omega) = 2\, d_{eff}\, E^2(\omega)$$

Number of Independent Elements for $\chi^{(2)}_{ijk}(\omega_3, \omega_2, \omega_1)$

324 complex quantities

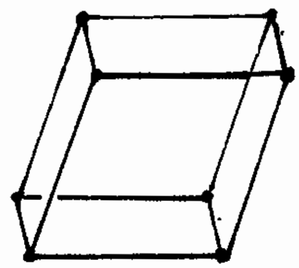
- Reality of field ⇒ ½ are independent
- Intrinsic Permutation Symmetry ⇒ 81 Independent
- Lossless medium ⇒ All real
- Full permutation ⇒ 27 Independent
- SHG + voit notation ⇒ 18 Independent
- Kleinman symmetry ⇒ 10 Indepent
- Crystaline Symmetry ⇒ Further reduction

Lecture 6 Crystal Structure & Nonlinear Optics

– Crystal Symmetry further reduces the number of independent elements of $\chi^{(2)}_{ijk}$

32 crystal classes

7 Crystal systems



Quartz

trigonal 32 (D_3)

Positive Uniaxial

Linear Properties

$$\chi^{(1)} \Rightarrow \begin{pmatrix} X & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & Z \end{pmatrix}$$

Nonlinear properties

$$xxx = -xyy = -yyx = -yxy$$
$$xyz = -yxz$$
$$xzy = -yzx$$
$$zxy = -zyx$$

Everything else zero

Use Kleiman Symmetry

$xxx \Rightarrow 11$ $\quad xyy \Rightarrow 12$ $\quad yyx \Rightarrow 26$

$xyz \Rightarrow 14$ $\quad yxz \Rightarrow 25$

$xzy \Rightarrow 14$ $\quad -yzx \Rightarrow 25$

$zxy \Rightarrow 36$ $\quad 3$

So

$d_{11} = -d_{12} = -d_{26}$

$d_{14} = -d_{25}$

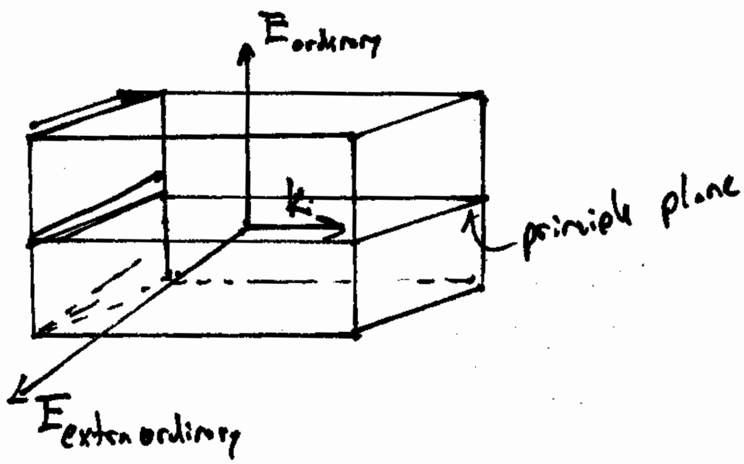
Catagories of birefringent crystals

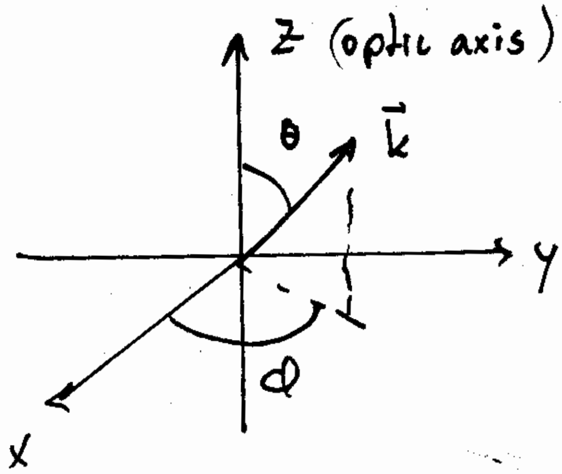
Uniaxial one optic axis $\Rightarrow$ the orientation where there is no birefringence

plane containing k + optic axis $\Rightarrow$ principle plane

ordinary beam $\Rightarrow$ polarization $\perp$ to principle plane

extraordinary beam $\Rightarrow$ polarization $\parallel$ to principle plane



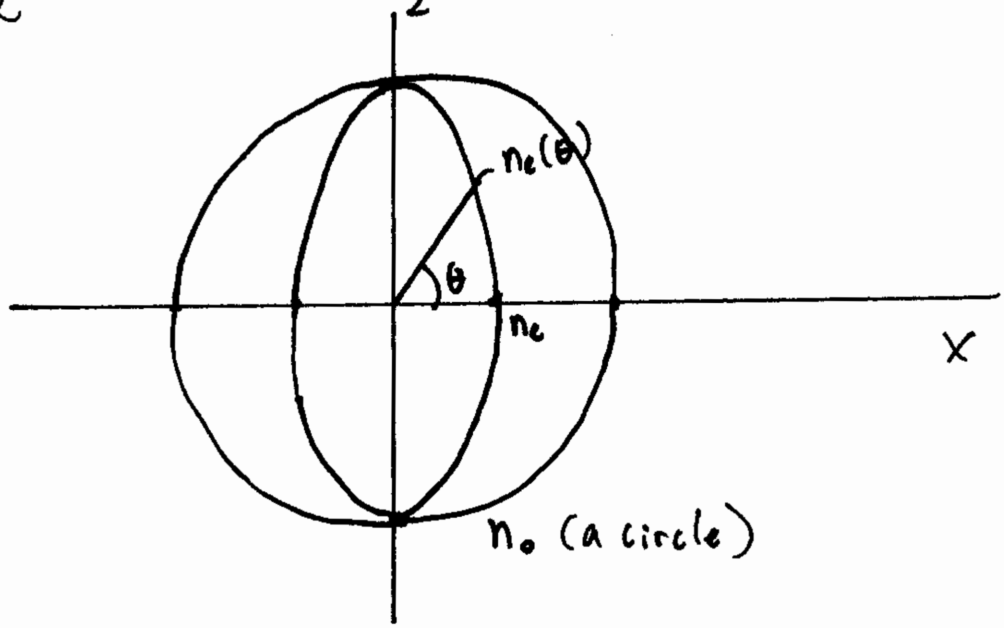


Draw index ellipsoids $n_o < n_e$ positive uniaxial

$n_o > n_e$ negative uniaxial

$n_o > n_e$

negative uniaxial



n_o does not depend on θ

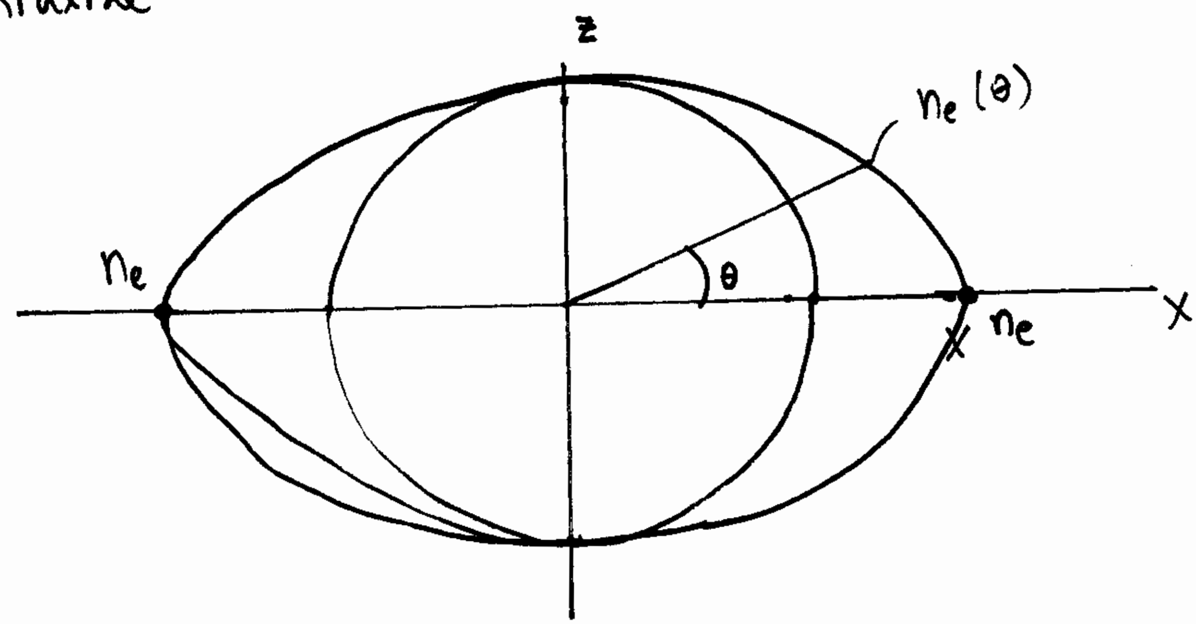
$$\boxed{\frac{1}{n_e^2(\theta)} = \frac{\sin^2\theta}{n_e^2} + \frac{\cos^2\theta}{n_o^2}}$$

OR

$$\boxed{n_e(\theta) = n_o \left[\frac{1 + \tan^2\theta}{1 + (n_o/n_e)^2 \tan^2\theta} \right]^{1/2}}$$

$n_o < n_e$

positive uniaxial



Extraordinary + Ordinary indices

$$\frac{1}{n_e^2(\theta)} = \frac{\sin^2\theta}{n_e^2} + \frac{\cos^2\theta}{n_o^2}$$

$$n_o(\theta) = n_o$$

$$n_e(0) = n_o$$

$$n_e(90^\circ) = n_e$$

$$\Delta n(0) = 0$$

$$\Delta n(90) = n_e - n_o$$

Class 32

Pictorial

Lithium Niobate (Elvis)

Class 3m (Trigonal)

Uniaxial crystal

KDP

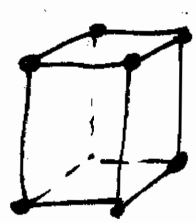
Class $\bar{4}2m$ (Tetragonal)

Uniaxial Crystal

Potassium Niobate

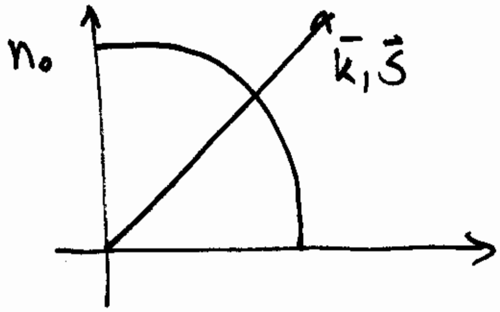
Class mm2 (orthorhombic)

Biaxial Crystal

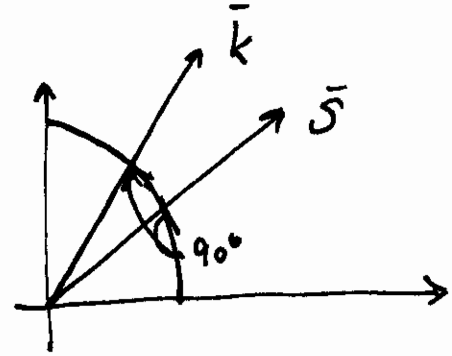


Walk off in ~~Birefringent~~ Uniaxial Crystals

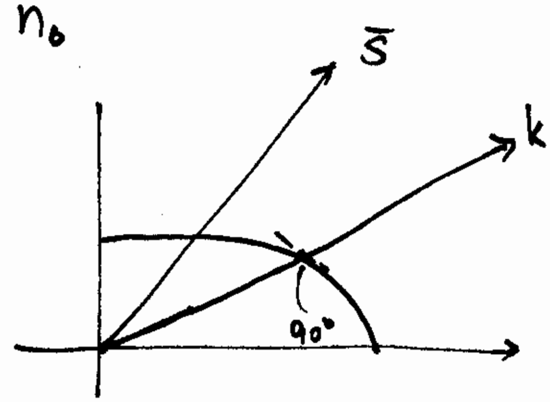
– The direction of $\vec{k}$ does <u>not</u> correspond to $\vec{S}$ in the crystal



$n_o > n_e$



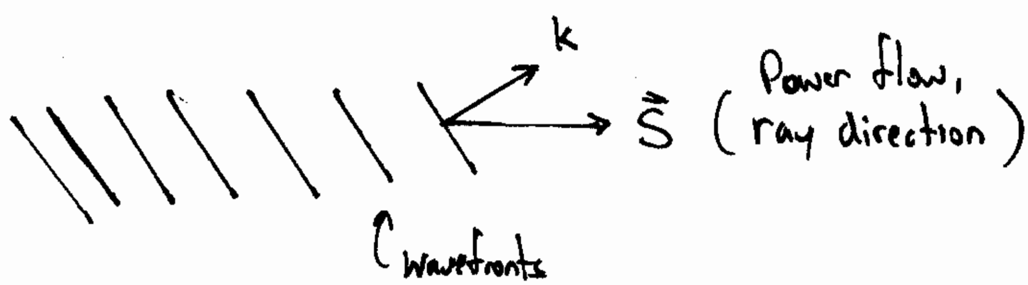
$n_e > n_o$



– $\vec{k}$ is $\perp$ to the surface of $n_e(\theta)$

$\bar{S}$ does not need to be $\perp$

What does a wave look like where $\vec{S} \nparallel \vec{k}$!?

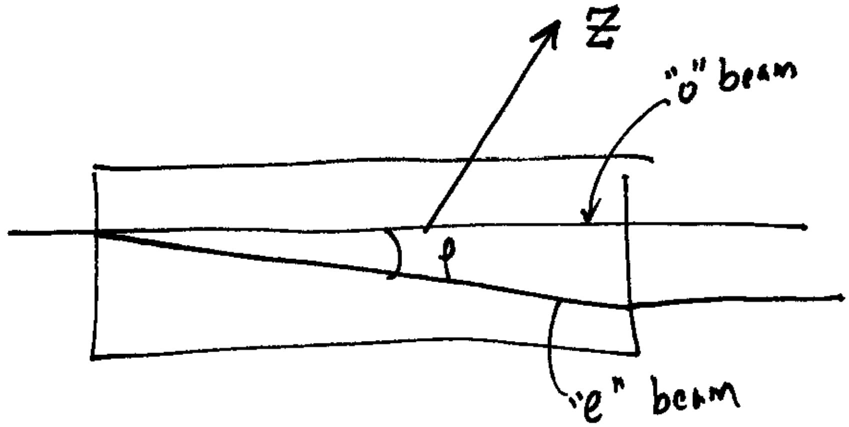


In the crystal the ordinary + extra ordinary beams separate or walk off

* k is $\perp$ to wavefronts *

Walk off angle $$\boxed{\rho = \pm \tan^{-1}\left(\left(\frac{n_o^2}{n_e^2}\right) \tan\theta \right) \mp \theta}$$ $\left(\begin{smallmatrix} - & \text{neg} \\ + & \text{pos} \end{smallmatrix}\right)$

Walk off

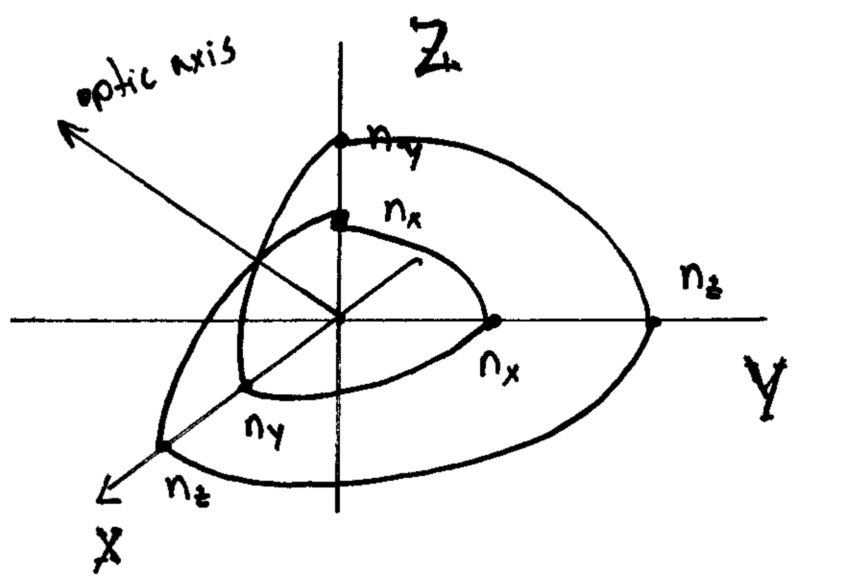


Biaxial Crystals

three separate indices n_x n_y n_z

Index surfaces

Principle planes XY YZ XZ



~~These will be two~~

Treat in plane $\Rightarrow$ polarization as circle & ellipse

Lecture 7 2nd order nonlinear effects

- Nonlinear polarization has higher frequency components
- These higher frequency components are source for new electric field at these frequencies.

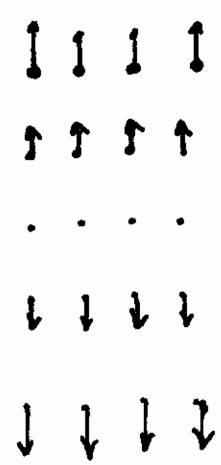
For efficient nonlinear wave generation the process must be phase matched

Consider N dipoles per unit volume

If relative phases of dipoles are correct each dipole will add up constructively.

$\Rightarrow$ Phased array of dipoles

Field will be N^2 larger



Nonlinear Wave equation

$$\nabla^2 \vec{E} - \frac{1}{c^2}\partial_t^2 \vec{\mathcal{E}} = \frac{4\pi}{c^2}\partial_t^2 \vec{\mathcal{P}} \qquad \text{(Gaussian)}$$

$$\vec{\nabla}^2 \vec{\mathcal{E}} - \frac{1}{c^2}\partial_t^2 \vec{\mathcal{E}} = \mu_0 \partial_t^2 \vec{\mathcal{P}} \qquad \text{(SI)}$$

Break up $\vec{\mathcal{P}} = \vec{\mathcal{P}}_L + \vec{\mathcal{P}}_{NL}$ { Write wave eq in terms of nonlinear polarization. }

Also $\quad \vec{D} = \epsilon_0 \vec{\mathcal{E}} + \vec{P}$

Rewrite the wave equation:

$$\vec{\nabla}^2 \vec{\mathcal{E}} - \frac{1}{c^2} \partial_t^2 \vec{\mathcal{E}} = \mu_0 \partial_t^2 (\vec{P}_L + P_{NL})$$

$$\vec{\nabla}^2 \mathcal{E} - \frac{1}{c^2} \partial_t^2 \vec{\mathcal{E}} - \mu_0 \partial_t^2 \vec{P}_L = \mu_0 \partial_t^2 \vec{P}_{NL}$$

but $\quad \vec{P}_L = \epsilon_0 \chi^{(1)} \vec{\mathcal{E}} \quad$ and $\quad c^2 = \frac{1}{\mu_0 \epsilon_0}$

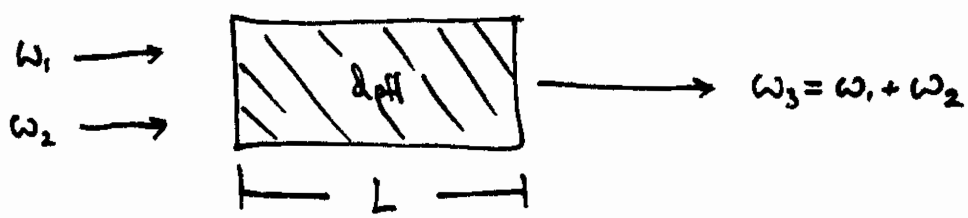
and $\quad \epsilon^{(1)} = 1 + \chi^{(1)} \Rightarrow$ Linear susceptibility $\chi^{(1)}$

+ $\quad n = \sqrt{\epsilon^{(1)}}$

So

$$\boxed{\vec{\nabla}^2 \vec{\mathcal{E}} - \frac{\epsilon^{(1)}}{c^2} \partial_t^2 \vec{\mathcal{E}} = \mu_0 \partial_t^2 \vec{P}_{NL}} \quad \text{(SI)}$$

Coupled wave equations for sum frequency generation



Wish to derive coupled differential equations to describe the field amplitudes for sum-frequency generation over time.

Above wave equation is valid for every frequency component.

For $\vec{P}_{NL} = 0 \quad$ solution $\quad \boxed{\bar{\mathcal{E}}_3(z,t) = A_3 \exp(i(k_3 z - \omega t)) + c.c.}$

$A_3 \equiv$ constant

For $\vec{P}_{NL} \neq 0$ we expect $\vec{\mathcal{E}}_3(z,t)$ to be a slowly varying function of z for small nonlinear source term.

$$\vec{P}_3 = \bar{P}_3 e^{-i\omega_3 t} + \text{c.c.}$$

where

$$\boxed{\bar{P}_3 = 4\varepsilon_0 d_{eff} E_1 E_2}$$

$d_{eff} \equiv$ Effective d_{il} parameter $\Rightarrow$ depends on crystal + phase matching

$d_{ijk} \Rightarrow$ summed over jk

For fields 1 + 2

$$\vec{\mathcal{E}}_i(z,t) = E_i e^{-i\omega_i t} + \text{c.c.}$$

$$E_i = A_i e^{i k_i z}$$

so

$$\boxed{P_3 = 4\varepsilon_0 d_{eff} A_1 A_2 e^{i(k_1+k_2)z}}$$

Subsitute $\bar{\mathcal{E}}_3$, $\bar{P}_3$ + P_3 into the wave equation:

$$\left(\frac{d^2 A_3}{dz^2} + 2ik_3\frac{dA_3}{dz} - k_3^2 A_3 + \frac{\varepsilon^{(1)}(\omega_3)\,\omega_3^2 A_3}{c^2}\right)\exp(ik_3 z - i\omega_3 t)$$

$$+ \text{c.c.} = -\mu_0\varepsilon_0 4 d_{eff} A_1 A_2 \exp(i(k_1+k_2-k_3)z)$$

Rewrite expression in ω only using $k_3 = \frac{n_3 \omega_3}{c}$ + $c^2 = \frac{1}{\mu_0 \varepsilon_0}$

$$\frac{dA_3}{dz} = \frac{2i\, d_{eff}\, \omega_3}{c\, n_3} A_1 A_2 \exp(i \Delta k z)$$

$$\frac{dA_2}{dz} = \frac{2i\, d_{eff}\, \omega_2}{c\, n_2} A_3 A_1^* \exp(-i \Delta k z)$$ (Without pump depletion)

$$\frac{dA_1}{dz} = \frac{2i\, d_{eff}\, \omega_1}{c\, n_1} A_3 A_2^* \exp(-i \Delta k z)$$

For $\Delta k = 0$ find A_3

$$A_3 = \int_0^L \frac{2i\, d_{eff}\, \omega_3}{n_3\, c} A_1 A_2 \exp(i \Delta k z)\, dz$$

$$= \frac{2i\, d_{eff}\, \omega_3}{n_3\, c} A_1 A_2 \left(\frac{\exp(i \Delta k L) - 1}{i \Delta k}\right)^2$$

$$= \quad '' \quad L^2 \left(\frac{\sin^2(\Delta k L)}{(\Delta k L)^2}\right)$$

Use $e^{x} = \cos x + i \sin x$ to express bracketed quantity^2 as

$$\frac{\sin(\Delta k L)}{\Delta k L} \equiv \mathrm{sinc}(\Delta k L)$$

Find the intensity $\boxed{I_j = 2\varepsilon_0 n_j c\, |A_j|^2}$

$$I_3 = 2\varepsilon_0 n_3 c\, |A_3|^2 = 2\varepsilon_0 n_3 c \left[\frac{4 d_{eff}^2\, \omega_3^2}{n_3^2\, c^2} |A_1|^2 |A_2|^2 L^2 \mathrm{sinc}^2(\Delta k L)\right]$$

But $|A_1|^2 = \frac{I_1}{2\varepsilon_0 n_1 c}$ and $|A_2|^2 = \frac{I_2}{2\varepsilon_0 n_2 c}$ and $\omega_3 = \frac{2\pi c}{\lambda_3}$

Invoke the slowly varying envelope approximation (SVEA)

$$\left| \frac{d^2 A_3}{dz^2} \right| \ll \left| k_3 \frac{dA_3}{dz} \right|$$

Ignore 2nd derivative

$$\boxed{\frac{dA_3}{dz} = \mu_0 \frac{2i\, d_{eff}\, \varepsilon_0 \omega_3^2}{k_3 \cancel{2}} A_1 A_2 \exp(i \Delta k z)} \quad k_3 = \frac{n_3 \omega_3}{c}$$

Where $\boxed{\Delta k = k_1 + k_2 - k_3}$: phase matching

Similar equations for A_1 + A_2

$$\frac{dA_1}{dz} = \varepsilon_0 \mu_0 \frac{2i\, d_{eff}\, \omega_1^2}{k_1 \cancel{2}} A_3 A_2^* \exp(-i \Delta k z)$$

$$\frac{dA_2}{dz} = \varepsilon_0 \mu_0 \frac{2i\, d_{eff}\, \omega_2^2}{k_2} A_3 A_1^* \exp(-i \Delta k z)$$

Phase matching considerations

Assume A_1 + A_2 are constants $\Rightarrow$ inputs are constants

For $\Delta k = 0$ (perfect phase matching)
$\left\{ \begin{array}{l} 1)\ A_3 \text{ increases linearly with } z \\ 2)\ I_3 \sim A_3 A_3^* \text{ increases quadratically with } z \end{array} \right.$

So

$$I_3 = c \, \frac{8 \, d_{eff}^2 \, \varepsilon_0 \, n_3 \, c \, \omega_3^2}{n_3^2 \, c^2} \; \frac{I_1}{2 \varepsilon_0 n_1 c} \; \frac{I_2}{2 \varepsilon_0 n_2 c} \; L^2 \, \mathrm{sinc}^2(\Delta k L)$$

$$I_3 = \frac{8 \, d_{eff}^2}{\varepsilon_0 \, n_1 n_2 n_3 \, c^3} \left(\frac{4\pi^2 c^2}{\lambda_3^2} \right) I_1 I_2 \, L^2 \, \mathrm{sinc}^2(\Delta k L)$$

$$\boxed{I_3 = \frac{8\pi^2 d_{eff}^2}{\varepsilon_0 \, n_1 n_2 n_3 \, c \, \lambda_3^2} \, I_1 I_2 \, L^2 \, \mathrm{sinc}^2(\Delta k L)}$$

where $\mathrm{sinc}\, x \equiv \dfrac{\sin x}{x}$

<u>Main Points</u> :

1) $I_3 \sim L^2$

2) Efficiency depends on phase mismatch

3) d_{eff}^2 dependence

4) $1/\lambda_3^2$ dependence

Table 3 Conversion Efficiency Formulas in the Infinite Plane Wave, Nondepleted Pump Approximation

SHG ($2\omega = \omega + \omega$)	$\eta_{2\omega} = \dfrac{\mathcal{P}_{2\omega}}{\mathcal{P}_{\omega}} = \eta^0_{2\omega}\dfrac{\sin^2(\Delta kL/2)}{(\Delta kL/2)^2}$		
	$\eta^0_{2\omega} = \dfrac{8\pi^2 d^2_{\rm eff} L^2 I_\omega}{\varepsilon_0 n^2_\omega n_{2\omega} c\lambda^2_\omega}$ (SI)	$\eta^0_{2\omega} = \dfrac{512\pi^5 d^2_{\rm eff} L^2 I_\omega}{n^2_\omega n_{2\omega} c\lambda^2_\omega}$	(cgs)
SFG ($\omega_s = \omega_{p1} + \omega_{p2}$)	$\eta_s = \dfrac{\mathcal{P}_s}{\mathcal{P}_{p2}} = \eta^0_s\dfrac{\sin^2(\Delta kL/2)}{(\Delta kL/2)^2}$		
	$\eta^0_s = \dfrac{8\pi^2 d^2_{\rm eff} L^2 I_{p1}}{\varepsilon_0 n_{p1} n_{p2} n_s c\lambda^2_s}$ (SI)	$\eta^0_s = \dfrac{512\pi^5 d^2_{\rm eff} L^2 I_{p1}}{n_{p1} n_{p2} n_s c\lambda^2_s}$	(cgs)
DFG ($\omega_d = \omega_{p1} - \omega_{p2}$)	$\eta_d = \dfrac{\mathcal{P}_d}{\mathcal{P}_{p2}} = \eta^0_d\dfrac{\sin^2(\Delta kL/2)}{(\Delta kL/2)^2}$		
	$\eta^0_d = \dfrac{8\pi^2 d^2_{\rm eff} L^2 I_{p1}}{\varepsilon_0 n_{p1} n_{p2} n_d c\lambda^2_d}$ (SI)	$\eta^0_d = \dfrac{512\pi^5 d^2_{\rm eff} L^2 I_{p1}}{n_{p1} n_{p2} n_d c\lambda^2_d}$	(cgs)

Table 19 Formulas for $d_{\rm eff}$ in Uniaxial Crystals for the Different Types of Angle Phase Matching

Crystal class	Type I (ooe)	Type II (oeo or eoo)
4, 6	$d_{31}\sin\theta$	$d_{15}\sin\theta$
422, 622	0	0
4*mm*, 6*mm*	$d_{31}\sin\theta$	$d_{15}\sin\theta$
$\bar{6}m2$	$-d_{22}\cos\theta\sin3\phi$	$-d_{22}\cos\theta\sin3\phi$
3*m*	$d_{31}\sin\theta - d_{22}\cos\theta\sin3\phi$	$d_{15}\sin\theta - d_{22}\cos\theta\sin3\phi$
$\bar{6}$	$(d_{11}\cos3\phi - d_{22}\sin3\phi)\cos\theta$	$(d_{11}\cos3\phi - d_{22}\sin3\phi)\cos\theta$
3	$(d_{11}\cos3\phi - d_{22}\sin3\phi)\cos\theta + d_{31}\sin\theta$	$(d_{11}\cos3\phi - d_{22}\sin3\phi)\cos\theta + d_{15}\sin\theta$
32	$d_{11}\cos\theta\cos3\phi$	$d_{11}\cos\theta\cos3\phi$
$\bar{4}$	$-(d_{31}\cos2\phi + d_{36}\sin2\phi)\sin\theta$	$-(d_{14}\sin2\phi + d_{15}\cos2\phi)\sin\theta$
$\bar{4}2m$	$-d_{36}\sin\theta\sin 2\phi$	$-d_{14}\sin\theta\sin2\phi$
	Type I (eeo)	Type II (eoe or oee)
4, 6	$-d_{14}\sin2\theta$	$d_{14}\sin\theta\cos\theta$
422, 622	$-d_{14}\sin2\theta$	$d_{14}\sin\theta\cos\theta$
4*mm*, 6*mm*	0	0
$\bar{6}m2$	$d_{22}\cos^2\theta\cos3\phi$	$d_{22}\cos^2\theta\cos3\phi$
3*m*	$d_{22}\cos^2\theta\cos3\phi$	$d_{22}\cos^2\theta\cos3\phi$
$\bar{6}$	$(d_{11}\sin3\phi + d_{22}\cos3\phi)\cos^2\theta$	$(d_{11}\sin3\phi + d_{22}\cos3\phi)\cos^2\theta$
3	$(d_{11}\sin3\phi + d_{22}\cos3\phi)\cos^2\theta - d_{14}\sin2\theta$	$(d_{11}\sin3\phi + d_{22}\cos3\phi)\cos^2\theta + d_{14}\sin\theta\cos\theta$
32	$d_{11}\cos^2\theta\sin3\phi - d_{14}\sin2\theta$	$d_{11}\cos^2\theta\sin3\phi + \frac{1}{2}d_{14}\sin2\theta$
$\bar{4}$	$(d_{14}\cos2\phi - d_{15}\sin2\phi)\sin2\theta$	$\frac{1}{2}[(d_{14}+d_{36})\cos2\phi - (d_{15}+d_{31})\sin2\phi]\sin2\theta$
$\bar{4}2m$	$d_{14}\sin2\theta\cos2\phi$	$\frac{1}{2}(d_{14}+d_{36})\sin2\theta\cos2\phi$

Source: Adapted from Ref. 12. [John Wiley & Sons, Inc., Copyright © 1973, reprinted by permission.]

positive and negative uniaxial crystals for both type I and type II phase matching and the various polarization combinations are given in Table 19.

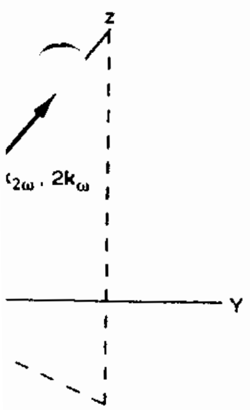
The formulas given in Table 19 are completely general for the various uniaxial crystal classes. When Kleinman symmetry prevails, which is quite often the case, the formulas are somewhat simpler. Kleinman symmetry will generally hold when there are no absorption bands in the vicinity of the interacting frequencies or in between the interacting frequencies. As discussed in Chapter 1, Kleinman symmetry implies that any permutation of the frequency arguments of the nonlinear susceptibility, *leaving the cartesian subscripts fixed*, leaves the tensor unchanged. Under this condition, it is irrelevant, from the standpoint of

sid g. 2, which illustrates SHG
se d harmonic wave polarized
present the laboratory coordinate
represent the principal axes of the
$d_{\text{eff}} = d_{xyy}$. However, in general
of the tensor components in the
ngles θ and ϕ as well. For most
or related to other components.
ses and for various optical wave

Waves

the nonlinear polarization. The
e form of these equations, for a
nits within a constant K, where

(28)



ry coordinate system (x, y, z) and

From "Handbook of Nonlinear Optics" Sutherland

SHG.

$$\frac{dA_{2\omega}}{dz} = iK\frac{2\omega}{n_{2\omega}c}d_{\text{eff}}A_\omega^2\exp(i\Delta kz) \tag{29}$$

$$\frac{dA_\omega}{dz} = iK\frac{2\omega}{n_\omega c}d_{\text{eff}}A_\omega^* A_{2\omega}\exp(-i\Delta kz) \tag{30}$$

SFG.

$$\frac{dA_s}{dz} = iK\frac{2\omega_s}{n_s c}d_{\text{eff}}A_{p1}A_{p2}\exp(i\Delta kz) \tag{31}$$

$$\frac{dA_{p1}}{dz} = iK\frac{2\omega_{p1}}{n_{p1}c}d_{\text{eff}}A_s A_{p2}^*\exp(-i\Delta kz) \tag{32}$$

$$\frac{dA_{p2}}{dz} = iK\frac{2\omega_{p2}}{n_{p2}c}d_{\text{eff}}A_s A_{p1}^*\exp(-i\Delta kz) \tag{33}$$

DFG.

$$\frac{dA_d}{dz} = iK\frac{2\omega_d}{n_d c}d_{\text{eff}}A_{p1}A_{p2}^*\exp(i\Delta kz) \tag{34}$$

$$\frac{dA_{p1}}{dz} = iK\frac{2\omega_{p1}}{n_{p1}c}d_{\text{eff}}A_d A_{p2}\exp(-i\Delta kz) \tag{35}$$

$$K = \begin{cases} 1 & (\text{SI}) \\ 4\pi & (\text{cgs}) \end{cases}$$

$$\frac{dA_{p2}}{dz} = iK\frac{2\omega_{p2}}{n_{p2}c}d_{\text{eff}}A_d^* A_{p1}\exp(i\Delta kz) \tag{36}$$

These equations were first solved by Armstrong et al. [3]. In general, both the modulus and phase of the complex field amplitudes are computed. However, to compute the output intensities of the generated waves, only the modulus is used. The intensity of a wave at some position z is given by

$$I_\alpha = 2\varepsilon_0 n_\alpha c|A_\alpha|^2 \tag{37}$$

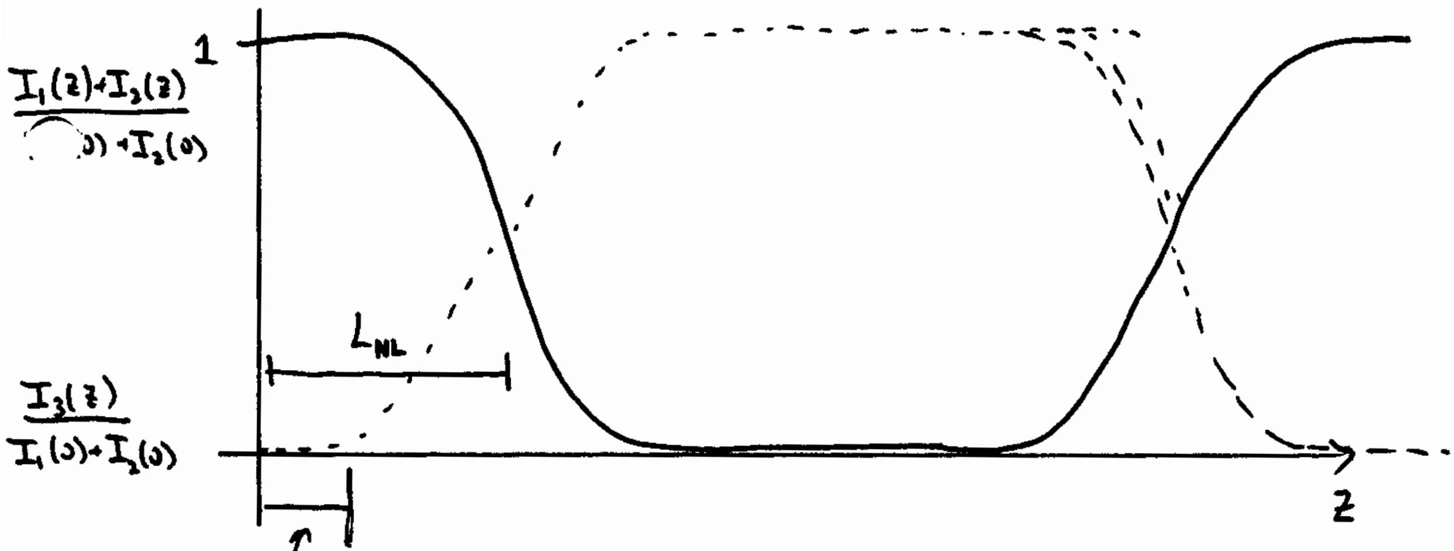
in SI units, and

$$I_\alpha = \frac{n_\alpha c}{2\pi}|A_\alpha|^2 \tag{38}$$

in cgs units. The optical power of a given wave is computed from

$$\mathcal{P} = \int_A I\,dA \tag{39}$$

For SHG (assuming pump depletion)



non pump depletion region

$A_1(z) \simeq A_1(0)$
$A_2(z) \simeq A_2(0)$

Analytic expression

$$\frac{I_3(z)}{I_1(0)+I_2(0)} = \tanh\left(L/L_{NL} \right)$$

where

$$L_{NL} = \frac{1}{4\pi d_{eff}} \sqrt{\frac{2\varepsilon_0 n^2(\omega)\, n(2\omega)\, c\, \lambda_1^2}{I_1(0)}}$$

$$\frac{I_3(z=z_{NL})}{I_1(0)+I_2(0)} \simeq 0.58 \qquad \text{at} \quad z = z_{NL}$$

Lecture 8: Phase matching in uniaxial crystals

$\Rightarrow$ Perfect Phasematching implies $\Delta k = 0$!!

However, Δk will be a function of λ + θ where θ is the angle with respect to the optic axis in a uniaxial crystal.

$$\boxed{\Delta k \equiv \text{phase mismatch}}$$

Associated with a uniaxial crystal are the ordinary + extraordinary Indices where

$$\boxed{\frac{1}{n_e^2(\theta)} = \frac{\sin^2\theta}{n_e^2} + \frac{\cos^2\theta}{n_o^2}}$$

OR

$$n_e(\theta) = n_o\left[\frac{1 + \tan^2\theta}{1 + (n_o/n_e)^2\tan^2\theta}\right]^{1/2}$$

To fulfill the $\Delta k = 0$ condition in a Second order process we need to have a proper orientation of the input electric fields

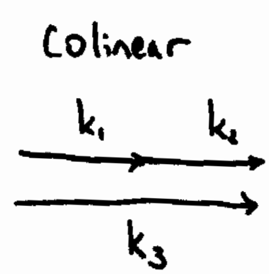
Phase matching

From before $\bar{k}_3 = \bar{k}_2 + \bar{k}_1$ $\qquad \Delta\bar{k} = \bar{k}_3 - \bar{k}_2 - \bar{k}_1$

For a colinear process we can treat $\bar{k}_i$ as scalars

$$k_3 = k_2 + k_1$$

$$\frac{\omega_3 n_3}{c} = \frac{\omega_2 n_2}{c} + \frac{\omega_1 n_1}{c}$$



Phase mismatch $= 0$ $\Rightarrow$ $\boxed{\omega_3 n_3 = \omega_2 n_2 + \omega_1 n_1}$

Specifically:

SHG

$$\boxed{n(2\omega) - n(\omega)}$$

where
$\omega_1 = \omega_2 = \omega$
$\omega_3 = 2\omega$

or

$$\boxed{n_3 = n_1}$$

SFG

$$\boxed{\omega_1 (n_3 - n_1) + \omega_2 (n_3 - n_2) = 0}$$

where
$\omega_3 = \omega_1 + \omega_2$

For normal isotropic materials the index increases with ω so it is not possible to obtain $n(2\omega) = n(\omega)$

However, we can use a uniaxial anisotropic material since it has <u>two</u> different indices of refraction. How we use these materials are called the type of phase matching

Type I$^{(-)}$: ooe phase matching (negative uniaxial)

$$\bar{k}_{o1} + \bar{k}_{o2} = \bar{k}_{3e} \quad \left\{ \begin{array}{l} E_1 \rightarrow \omega_1 \rightarrow n_o(\omega_1) \\ E_2 \rightarrow \omega_2 \rightarrow n_o(\omega_1) \\ E_3 \rightarrow \omega_3 \rightarrow n_e(\omega_3) \end{array} \right.$$

For SHG $\quad n_e(2\omega) = n_o(\omega)$

Type I $^{(+)}$ eeo phase matching (positive uniaxial)

$$\bar{k}_{1e}(\theta) + \bar{k}_{2e}(\theta) = \bar{k}_{o3}$$

For SHG $$\boxed{n_e(\omega) = n_o(2\omega)}$$

Type II $^{(-)}$ oee phasematching (negative uniaxial)

$$\bar{k}_{o1} + \bar{k}_{e2}(\theta) = \bar{k}_{e3}(\theta)$$

Type II $^{(+)}$ eoe phase matching (negative uniaxial)

$$\bar{k}_{1e}(\theta) + \bar{k}_{o2} = \bar{k}_{3e}(\theta)$$

Type II $^{(+)}$ oeo phasematching (positive uniaxial)

$$\bar{k}_{o1} + \bar{k}_{e2}(\theta) = \bar{k}_{o3}$$

Type II $^{(+)}$ eoo phasematching (positive uniaxial)

$$\bar{k}_{1e}(\theta) + \bar{k}_{o2} = \bar{k}_{o3}$$

How to compute the phase mismatch? How to compute phase matching angle?

consider an example of SHG using Type $I^{(-)}$ phasematching (ooe)

$$\Delta k = k_3 - k_1 - k_2$$

For ooe $\begin{cases} E_1 \to \text{ordinary axis} \\ E_2 \to \text{ordinary axis} \\ E_3 \to \text{extraordinary axis} \end{cases}$

But

$$k_3 = \frac{2\pi n_3}{\lambda_3} = \frac{2\pi}{(\lambda/2)} n_e(\theta, \lambda/2)$$

$$k_2 = \frac{2\pi n_2}{\lambda_2} = \frac{2\pi}{\lambda} n_o(\lambda)$$

$$k_1 = \frac{2\pi n_1}{\lambda_1} = \frac{2\pi}{\lambda} n_o(\lambda)$$

So

$$\Delta k(\theta, \lambda) = \frac{2\pi}{(\lambda/2)} n_e(\theta, \lambda/2) - 2\, \frac{2\pi}{\lambda} n_o(\lambda)$$

$$\boxed{\Delta k(\theta, \lambda) = \frac{4\pi}{\lambda}\left[n_e(\theta, \lambda/2) - n_o(\lambda) \right]}$$

$\lambda \equiv$ fundament wavelength

$\frac{\lambda}{2} = \text{SHG}$

For a given λ, $\Delta k(\theta, \lambda) = 0$ for $\theta = \theta_{pm}$ the phase mismatch angle. How to find θ_{pm}?

1) Solve for θ_{pm} where $n_e(\theta_{pm}, \lambda/2) = n_o(\lambda)$ for the given n_e + n_o for a crystal

2) For ooe process

$$\sin^2\theta_{pm} = \left(\frac{n_e(2\omega)}{n_o(\omega)}\right)^2 \left[\frac{(n_o(2\omega))^2 - (n_o(\omega))^2}{(n_o(2\omega))^2 - (n_e(2\omega))^2} \right]$$

Methods of phase matching

Angle tuning $\Rightarrow$ use n_o + $n_e(\theta, \lambda)$

Temperature tuning $\Rightarrow$ use $n_o(T)$ + $n_e(T, \lambda)$

Problem with Angle tuning : **Walkoff!!**

Birefringence is temperature dependent. Use temperature tuning.

OR Quasiphasematching

Properties of SHG for non-zero phase matching : Non depletion

Non pump depleted case for cristal of length L

Define conversion efficiency

$$\eta \equiv \frac{I_3(z)}{I_1(0) + I_2(0)} \qquad \left(\frac{d_{eff}^2}{n_1 n_2 n_3} \right)$$

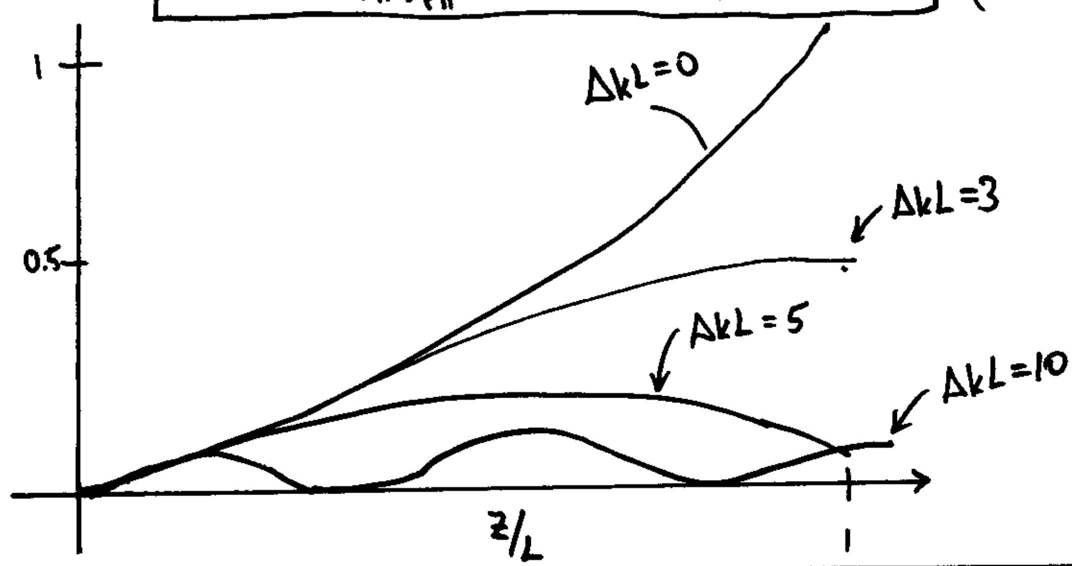
Define nonlinear length at $\Delta k = 0$

$$\boxed{L_{NL} = \frac{1}{4\pi d_{eff}} \sqrt{\frac{2\varepsilon_0 n_1 n_2 n_3 c \lambda_1}{I_1(0)}}}$$

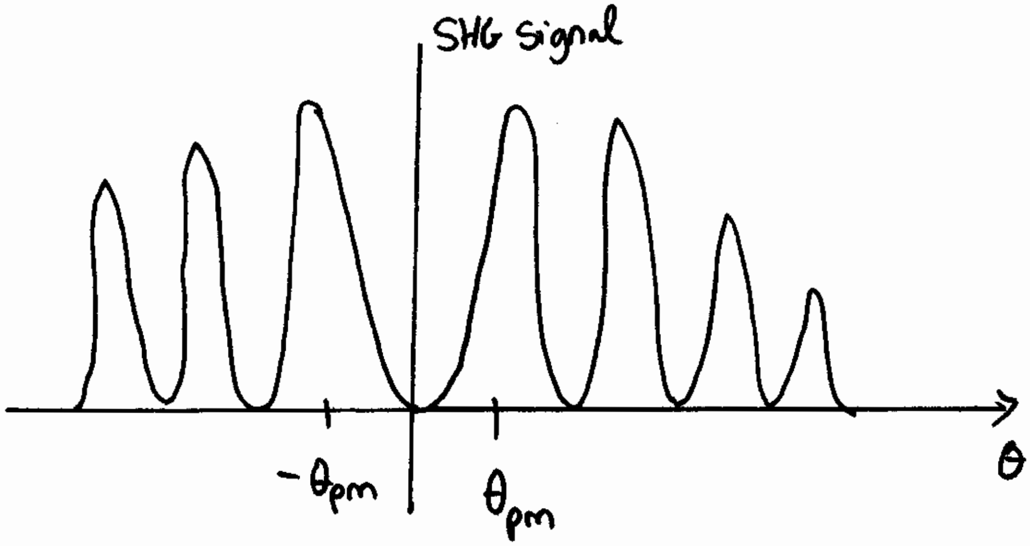
For SHG
$$\left\{ \begin{array}{l} \lambda_1 = \lambda_2 = \lambda \\ \lambda_3 = \lambda/2 \\ n_j \equiv n(\lambda_j) \end{array} \right.$$

Look at conversion efficiency for

$\Delta k = 0, 3, 5, 10$

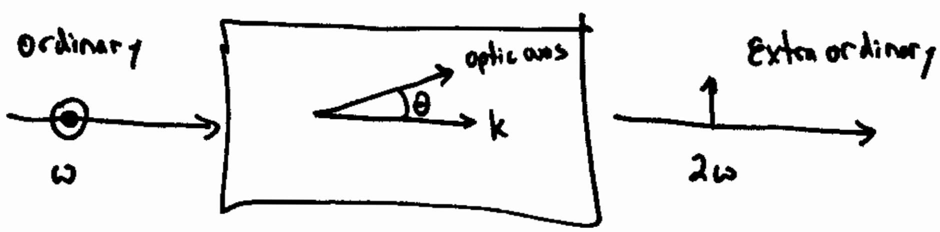


Angle tuning : SHG intensity vs angle.



<u>Orientation of beams into crystal for SHG</u>

Type $I^{(-)}$ ooe phase matching



$$n_e(2\omega, \theta) = n_o(\omega)$$

- SHG is along extraordinary axis
- Fundamental is along ordinary axis

SHG + Fundamental are polarized orthogonal to each other!

Table 10 Angle Phase Matching Formulas for DFG in Uniaxial Crystals

Type I	
ooe	$\sin^2\theta_{pm} = \dfrac{(n_d^e)^2}{(n_d^e)^2-(n_d^o)^2}\,\dfrac{[n_{p1}^o-(\lambda_{p1}/\lambda_{p2})n_{p2}^o]^2-(\lambda_{p1}/\lambda_d)^2(n_d^o)^2}{[n_{p1}^o-(\lambda_{p1}/\lambda_{p2})n_{p2}^o]^2}$
eeo	$\dfrac{n_{p1}^o}{\sqrt{1+\left[\dfrac{(n_{p1}^o)^2}{(n_{p1}^e)^2}-1\right]\sin^2\theta_{pm}}} - \dfrac{(\lambda_{p1}/\lambda_{p2})n_{p2}^o}{\sqrt{1+\left[\dfrac{(n_{p2}^o)^2}{(n_{p2}^e)^2}-1\right]\sin^2\theta_{pm}}} = (\lambda_{p1}/\lambda_d)$
Type II	
oee	$\dfrac{(\lambda_{p1}/\lambda_d)n_d^o}{\sqrt{1+\left[\dfrac{(n_d^o)^2}{(n_d^e)^2}-1\right]\sin^2\theta_{pm}}} + \dfrac{(\lambda_{p1}/\lambda_{p2})n_{p2}^o}{\sqrt{1+\left[\dfrac{(n_{p2}^o)^2}{(n_{p2}^e)^2}-1\right]\sin^2\theta_{pm}}} = n_{p1}^o$
eoe	$\dfrac{n_{p1}^o}{\sqrt{1+\left[\dfrac{(n_{p1}^o)^2}{(n_{p1}^e)^2}-1\right]\sin^2\theta_{pm}}} - \dfrac{(\lambda_{p1}/\lambda_d)n_d^o}{\sqrt{1+\left[\dfrac{(n_d^o)^2}{(n_d^e)^2}-1\right]\sin^2\theta_{pm}}} = (\lambda_{p1}/\lambda_{p2})n$
eoo	$\sin^2\theta_{pm} = \dfrac{(n_{p1}^e)^2}{[(\lambda_{p1}/\lambda_d)n_d^o+(\lambda_{p1}/\lambda_{p2})n_{p2}^o]^2} \times \left(\dfrac{(n_{p1}^o)^2-[(\lambda_{p1}/\lambda_d)n_d^o+(\lambda_{p1}/\lambda_{p2})n_{p2}^o]^2}{(n_{p1}^o)^2-(n_{p1}^e)^2}\right)$
oeo	$\sin^2\theta_{pm} = \dfrac{(n_{p2}^e)^2}{(n_{p2}^e)^2-(n_{p2}^o)^2}\,\dfrac{[n_{p1}^o-(\lambda_{p1}/\lambda_d)n_d^o]^2-(\lambda_{p1}/\lambda_{p2})^2(n_{p2}^o)^2}{[n_{p1}^o-(\lambda_{p1}/\lambda_d)n_d^o]^2}$

It is noted that for some cases, analytical results for θ_{pm} cannot be obtaine In these situations, the phase matching angle must be calculated numericall This is very straightforward using available software packages.

A simple example is given using the *root* function of Mathcad®.* Type SHG is potassium dihydrogen phosphate (KDP), a negative uniaxial crystal, considered. The fundamental wavelength is 800 nm and the second harmoı wavelength is 400 nm, for which $n_\omega^o = 1.501924$, $n_\omega^e = 1.463708$, $n_{2\omega}^o$ 1.524481, and $n_{2\omega}^e = 1.480244$ [7]. The computation takes only a few secon and the computed angle, 70.204°, is accurate to <0.1%.

*Mathcad is a registered trademark of MathSoft, Inc., Cambridge, MA.

Lecture 9 Analytic results for SHG + SFG

- Wish to look at two cases

1) SFG : For ~~unamplified~~ depleted pumps where $I_2 \gg I_1$, $\Delta k \neq 0$

2) SHG : For depleted pumps + $\Delta k \neq 0$.

Can always solve coupled differential equations numerically.

Case 1) Rewrite out coupled DE's

$$\boxed{\frac{dA_1}{dz} = K_1 A_3 \exp(-i\Delta k z) \quad (1) \qquad \frac{dA_3}{dz} = K_3 A_1 \exp(+i\Delta k z) \quad (2) \qquad \frac{dA_2}{dz} = 0}$$

Where

$$K_1 \equiv \frac{2i\omega_1 d_{eff}}{n_1 c} A_2^* \qquad K_3 \equiv \frac{2i\omega_3 d_{eff}}{n_3 c} A_2$$

Seek solutions of the form

$$A_1(z) = \left[A_{1+} \exp(igz) + A_{1-} \exp(-igz)\right] \exp(-i\Delta k z/2)$$

$$A_3(z) = \left[A_{3+} \exp(igz) + A_{3-} \exp(-igz)\right] \exp(-i\Delta k z/2)$$

- $g \equiv$ rate of spatial variation of A_1 + A_3

Same rate since A_1 + A_3 are coupled via energy conservation.

Subsotute into $\partial A_1/\partial z$ (1)

The derivative

$$\frac{\partial A_1}{\partial z} = \left(iA_{1+}\, g \exp(igz) - iA_{1-}\, g \exp(-igz)\right)\exp(-i\Delta k z/2)$$

$$- A_1(z)\, i\,\Delta k/2$$

<u>Sub In</u>

$$\left(igA_{1+} \exp(igz) - igA_{1-}\exp(-igz)\right)\exp(-i\Delta k/_2 z)$$

$$-\left(A_{1+}\exp(igz) + A_{1-}\exp(-igz)\right) i\,\Delta k/_2 \exp(-i\Delta k/_2 z)$$

$$= K_1\left[A_{3+}\exp(igz) + A_{3-}\exp(-igz)\right]\exp(-i\Delta k z/_2 z)$$

Equation must hold for all z, $\exp(igz)$ + $\exp(-igz)$ must maintain equality separately. Separate these terms:

$$\boxed{\begin{aligned} A_{1+}\left(ig - \tfrac{1}{2}i\Delta k\right) &= K_1 A_{3+} \qquad (3)\\ -A_{1-}\left(ig + \tfrac{1}{2}i\Delta k\right) &= K_1 A_{3-} \qquad (4)\end{aligned}}$$

Now subsitute solutions in $\partial A_3/\partial z$ + get similar eqs.

$$\boxed{\begin{aligned} A_{3+}\left(ig + \tfrac{1}{2}i\Delta k\right) &= K_3 A_{1+} \qquad (5)\\ -A_{3-}\left(ig - \tfrac{1}{2}i\Delta k\right) &= K_3 A_{1-} \qquad (6)\end{aligned}}$$

Eqs. (3) + (5) are simultaneous eqs for A_{1+} + A_{3-}

$$\begin{pmatrix} i(g + \frac{1}{2}\Delta k) & -K_1 \\ -K_3 & i(g+\frac{1}{2}\Delta k) \end{pmatrix} \begin{pmatrix} A_{1+} \\ A_{3+} \end{pmatrix} = 0$$

A unique solution exists iff the determinant of the matrix vanishes

$$K_3 K_1 = (g - \tfrac{1}{2}\Delta k)(g + \tfrac{1}{2}\Delta k)$$

$$\boxed{g^2 = -K_1 K_3 + \frac{1}{4}\Delta k^2}$$

$$g = +\sqrt{-K_1 K_2 + \tfrac{1}{4}\Delta k^2} \qquad \text{(positive root)}$$

Use initial conditions in order to solve for $A_1(z)$ + $A_3(z)$

$$A_1(0) = A_{1+} + A_{1-} \qquad (7)$$

$$A_3(0) = A_{3+} + A_{3-} \qquad (8)$$

Using $(7)(8)$ + $(3)(4)$ we have for equations for the four unknowns A_{1+} A_{1-} A_{3+} A_{3-} ~~we can then solve~~. Also let us assume there is no initial sum frequency generation

so

$$A_3(0) = 0 \Rightarrow A_{3+} = -A_{3-}$$

Solution :

$$A_1(z) = A_1(0)\left(\cos(gz) + \frac{i\Delta k}{g}\sin(gz)\right) \exp(-i\Delta k z/2)$$

$$A_3(z) = A_1(0)\,\frac{K_3}{g}\,\sin(gz)\exp(-i\Delta k z/2)$$

For intensities

$$\boxed{\begin{aligned} I_1(z) &= 2n_1\varepsilon_0 c\, I_1(0)\left(\cos^2(gz) + \frac{\Delta k^2}{4g^2}\sin^2(gz)\right) \\ I_3(z) &= 2n_3\varepsilon_0 c\, I_1(0)\,\frac{|K_3|^2}{g^2}\sin^2(gz) \end{aligned}}$$

Characteristic Length g^{-1}

$\Rightarrow$ g^{-1} decreases as Δk increases

$\Rightarrow$ SHG Intensity as $1/g^2$

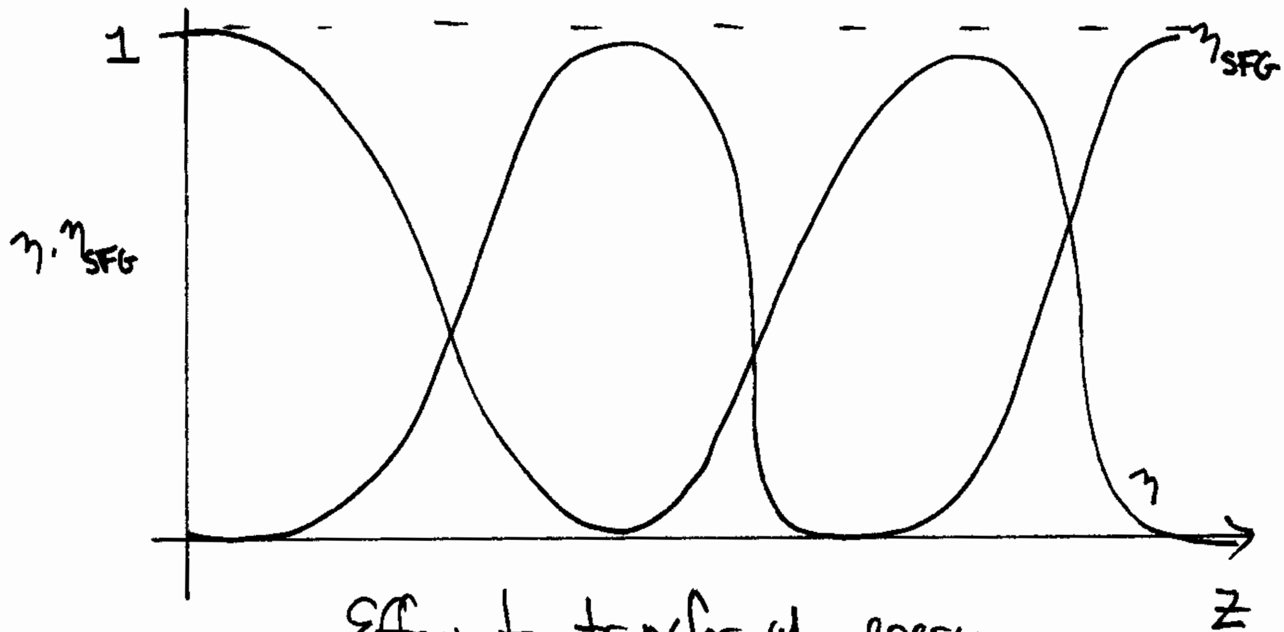
Conversion efficiency $\quad \eta_{SFG} = \dfrac{I_3(z)}{I_1(0)+I_2(0)} \simeq \dfrac{I_3(z)}{I_2(0)}$

Equal to 1 only if $\Delta k L = 0$

$$\eta \equiv \frac{I_1(z)}{I_1(0)}$$

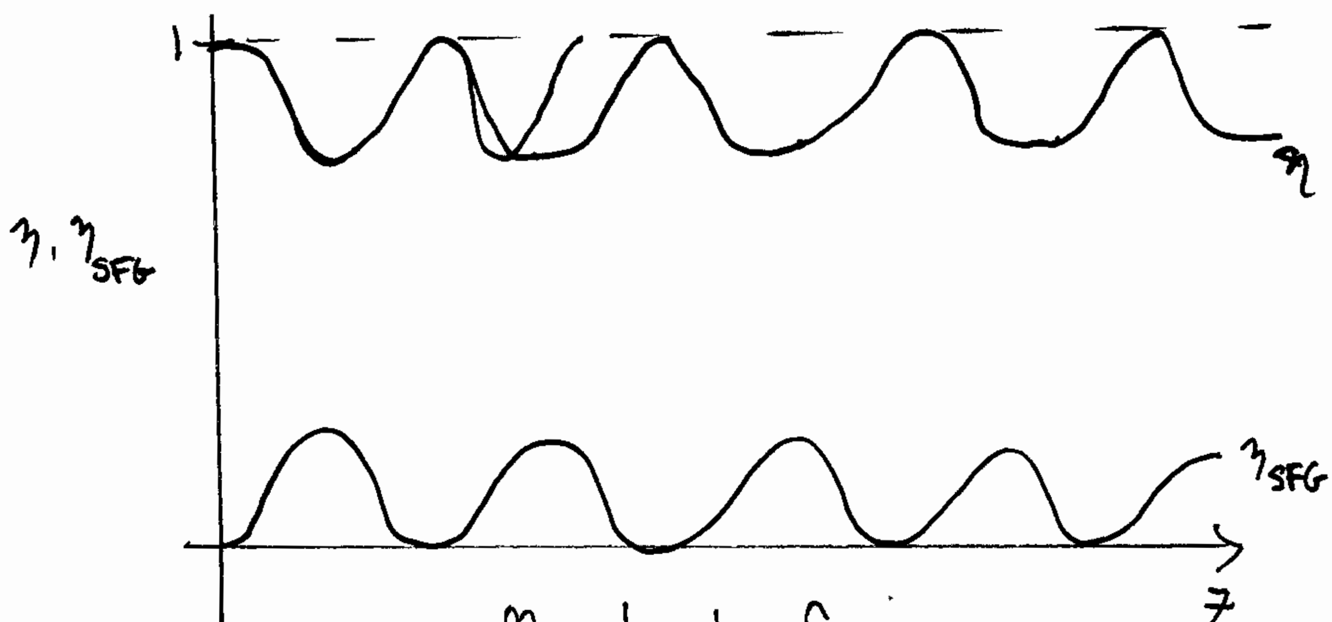
Two Cases

Perfect Phase matching $\Delta k = 0$ (or $\Delta k L = 0$)



Effecient transfer of energy

Non perfect phase matching $\Delta k L = 5$



inefficient transfer of energy.

Case II SHG generation with depleted pumps

$I_1 = I_2$ at $\omega = \omega_1$

I_3 at $2\omega = \omega_3$

$\Delta k = 2k_1 - K_3$

Solve coupled differential equations using a similar but complicated manner as in case II. Assume also $A_3(0) = 0$

~~Solution:~~

$$\boxed{\begin{aligned} \frac{dA_1}{dz} &= \frac{2i\omega_1}{n_1 c} d_{eff} A_1^* A_3 \exp(-i\Delta k z) \\ \frac{dA_3}{dz} &= \frac{2i\omega_3}{n_3 c} d_{eff} A_1^2 \exp(+i\Delta k z) \end{aligned}}$$

where $$\frac{I_1(z) + I_3(z)}{I_1(0)} = 1 \quad \text{and} \quad I_3(0) = 0$$

Solution for $\Delta k \neq 0 \Rightarrow$ solutions in terms of elliptic integrals

For $\Delta k = 0$ $$\boxed{I_3(z) = I_1(0) \tanh^2(z/L_{NL})}$$

where $$\boxed{L_{NL} = \frac{1}{4\pi d_{eff}} \sqrt{\frac{2\varepsilon_0 n_1^2 n_3 c \lambda_1^2}{I_1(0)}}}$$

For $\Delta k \neq 0$

$$I_3(z) = I_1(0)\,\gamma\,\mathrm{sn}^2\left\{\left[\sqrt{1+\left(\frac{\Delta k z}{4}\right)^2\left(\frac{L_{NL}}{z}\right)^2} + \frac{\Delta k z}{4}\left(\frac{L_{NL}}{z}\right)\right]\left(\frac{z}{L_{NL}}\right),\ \gamma\right\}$$

Where

$$\gamma \equiv \sqrt{1+\left(\frac{\Delta k z}{4}\right)^2\left(\frac{L_{NL}}{z}\right)^2} - \left(\frac{\Delta k z}{2}\right)\left(\frac{L_{NL}}{z}\right)^2$$

and

$$\mathrm{sn}\{-,\ \gamma\} \equiv \text{Jacobi elliptic sine function}$$

Look again at $g(\Delta k)$

$$\boxed{g = \sqrt{-K_1 K_3 + \tfrac{1}{4}\Delta k^2}}$$

g is the smallest when $\Delta k = 0$

also $-K_1 K_3$ is a positive #

$$-K_1 K_3 = -\left(\frac{2i\omega_1 d_{eff}}{n_1 c} A_2^*\right)\left(\frac{2i\omega_3 d_{eff}}{n_3 c} A_2\right)$$

$$= \frac{4 d_{eff}^2 \omega_1 \omega_3}{n_1 n_3 c^2} I_2 \frac{1}{2\varepsilon_0 n_2 c} = \frac{2 d_{eff}^2 \omega_1 \omega_3}{\varepsilon_0 n_1 n_2 n_3 c^3} I_2$$

Which is a positive quantity.

At $\Delta k = 0$

$$g = \sqrt{-K_1 K_3} = \left(\frac{2 d_{eff}^2 \omega_1 \omega_3}{\varepsilon_0 n_1 n_2 n_3 c^3} I_2\right)^{1/2} \quad \left\{ \begin{aligned} \omega_1 &= \frac{2\pi c}{\lambda_1} \\ \omega_3 &= \frac{2\pi c}{\lambda_1} \end{aligned} \right.$$

$$\frac{1}{g} = \left(\frac{\varepsilon_0 n_1 n_2 n_3 c^3}{2 d_{eff}^2 \omega_1 \omega_3} I_2^{-1}\right)^{1/2} = \left(\frac{\varepsilon_0 n_1 n_2 n_3 c^3}{2 d_{eff}^2 (2\pi c)^2} \lambda_1 \lambda_3 I_2^{-1}\right)^{1/2}$$

$$= \frac{1}{4\pi d_{eff}} \sqrt{\frac{2\varepsilon_0 n_1 n_2 n_3 c \lambda_1 \lambda_3}{I_2(0)}} = L_{NL}$$

General L_{NL}

$$L_{NL} = \frac{1}{4\pi d_{eff}} \sqrt{\frac{2\varepsilon_0 n_1 n_2 n_3 \lambda_2 \lambda_3}{I_1(0)}} = \frac{1}{\sqrt{-K_1 K_3}}$$

Has different forms for SHG and SFG

- Can rewrite g for SFG

$$\boxed{g(\Delta k) = \frac{1}{L_{NL}} \sqrt{1 + \frac{\Delta k^2 L_{NL}^2}{4}}}$$

From
Handbook of Nonlinear
Optics

Sutherland

Table 4 Frequency Conversion Efficiency Formulas in the Infinite Plane Wave Approximation, Including Pump Depletion

SHG	$\eta_{2\omega} = \tanh^2(L/L_{NL})$	
	$L_{NL} = \dfrac{1}{4\pi d_{eff}}\sqrt{\dfrac{2\varepsilon_0 n_\omega^2 n_{2\omega} c\lambda_\omega^2}{I_\omega(0)}}$ (SI)	$L_{NL} = \dfrac{1}{16\pi^2 d_{eff}}\sqrt{\dfrac{n_\omega^2 n_{2\omega} c\lambda_\omega^2}{2\pi I_\omega(0)}}$ (cgs)
SFG	$\eta_s = \dfrac{\lambda_{p2}}{\lambda_s}\,\mathrm{sn}^2[(L/L_{NL}), \gamma]$	$\gamma^2 = \dfrac{\lambda_{p2}\mathcal{P}_{p2}(0)}{\lambda_{p1}\mathcal{P}_{p1}(0)}$
	$L_{NL} = \dfrac{1}{4\pi d_{eff}}\sqrt{\dfrac{2\varepsilon_0 n_{p1} n_{p2} n_s c\lambda_{p2}\lambda_s}{I_{p1}(0)}}$ (SI)	$L_{NL} = \dfrac{1}{16\pi^2 d_{eff}}\sqrt{\dfrac{n_{p1} n_{p2} n_s c\lambda_{p2}\lambda_s}{2\pi I_{p1}(0)}}$ (cgs)
DFG	$\eta_d = -\dfrac{\lambda_{p2}}{\lambda_d}\,\mathrm{sn}^2[i(L/L_{NL}), i\gamma]$	$\gamma^2 = \dfrac{\lambda_{p2}\mathcal{P}_{p2}(0)}{\lambda_{p1}\mathcal{P}_{p1}(0)}$
	$L_{NL} = \dfrac{1}{4\pi d_{eff}}\sqrt{\dfrac{2\varepsilon_0 n_{p1} n_{p2} n_d c\lambda_{p2}\lambda_d}{I_{p1}(0)}}$ (SI)	$L_{NL} = \dfrac{1}{16\pi^2 d_{eff}}\sqrt{\dfrac{n_{p1} n_{p2} n_d c\lambda_{p2}\lambda_d}{2\pi I_{p1}(0)}}$ (cgs)

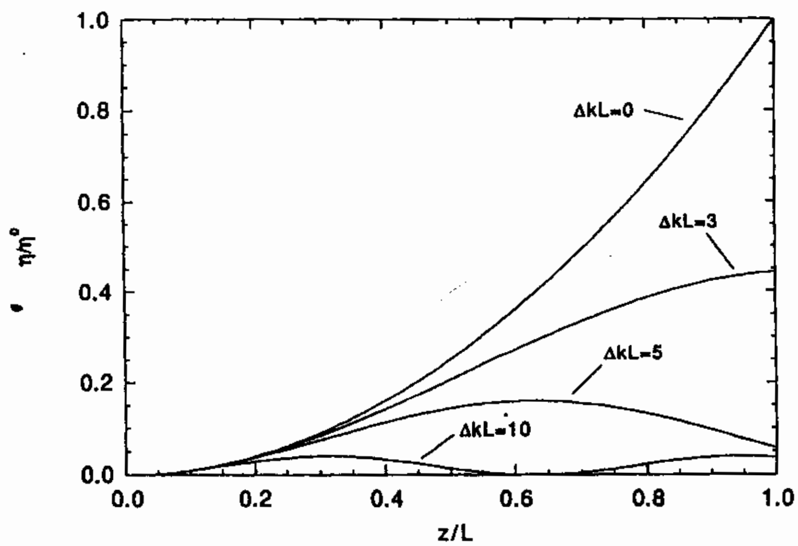


Figure 6 Normalized conversion efficiency as a function of position in a nonlinear medium for various values of phase mismatch for SHG, SFG, and DFG.

Table 6 Limiting Forms of the DFG Efficiency in the Infinite Plane Wave Approximation, Including Pump Depletion

DFG ($\gamma \ll 1$)	$\eta_d = \dfrac{\lambda_{p2}}{\lambda_d} \sinh^2 \dfrac{L}{L_{NL}}$
DFG ($\gamma = 1$)	$\eta_d = \dfrac{\lambda_{p2}}{\lambda_d} \dfrac{\mathrm{sn}^2[\sqrt{2}(L/L_{NL}), 1/\sqrt{2}]}{2 - \mathrm{sn}^2[\sqrt{2}(L/L_{NL}), 1/\sqrt{2}]}$
	$L_{NL} = \dfrac{1}{4\pi d_{eff}} \sqrt{\dfrac{2\varepsilon_0 n_{p1} n_{p2} n_d c \lambda_{p2} \lambda_d}{I_{p1}(0)}}$ (SI) $\quad L_{NL} = \dfrac{1}{16\pi^2 d_{eff}} \sqrt{\dfrac{n_{p1} n_{p2} n_d c \lambda_{p2} \lambda_d}{2\pi I_{p1}(0)}}$ (cgs)

photon flux originally in pump wave 2. Formulas for the limiting cases in SFG when $\gamma = 1$ and $\gamma \ll 1$ are given in Table 5.

Table 5 Limiting Forms of SFG Efficiency Formulas in the Infinite Plane Wave Approximation, Including Pump Depletion

SFG ($\gamma \ll 1$)	$\eta_s = \dfrac{\lambda_{p2}}{\lambda_s} \sin^2 \dfrac{L}{L_{NL}}$
SFG ($\gamma = 1$)	$\eta_s = \dfrac{\lambda_{p2}}{\lambda_s} \tanh^2 \dfrac{L}{L_{NL}}$
	$L_{NL} = \dfrac{1}{4\pi d_{eff}} \sqrt{\dfrac{2\varepsilon_0 n_{p1} n_{p2} n_s c \lambda_{p2} \lambda_s}{I_{p1}(0)}}$ (SI) $\quad L_{NL} = \dfrac{1}{16\pi^2 d_{eff}} \sqrt{\dfrac{n_{p1} n_{p2} n_s c \lambda_{p2} \lambda_s}{2\pi I_{p1}(0)}}$ (cgs)

Table 7 Frequency Conversion Efficiency Formulas in the Infinite Plane Wave Approximation, Including Pump Depletion and the Effects of Phase Matching

SHG	$\eta_{2\omega} = \gamma \mathrm{sn}^2\{[\sqrt{1 + (\Delta k L/4)^2 (L_{NL}/L)^2} + (\Delta k L/4)(L_{NL}/L)](L/L_{NL}), \gamma$
	$\gamma = [\sqrt{1 + (\Delta k L/4)^2 (L_{NL}/L)^2} - (\Delta k L/4)(L_{NL}/L)]^2$
SFG	$\eta_s = \dfrac{\lambda_{p2}}{\lambda_s} \dfrac{(1 + \gamma_0^{-2})}{2} p_- \mathrm{sn}^2[\sqrt{\tfrac{1}{2}(1 + \gamma_0^2) p_+}(L/L_{NL}), \gamma]$
	$\gamma^2 = \dfrac{p_-}{p_+} \qquad \gamma_0^2 = \dfrac{\lambda_{p2} \mathcal{P}_{p2}(0)}{\lambda_{p1} \mathcal{P}_{p1}(0)}$
	$p_\pm = 1 + \dfrac{(\Delta k L/2)^2 (L_{NL}/L)^2}{1 + \gamma_0^2} \pm \sqrt{\left[1 + \dfrac{(\Delta k L/2)^2 (L_{NL}/L)^2}{1 + \gamma_0^2}\right]^2 - \left(\dfrac{2\gamma_0}{1 + \gamma_0}\right)^2}$
DFG	$\eta_d = -\dfrac{\lambda_{p2}}{\lambda_d} \dfrac{(1 - \gamma_0^{-2})}{2} p_- \mathrm{sn}^2[i\sqrt{\tfrac{1}{2}(1 - \gamma_0^2) p_+}(L/L_{NL}), i\gamma]$
	$\gamma^2 = -\dfrac{p_-}{p_+} \qquad \gamma_0^2 = \dfrac{\lambda_{p2} \mathcal{P}_{p2}(0)}{\lambda_{p1} \mathcal{P}_{p1}(0)}$
	$p_\pm = 1 - \dfrac{(\Delta k L/2)^2 (L_{NL}/L)^2}{1 - \gamma_0^2} \pm \sqrt{\left[1 - \dfrac{(\Delta k L/2)^2 (L_{NL}/L)^2}{1 - \gamma_0^2}\right]^2 + \left(\dfrac{2\gamma}{1 - \gamma_0}\right)^2}$
($\gamma \ll 1$)	$\eta_d = \dfrac{\lambda_{p2}}{\lambda_d} \dfrac{1}{1 - (\Delta k L/2)^2 (L_{NL}/L)^2} \sinh^2[\sqrt{1 - (\Delta k L/2)^2 (L_{NL}/L)^2} \; (L/L_{NL})]$

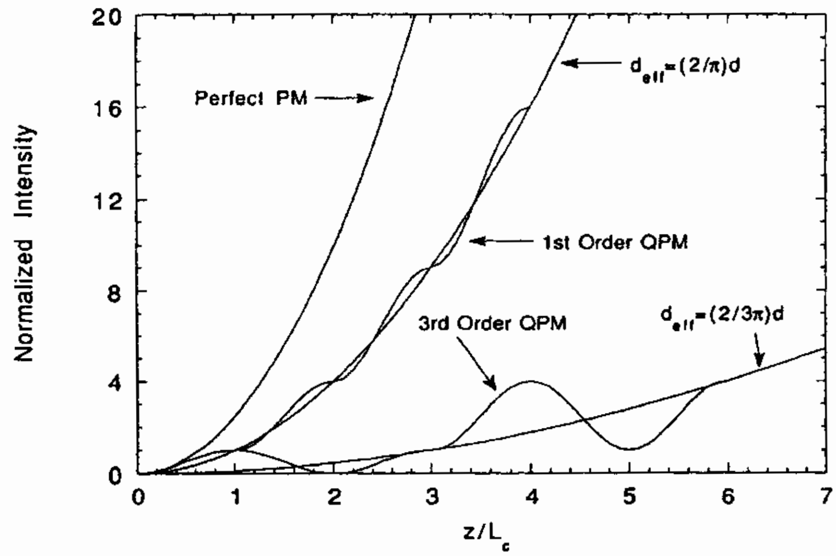


Figure 25 Normalized SHG intensity as a function of position in perfectly pha matched, first order quasi-phase matched, and third order quasi-phase matched nonline media.

analysis is a scaled dimensionless wave vector mismatch Δs, which proportional to Δk and inversely proportional to the pump intensity. Rustagi et a determined that in an ideal stack of plates a relative phase change of π radians c propagation through each plate is required for proper QPM. Note that this is th same requirement determined in the nondepleted pump regime. Howeve

Table 35 Frequency Conversion Efficiencies in the Infinite Plane Wave, Nondepleted Pump Approximation for mth order Quasi-Phase Matched Interactions in a Stack of N Plates

	SI	cgs
SHG	$\eta_{2\omega} = \dfrac{8\pi^2(2/m\pi)^2 d_{\text{eff}}^2 (NL_c)^2 I_\omega}{\varepsilon_0 n_\omega^2 n_{2\omega} c\lambda_\omega^2}$	$\eta_{2\omega} = \dfrac{512\pi^5(2/m\pi)^2 d_{\text{eff}}^2 (NL_c)^2 I_\omega}{n_\omega^2 n_{2\omega} c\lambda_\omega^2}$
SFG	$\eta_s = \dfrac{8\pi^2(2/m\pi)^2 d_{\text{eff}}^2 (NL_c)^2 I_{p1}}{\varepsilon_0 n_{p1} n_{p2} n_s c\lambda_s^2}$	$\eta_s = \dfrac{512\pi^5(2/m\pi)^2 d_{\text{eff}}^2 (NL_c)^2 I_{p1}}{n_{p1} n_{p2} n_s c\lambda_s^2}$
DFG	$\eta_d = \dfrac{8\pi^2(2/m\pi)^2 d_{\text{eff}}^2 (NL_c)^2 I_{p1}}{\varepsilon_0 n_{p1} n_{p2} n_d c\lambda_d^2}$	$\eta_d = \dfrac{512\pi^5(2/m\pi)^2 d_{\text{eff}}^2 (NL_c)^2 I_{p1}}{n_{p1} n_{p2} n_d c\lambda_d^2}$

Lecture 10 : Difference frequency generation and OPO

– Consider process $\omega_2 = \omega_3 - \omega_1$

Here ω_2 is the generated output.

$$\frac{dA_1}{dz} = \frac{2i\omega_1 d_{eff}}{n_1 c} A_3 A_2^* \exp(i\Delta k z)$$

$$\frac{dA_3}{dz} = \frac{2i\omega_3 d_{eff}}{n_3 c} A_1 A_2 \exp(-i\Delta k z)$$

$$\frac{dA_2}{dz} = \frac{2i\omega_2 d_{eff}}{n_2 c} A_3 A_1^* \exp(i\Delta k z)$$

Solution for $\Delta k \neq 0$ and assuming $A_3(z) \simeq A_3(0)$

$$A_1(z) = \left[A_1(0)\left(\cosh(gz) - \frac{i\Delta k}{2g} \sinh(gz) \right) + \frac{K_1}{g} A_2^*(0) \sinh(gz) \right] e^{i\Delta k/2 z}$$

$$A_2(z) = \left[A_2(0)\left(\cosh gz - \frac{i\Delta k}{2g} \sinh gz \right) + \frac{K_2}{g} A_1^*(0) \sinh gz \right] e^{i\Delta k/2 z}$$

Where $g \equiv \left(K_1 K_2 - \frac{\Delta k^2}{4} \right)^{1/2}$ $\quad K_j \equiv \frac{2i\omega_j d_{eff}}{n_j c} A_3(0)$

OR for $A_2(0) = 0$

$$A_1(z) = A_1(0)\left(\cosh gz - \frac{i\Delta k}{2g}\sinh gz\right)\exp\left(i\Delta k/2\, z\right)$$

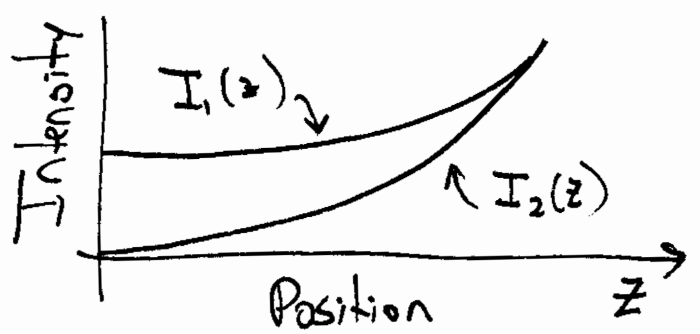
$$A_2(z) = \frac{K_2}{g} A_1(0)\sinh(gz)\exp\left(i\Delta k z/2\right)$$

<u>For $\Delta k = 0$</u>

$$\begin{cases} A_1(z) = A_1 \cosh(z/L_{NL}) \\ A_2(z) = i\left(\dfrac{n_1\omega_1}{n_2\omega_2}\right)^{1/2} \dfrac{A_3}{|A_3|} A_1^*(0)\sinh(z/L_{NL}) \end{cases}$$

<u>Important Points</u>

- Fields & Intensities are <u>not</u> harmonic in z
- Monotonic growth of difference frequency



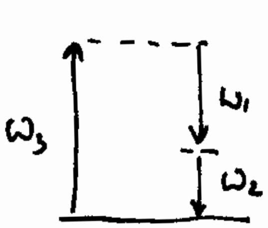
ω_1 is amplified by this process

$\Rightarrow$ This process is known as

<u>Parametric Amplification</u>

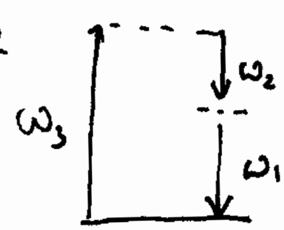
- ω_3 (pump)
- ω_1 is amplified by the process (signal wave)
- ω_2 is created by the process (idler wave)

Consider two cases



(a)

OR



(b)

(a) Presence of ω_1 causes a transition of ω_2

(b) ω_2 stimulates a transition of ω_1.

Both of these process lead to exponential growth.

Optical Parametric Amplifier (OPA)

Single pass thru crystal

No feedback

Modest gain for single pass

Very useful for tunable wavelength conversion of a CW laser

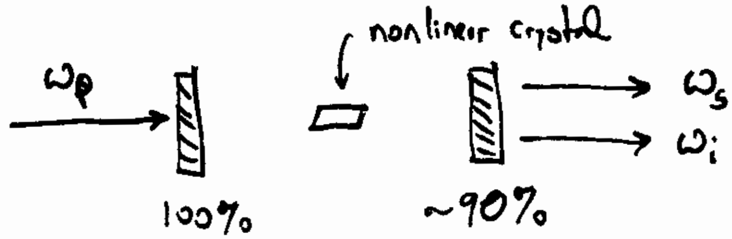
$$\text{gain} \simeq \left(L/L_{NL}\right)^2 \sinh^2\left(\frac{\left(L/L_{NL}\right)^2 - \left(\Delta k L/2\right)^2}{\left(L/L_{NL}\right)^2 - \left(\Delta k L/2\right)^2}\right)$$

$$L_{NL} = \frac{1}{4\pi d_{eff}} \sqrt{\frac{2\varepsilon_0 n_p n_s n_i c \lambda_s \lambda_i}{I_p(0)}} \quad \left\{ \begin{array}{l} p \equiv \text{pump} \\ s \equiv \text{signal} \\ i \equiv \text{idler} \end{array} \right.$$

For $\Delta k = 0$ $\quad \text{gain} \simeq \left(L/L_{NL}\right)^2$

Optical Parametric Oscillators (OPO)

Put the nonlinear process in a cavity with mirrors that are highly reflecting a ω_1 or ω_2



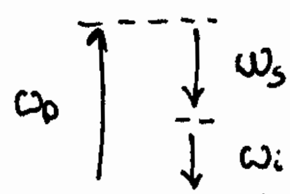
$$\left\{ \begin{array}{l} \omega_p \Rightarrow \omega_3 \\ \omega_s \Rightarrow \omega_1 \\ \omega_i \Rightarrow \omega_2 \end{array} \right.$$

Difference frequency generation leads to the amplification of the lower frequency input field.

$$\omega_i = \omega_p - \omega_s$$

The gain associated with parametric amplification can in the presence of feedback provide an oscillation.

Mirrors can reflect both ω_i and/or ω_s



$\omega_i < \omega_s$

(also called Parametric Down Conversion)

For the case where $\Delta k = 0$ $\quad A_2(0) = 0$ and $A_3(z) \simeq A_3(0)$

$$A_1(z) = A_1(0)\cosh(gz) \quad \text{(an exponential function)}$$

$$A_2(z) = i\left(\frac{n_1\omega_2}{n_2\omega_1}\right)^{1/2} \frac{A_3(0)}{|A_3(0)|} A_1^*(0)\sinh(gz)$$

Both signal + idler experience exponential growth.

** But is an OPO a laser ?! **

Is an OPO a laser?

An OPO Has:

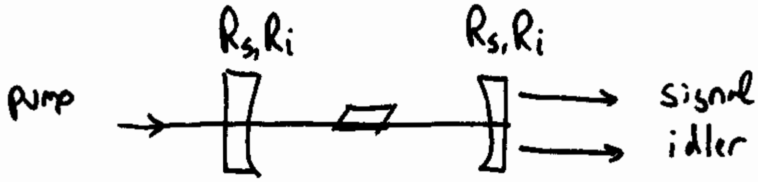
1) A pump source
2) A cavity for feedback
3) Gain at a specific freqeuncy

Answer: No!

A laser has a population inversion caused by the pump. An OPO does not have a population inversion. So it is technically <u>not</u> a laser.

The problem with a laser is saturation when the upper population gets too large. An OPO does not have this problem!

Threshold for Parametric oscillation for a Doubly Resonant OPO



$R_i \equiv$ idler reflectivity
$R_s \equiv$ signal ref.

Threshold of oscillation

$$\text{Gain per pass} \equiv \text{loss per pass}$$

$$\left(\exp(2L/L_{NL}) - 1\right) = (1 - R_s)(1 - R_i)$$

For low loss

$$\left(L/L_{NL}\right)^2 \simeq (1 - R_s)(1 - R_i)$$

For single resonant OPO

$$\left(L/L_{NL}\right)^2 \sim 2(1 - R_s)$$

Tuning and Bandwidth

Wavelength tuning is typically done by temperature tuning.

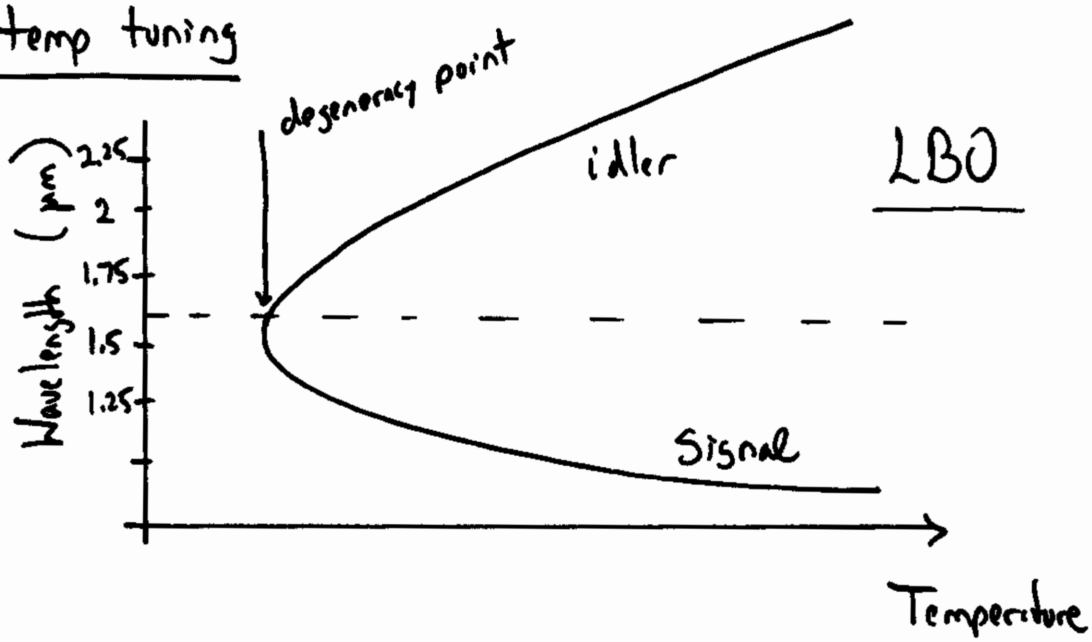
Must satisfy momentum + energy conservation

$$\omega_s + \omega_i = \omega_p \qquad \vec{k}_s + \vec{k}_i = \vec{k}_p$$

Phase Matching $\Rightarrow \omega_p\left[n(\omega_p) - n(\omega_p - \omega_s)\right] = \omega_s\left[n(\omega_s) - n(\omega_p - \omega_s)\right]$

Can also be done using angle tuning

Example of temp tuning



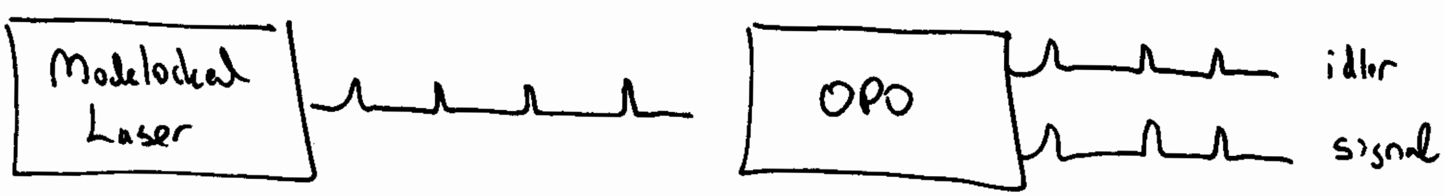
Pump $\lambda_p = 800nm$

Huge tuning range $\begin{cases} \text{signal} & 1 - 1.5\ \mu m \\ \text{idler} & 1.5 - 2.25\ \mu m \end{cases}$

Synchronously Pumped OPO

Use a pulsed laser for the pump

Get tunable wavelength pulsed light $\Rightarrow$ signal + idler

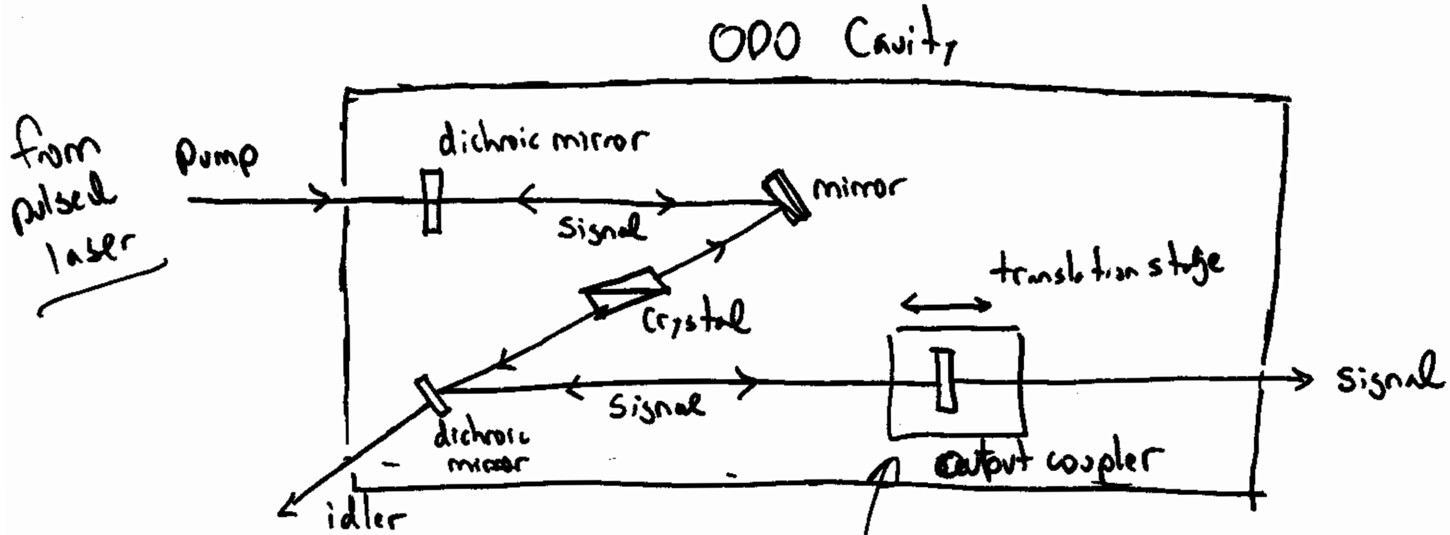


The second order process is modest ~~so~~ even for $\Delta k = 0$. ~~[illegible]~~ For efficiency signal + idler generation, the round trip time of the OPO cavity must be matched to the mode locked laser repetition rate.

$$\tau_{RT} = \frac{2L_c}{c} N_{crystal}(\lambda) + \frac{2L_F}{c} = \frac{1}{F} \quad \begin{cases} L_c \equiv \text{crystal length} \\ L_F \equiv \text{cavity length} \end{cases}$$

To match the cavity length & Rep Rate

Use following ~~mirror~~ cavity



this translation stage with μm resolution is used to ~~get~~ match the cavity length to the repetition rate.

Output coupler (~99% for ωs) is on translation stage

$$\tau_{RT} = \frac{2L_c}{c} N(\lambda) + \frac{2L_F}{c}$$

$$= \frac{1}{F} = T$$

Lecture 11 Quasi Phase matching

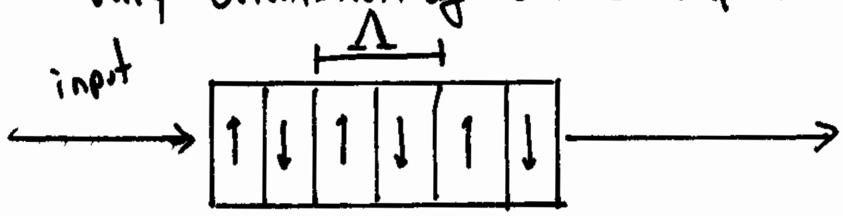
Problems with angle tuning for perfect phase matching

- at some angles propagation is difficult thru the crystal
- walk off
- cannot angle tune cubic crystals, which typically have larger d_{eff}
- cannot use d_{ii} terms, which are typically larger than off diagonal terms (d_{13} etc)
- cannot be used with crystals without birefringence
  e.g. Gallium Arsenide (GaAs)
- birefringence decreases with increasing ω.

— Use quasi phasematching instead with periodically poled materials

Periodically poled materials

Vary orientation of c axis (~~optical axis~~) in crystal



$\Rightarrow$ fabricated structure, not natural

Inversion of c axis changes the sign of d_{eff}

$$+d_{eff}\ \ -d_{eff}\ \ +d_{eff}\ \ -d_{eff}\ \ -\ \cdots$$

The variation of d_{eff} can compensate for $\Delta k \neq 0$!

$$\boxed{d_{n,\text{eff}} = (-1)^{n-1} |d_{\text{eff}}| \frac{2}{\pi m}}$$

$m \equiv$ order of phase matching

Write down coupled eqs

$(m=1)$

$$\frac{dA_1}{dz} = \frac{2i\omega_1 d_{\text{eff},n}}{n_1 c} A_3 A_2^* \exp(\Delta k_n z)$$

$$\frac{dA_2}{dz} = \frac{2i\omega_2 d_{\text{eff},n}}{n_2 c} A_1^* A_3 \exp(\Delta k_n z)$$

$$\frac{dA_3}{dz} = \frac{2i\omega_3 d_{\text{eff},n}}{n_3 c} A_1 A_2 \exp(\Delta k_n z)$$

where $\Delta k_n = k_1 + k_2 - k_3 - \frac{2\pi}{\Lambda}$

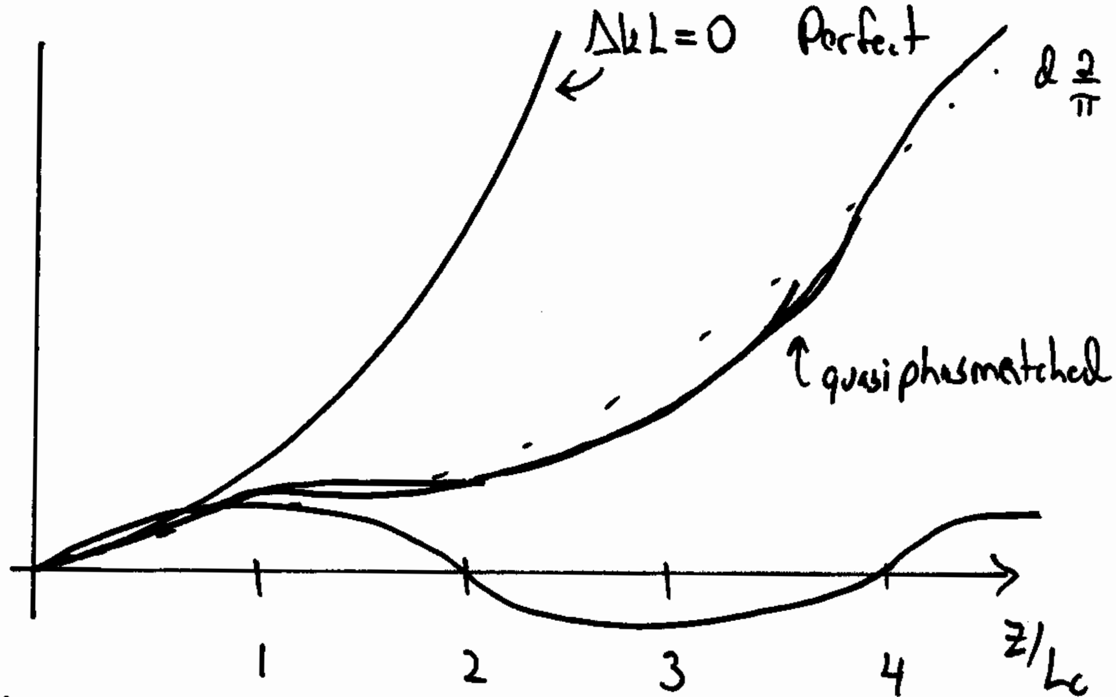
We can determine the optimal period Λ

$$\boxed{\Lambda = \frac{2\pi}{k_1 + k_2 - k_3} \equiv 2L_c}$$

for Lithium niobate $L_c = 3.4\,\mu m$ for $\lambda = 1.06\,\mu m$

Define $L_c = \frac{\Lambda}{2}$

Plot Phase matching



From this plot we see ....
QPM is not as efficient as Perfect phase matching but the disadvantage is compensated because with QPM larger die values can be used

$LiNbO_3$: Angle phase matching Type $I^{(-)}$ $\Rightarrow d_{31}$

QPM $\Rightarrow d_{33}$

$$\left(\frac{d_{33}}{d_{31}}\right)^2 \left(\frac{2}{\pi}\right)^2 \simeq 15!$$

Efficiency of QPM is 15 times larger!

Temperature tuning

One can change the output wavelength by heating the crystal

$$L_c = L_c(T) \qquad T \equiv \text{temperature}$$

Heating the crystal increases Λ thus decreasing the generated wavelength λ_3. It also changes the index. Calculate phase matching λ using SNLO program.

How does one make a periodically poled crystal?

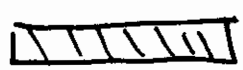
Quasiphase match is an old idea but at the time there was not a method to create the periodic poling. (Circa 196[illegible])

Exposing a crystal to a strong electric field inverts magnetic domains which changes the orientation of deff. (1993)

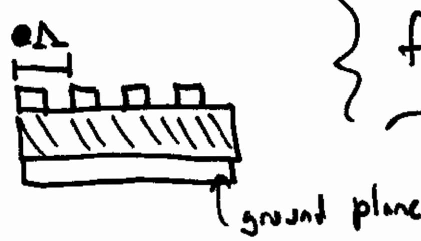
The crystal needs to be ferroelectric

Procedure

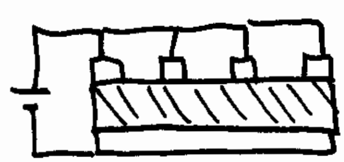
1) Lithium Niobate



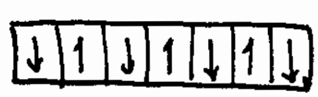
2) Deposit metal mask with desired periodicity



3) Apply 21 kV/mm electric field



4) Only material under electrodes get the domain reversal



Change domain with $\vec{E}$

No iron (ferro)

Advantages for QPM

- Use d_{ii} terms instead of d_{ij} terms. $d_{ii} > d_{ij}$ in general.
- Can be used for crystals that are <u>not</u> birefringent.
- Less walk off than for birefringent phase matching.
- Easier alignment
- Waveguide geometry $\Rightarrow$ Guide the fundamental!!

Disadvantages

- Periodic poling with strong electric fields can only be done in ferroelectric materials.

Lecture 12 : SHG with ultrashort pulse

So far we discussed SHG for a monochromatic source (a CW laser). For ultrashort pulses, which a comprised a bandwidth of spectral components, SHG occurs for all components. However, perfect phasematching $\Delta k = 0$ only occurs for <u>one</u> spectral component.

For pulses we discussed the group velocity + group index

$$N(\lambda) = n - \lambda \frac{dn}{d\lambda} \qquad v_g = \frac{c}{N(\lambda)} = \frac{dk}{d\omega}$$

We can define a group velocity for the fundamental ω and SHG 2ω.

$$N_\omega \quad N_{2\omega} \quad \Big\} \quad V_{g\omega} \quad V_{g2\omega}$$

In-general the two group velocities will <u>not</u> be the same. This is called the <u>group velocity mismatch</u> (GVM)

$$\boxed{\Delta v_{GVM} = -V_{g,2\omega} + V_{g,\omega}} = -\frac{c}{N(\lambda/2)} + \frac{c}{N(\lambda)}$$

It is a measure of the delay between the fundamental and SHG pulse. This mismatch leads to a finite phase matching bandwidth between the fundamental + SHG.

Example of GVM

0.3 mm KDP crystal

$\lambda_0 = 620\,nm \qquad \lambda_0/2 = 310\,nm$

$$\text{group delay mismatch} = \frac{1}{\Delta v_g} L = 56\,fs$$

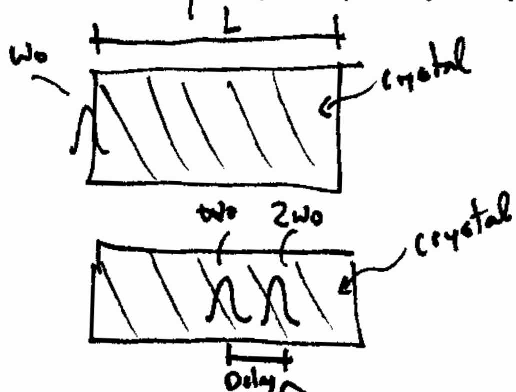
We wish to see how the effect of GVM changes the generated second harmonic spectrum for SHG.

$$I_{SHG}(\omega) = \,?$$

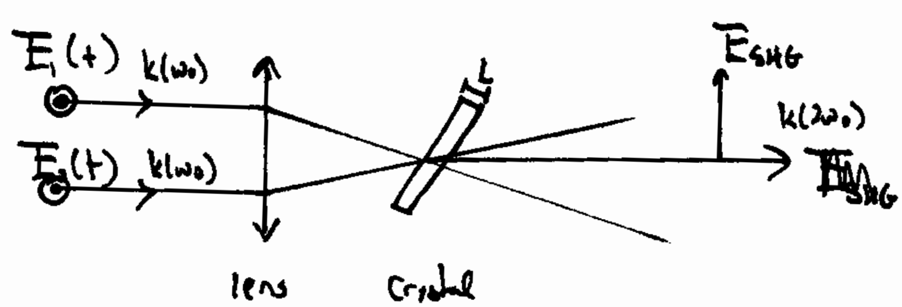
However, due to GVM the fundamental + SHG pulse will not maintain temporal synchronicity for the entire length of the crystal.

$t=0$

$t = $ later



We wish to derive an expression for $I_{SHG}(\omega)$ for a noncolinear coupling into the crystal.



$E_1 + E_1$ along ordinary axis. E_{SHG} along extraordinary. Note $E_2(t) = E_1(t-\tau)$

The result for $I_{SHG}(\omega)$ is given by:

$$\boxed{I_{SHG}(\omega) = \frac{\sin^2(\Delta k L/2)}{(\Delta k L/2)} \; I(\omega) \otimes I(\omega)}$$

$\Delta k \equiv$ total phase mismatch. Write the total mismatch for SHG with a spectral bandwidth.

<u>For SHG</u> with pulse. We need to Taylor expand $\Delta k(\omega)$

$$\Delta k(\omega) = \underbrace{\left[k(\omega_0) + k(\omega_0) - k(2\omega_0) \right]}_{\text{phase matching for } \omega_0 + 2\omega_0 \text{ only}}$$

phase velocity mismatch

$$+ \underbrace{\left[\frac{\partial k}{\partial \omega}\Big|_{\omega_0} - \frac{\partial k}{\partial \omega}\Big|_{2\omega_0} \right] \omega}_{\text{Group velocity mismatch}}$$

$$+ \underbrace{\frac{1}{2}\omega^2 \left[\frac{\partial^2 k}{\partial \omega^2}\Big|_{\omega_0} - \frac{\partial^2 k}{\partial \omega^2}\Big|_{2\omega_0} \right]}_{\text{group delay dispersion}}$$

This expansion gives terms that will be nonzero at the phase matching angle. This implies that perfect phasematching cannot be applied for all spectral components of the pulse.

Consider only phase + group velocity mismatch

$$\boxed{\Delta k \cong \Delta k_0 + \frac{1}{\Delta V_g} \omega}$$

GVM

But for Type I ooe phase matching

$$\boxed{\Delta k(2\omega,\theta) = \frac{2\omega}{c}\left[n_e(2\omega,\theta) - n_o(\omega)\right]}$$

We can also write the GVM as

$$\boxed{\Delta V_g = \frac{c}{N(2\omega)} - \frac{c}{N(\omega)}}$$

SHG Spectral filtering

The group velocity mismatch leads to a finite spectral bandwith for SHG. Find this bandwith.

$$E_1(z,t) = A_1\left(t - \frac{1}{V_g(\omega_0)} z\right) \exp(-i(\omega_0 t - k(\omega_0)z))$$

$$E_2(z,t) = E_1(z, t-\tau) \Leftarrow \text{Delayed version of } E_1$$

$$E_{SHG}(z,t) = A_{SHG}\left(z,\ t - \frac{1}{V_g(2\omega_0)} z\right) \exp(-i(2\omega_0 t - k(2\omega_0)z))$$

$$P_{NL}(z,t) = \epsilon_0 \chi^{(2)} E_1 E_2 \quad \Rightarrow \text{assume } \chi \text{ is fast.}$$

Should be

$$P_{NL}(z,t) = \epsilon_0 \iint \chi^{(2)}(t-t', t-t'') E_1(z, t-t') E_2(z, t-t'-\tau)\, dt'\, dt''$$

Write each in terms of its spectrum

$$E_1(z,t) = \mathcal{F}^{-1}\{A_1(\omega)\}\exp(-i\ \omega_o t - k(\omega_o)z)$$

$$E_2(z,t) =$$

$$P_{NL}(z,t) = \epsilon_o \chi^{(2)} \exp(-i(2\omega_o t - 2k(\omega_o)z))e^{-i\omega_o\tau}$$

$$* \mathcal{F}\left\{ \int A_1(\omega-\omega')A_1(\omega')\exp(-i\omega'\tau)\right\}$$

Differential Eq using the SVEA

$$\boxed{\partial_z A_{SHG}(z,t) = -\frac{i2\omega_o\mu_o c}{2n} P_{NL}(z,t)\exp(i\Delta k_o z)}$$

where $\Delta k_o = 2k(\omega_o) - k(2\omega_o)$

Write DE in frequency domain

$$\boxed{\begin{aligned}\partial_z A_{SHG}(z,\omega) &= \frac{-i\omega_o\chi^{(2)}}{nc\,2\pi}\exp(-i\omega_o\tau)\exp(i\Delta k z)\\ &* \int A_1(\omega-\omega')A(\omega')e^{-i\omega'\tau}d\omega'\end{aligned}}$$

where $\Delta k = \Delta k_o + \frac{1}{\Delta v_g}\omega$

Integrate from 0 to L

$$\boxed{\hat{A}_{SHG}(\omega, L) \simeq e^{-i\omega_0\tau}\left[\exp(i\Delta k L/2)\right]\left[\frac{\sin(\Delta k L/2)}{\Delta k L/2}\right] \times \int A_1(\omega-\omega')A(\omega')e^{-i\omega'\tau}\,d\omega'}$$

We want the intensity $I_{SHG}(\omega, L) \simeq \left|A_{SHG}(\omega, L)\right|^2$ over all τ.

$$\hat{I}_{SHG}(\omega) = \frac{\sin^2(\Delta k L/2)}{(\Delta k L/2)^2}\left[\int \left[\int A(\omega-\xi)A(\xi)e^{-i\xi\tau}d\xi\right] \times \left[\int A^*(\omega-\eta)A(\eta)e^{-i\eta\tau}d\eta\right] d\tau\right]$$

But $\delta(\eta-\xi) = \int \exp(i(\eta-\xi)\tau)\,d\tau$

(FT of a delta function). Which allows us to do the integral over τ to get rid of ξ.

$$\hat{I}_{SHG}(\omega) = \text{sinc}(\Delta k L/2)\int A^*(\omega-\eta)A(\omega-\eta)A^*(\eta)A(\eta)\,d\eta$$

But $A^*(\,)A(\,) = I(\,)$

So

$$\boxed{\begin{aligned} I_{SHG}(\omega) &= \operatorname{sinc}^2(\Delta k L/2) \int I_1(\omega-\eta)\, I(\eta)\, d\eta \\ &\text{OR} \\ I_{SHG}(\omega) &= \operatorname{sinc}^2(\Delta k L/2)\; I_1(\omega) \otimes I_1(\omega) \\ &\text{OR} \\ I_{SHG}(\omega) &= H(2\omega)\; I_1(\omega) \otimes I_1(\omega) \end{aligned}}$$

where $$H(2\omega) = \frac{\sin^2\left(\Delta k(2\omega,\theta)\, L/2\right)}{\left(\Delta k(\omega) L/2\right)^2}$$

The filter function $H(2\omega)$ is evaluated for the case of perfect phase matching

$$\theta = \theta_{pm} \quad \text{where} \quad \Delta k_0 = 0$$

Major Points

1) The width of the SHG spectrum is related to the autoconvolution of the fundamental spectrum.

2) Width of $H(\omega)$ depends on GVM $\Rightarrow \Delta v_g^{-1} L$ and the crystal length.

3) Center of $H(\omega)$ depends on Δk_0

How to calculate the spectral filtering ?

1) Find $I_1(\omega)$ and $\Delta\lambda$ of the fundamental pulse

2) Determine $\Delta k(2\omega, \theta)$ given $n_e(\lambda)$ + $n_o(\lambda)$ of your given crystal. Set $\theta = \theta_{pm}$ (perfect phase matching)

3) Find $$H(2\omega) = \frac{\sin^2(\Delta k(2\omega,\theta_{pm}) L/2)}{(\Delta k(2\omega,\theta_{pm}) L/2)}$$

4) Determine the SHG spectrum from $I_1(\omega) \otimes I_1(\omega)$. Find its spectral width $\Delta\lambda_{SHG}$

5) Determine the filtered SHG spectrum using

$$H(2\omega)(I(\omega) \otimes I(\omega))$$

6) Make sure the filtered spectrum $\Delta\lambda_{SHG,\,filtered}$ is not significantly different than $\Delta\lambda_{SHG}$

Lecture 15 Applications for SHG

Back to $I_{SHG}(\omega) \Rightarrow$ Problem with Weiner result for chirped pulses

$$\boxed{I_{SHG}(\omega) \simeq \mathrm{sinc}^2(\Delta k L/2) \left| \int E_1(q)\, E_1(\omega - q)\, dq \right|^2}$$

for transform-limited pulses (or nearly transform-limited pulses)

$$I_{SHG}(\omega) \simeq \mathrm{sinc}^2(\Delta k L/2) \left[I_1(\omega) \otimes I_1(\omega) \right]$$

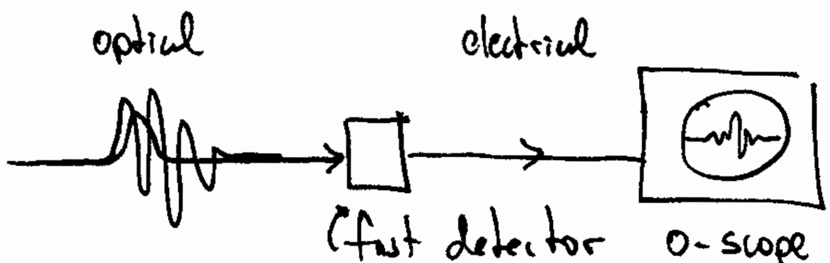
The presence of a phase distortion does not modify the spectral width but does modify the spectral shape

- What about $I_{SHG}(t)$?

  The SHG intensity will also be a function of chirp.

Applications Pulse measurement

We really want



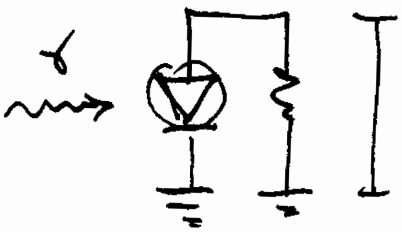
- Why can't we do this! Motion of electrons in the detection cannot follow the oscillations of the optical electric field.

$$V(t) = h(t) \otimes I(t)$$

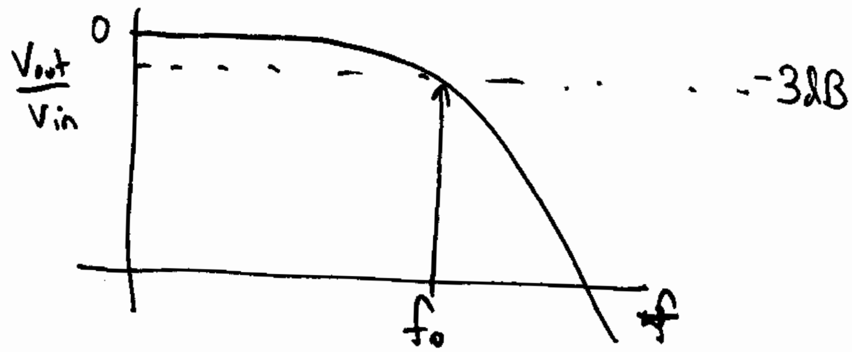
↑ detector response

Really Fast optical detector

Converts optical power to a current, or a voltage



"Bandwidth" $f_0 \sim 100$ GHz



$\frac{1}{f_0} \sim 0.1$ ps $\Leftarrow$ not fast enough to resolve the oscilations of the pulse's electric field oscillations

(really more like 1ps)

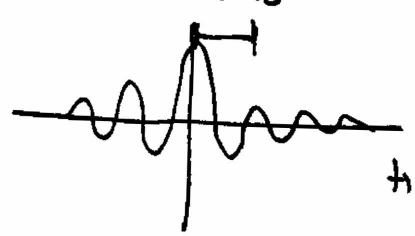
Example

$\lambda_0 = 800$ nm

$\omega_0 = \frac{2\pi c}{\lambda_0} = 2.35\ {}^{1}/_{fs}$

$f_0 = 0.375\ {}^{1}/_{fs}$

${}^{1}/_{f_0} = 2.7$ fs (single cycle)



We need a better method

What do we want to measure?

$$\boxed{E(t) \cong \sqrt{I(t)}\ \exp(\,i\,\phi(t)\,)}$$

$I(t) \equiv$ Temporal Intensity $\quad \phi(t) \equiv$ Temporal Phase

OR we can write

$$\boxed{E(\omega) \cong \sqrt{I(\omega)}\ \exp(\,i\,\phi(\omega)\,)}$$

$I(\omega) \equiv$ spectral intensity $\quad \phi(\omega) \equiv$ spectral phase

Can we measure $E(\omega)$ + $E(t)$ directly $\Rightarrow$ Difficult.

earlier measurements provided limited information about $E(t)$

<u>Intensity Autocorrelation</u>

Gives an <u>estimate</u> of the pulse duration and shape

Really a "guess-estimate"

Measure $\underbrace{I_{ac}(\tau)}_{\text{autocorrelation}} = \int I(t)\, I(t-\tau)\, dt$

How to do this $\Rightarrow$ use SHG generation

Prove

$$\mathcal{F}\{\Gamma^{(1)}(\tau)\} = \mathcal{F}\{\int E(t)\,E^*(t-\tau)\,dt\} = I(\omega)$$

$$\mathcal{F}\{\int E(t)\,E^*(t-\tau)\,dt\} = \mathcal{F}\{E(t)\overset{\text{convolution}}{\otimes} E(t)\}$$

From the convolution thm $\mathcal{F}\{g\otimes f\} = \mathcal{F}\{g\}\mathcal{F}\{f\}$

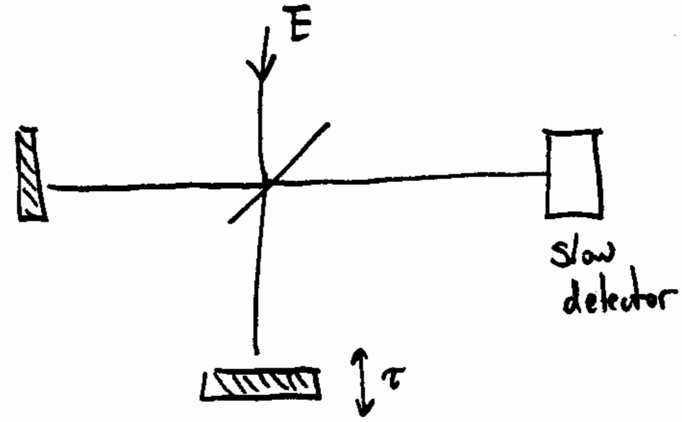
So

$$= \mathcal{F}\{E(t)\}\;\mathcal{F}\{E^*(t)\}$$

$$= E(\omega)\,E^*(\omega)$$

$$= I(\omega)$$

Can we use a Michelson Interferometer?!



Measure interferogram

$$I_M(\tau) \sim \int \left| E(t) - E(t-\tau) \right|^2 dt$$

$$I_m(\tau) \sim \underbrace{2\int |E(t)|^2 dt}_{\text{pulse intensity}} \; - \underbrace{2\,\mathrm{Re}\left\{ \int E(t)\, E^*(t+\tau)\, dt \right\}}_{\substack{\text{interferogram} \\ \text{Field autocorrelation} \\ (\Gamma^{(2)}(t))}}$$

But $\mathcal{F}\left\{ \Gamma^{(2)}(t) \right\} = I(\omega)$
(the spectrum!)

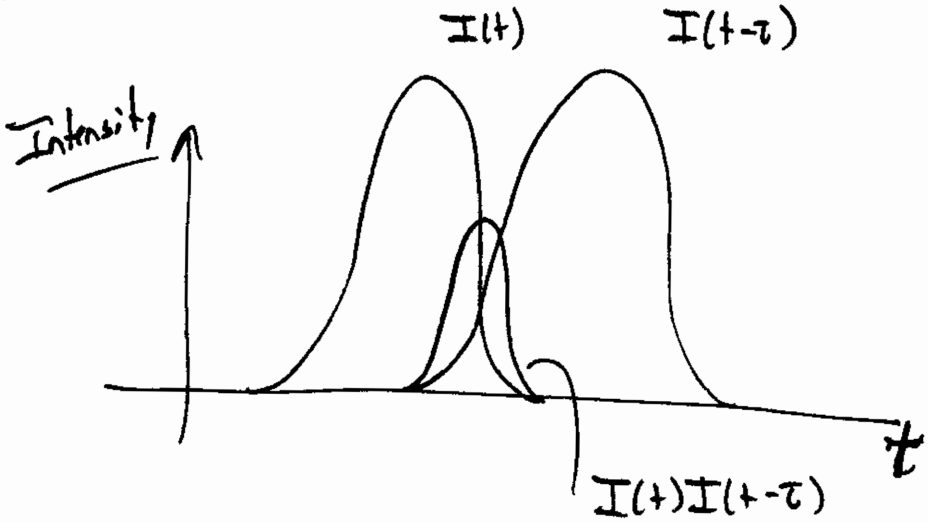
Measuring interferometer is same as measuring the spectrum.

→ If you do not have a detector or modulator that is fast compared to the pulse you cannot measure the pulse intensity + phase

Need something that has a faster response $\Rightarrow$ $\chi^{(2)}$ effects!!

Pulse measurement in time domain: Intensity AC

Wish to overlap two pulses in a crystal as a function of delay τ



$$I_{AC}(\tau) \equiv \int I(t)\, I(t-\tau)\, dt$$

The second harmonic intensity is a function of τ and the temporal overlap in the nonlinear crystal

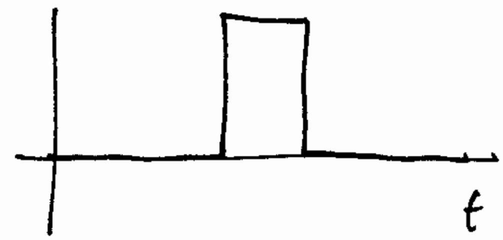
The response of the medium provides the fast temporal resolution to measure the pulse duration.

Note that $I_{AC}(\tau) = I_{AC}(-\tau)$

Mathematical Picture of an autocorrelation.

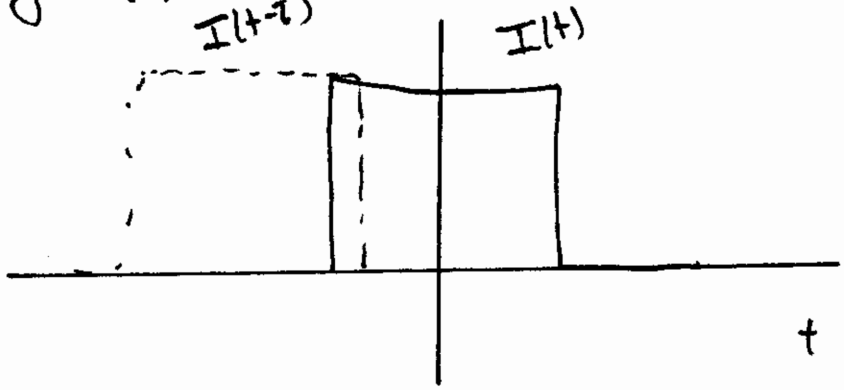
Pulse

$$I(t) = \begin{cases} 1 & |t| \leq \Delta t/2 \\ 0 & \text{elsewhere} \end{cases}$$



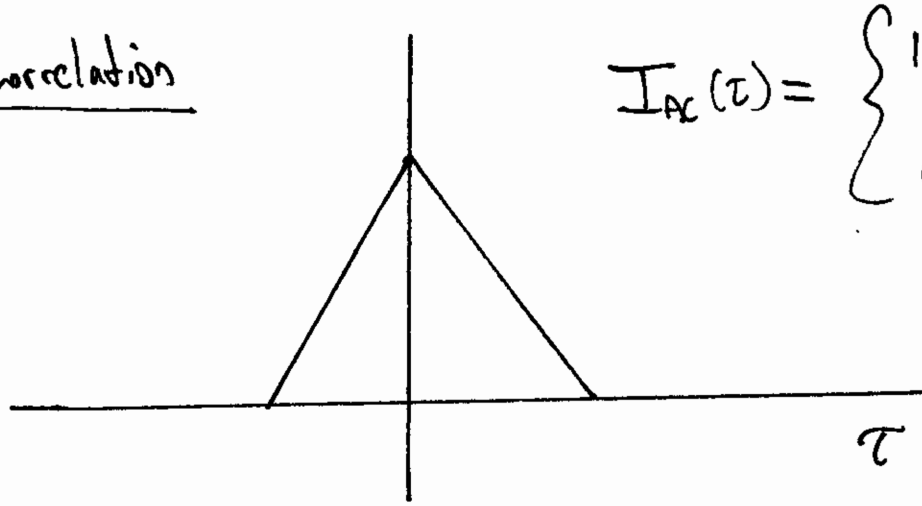
Autocorrelation

Scan a replica of the pulse $I(t-\tau)$ for all values of t.



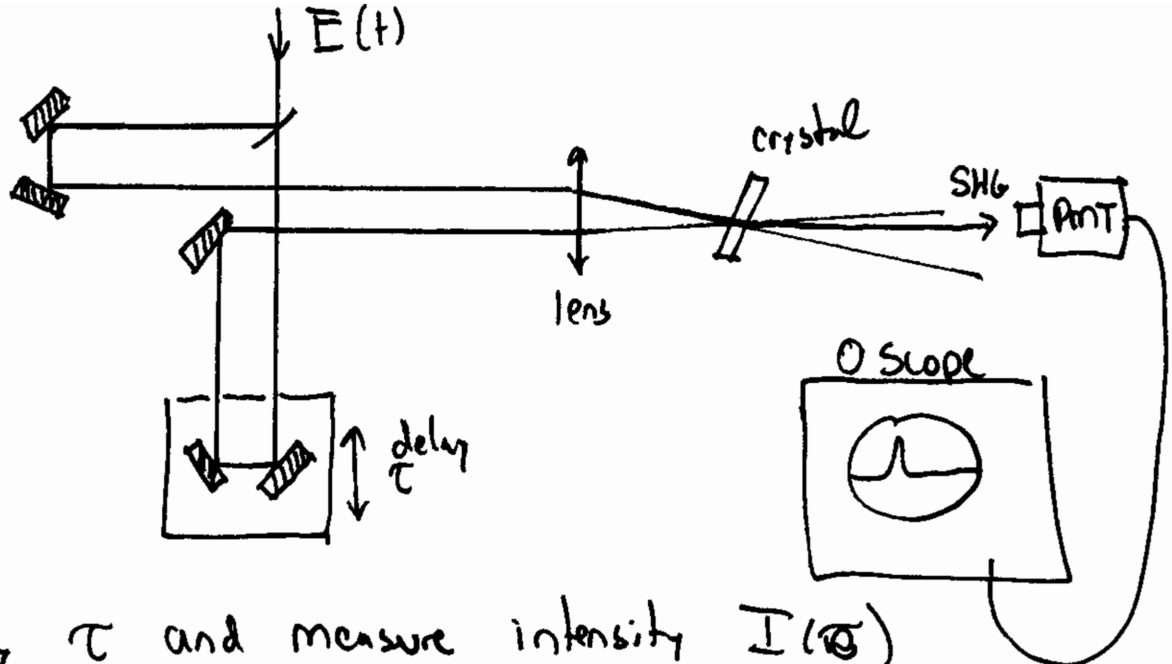
Autocorrelation

$$I_{AC}(\tau) = \begin{cases} 1 - \left|\dfrac{\tau}{\Delta t_{AC}}\right| & \tau < \Delta\tau_{AC} \\ 0 & \end{cases}$$



Where $\Delta\tau_{AC} = \Delta t$

Setup



Scan delay τ and measure intensity $I(\tau)$

Due to the non colinear focusing in the crystal, SHG into the PMT (photomultipler tube) will be detected for an temporal overlap. The SHG intensity as a function of delay will be the autocorrelation of the fundamental intensities.

$$I_{AC}(\tau) \sim \int I(t)\, I(t-\tau)\, d\tau$$

Advantages

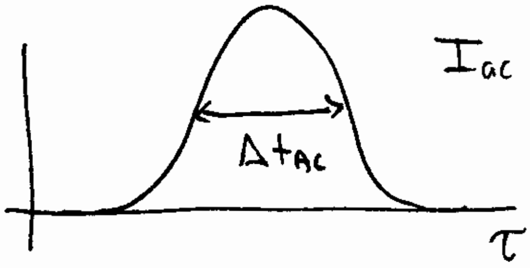
- Can scan τ at any rate
- gives a "pretty good" estimate of maximum Δt.

Disadvantages

- Need to guess form of $E(t)$ to get estimate of Δt
- Does not measure $E(t)$ or even $I(t)$
- Autocorrelations are not unique $\Rightarrow$ multiple $E(t)$ will produce the same autocorrelation.
- Cannot get $\phi(t)$ or $\phi(\nu)$

Example

Measure $I_{AC}(\tau)$



$\Delta t_{AC} = 100\,fs$

want pulse duration $I(t)$.

Guess shape $\Rightarrow$ $\mathrm{sech}^2(t)$

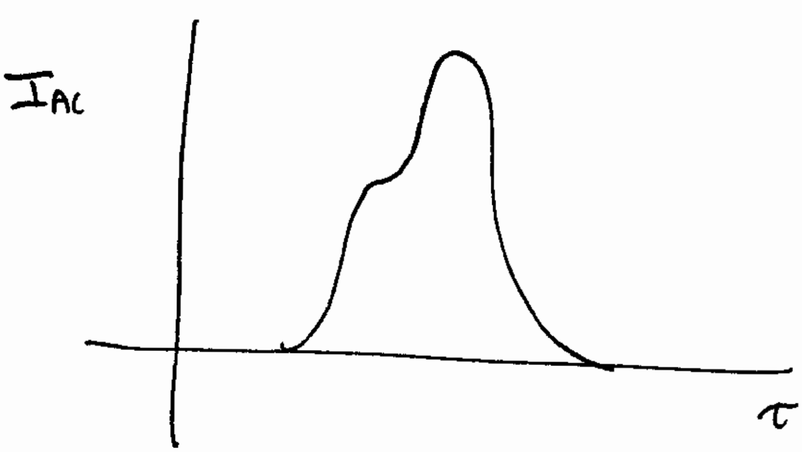
So $\quad \dfrac{\Delta t}{\Delta t_{AC}} = 0.6482 \quad$ for $\mathrm{sech}^2(\ )$

$$\boxed{\Delta t \simeq 64.8\,fs}$$

Autocorrelation of $\mathrm{sech}^2(t/T)$ $\qquad T \equiv \Delta t\left(\dfrac{1}{2\,\mathrm{sech}^{-1}(0.5)}\right)$

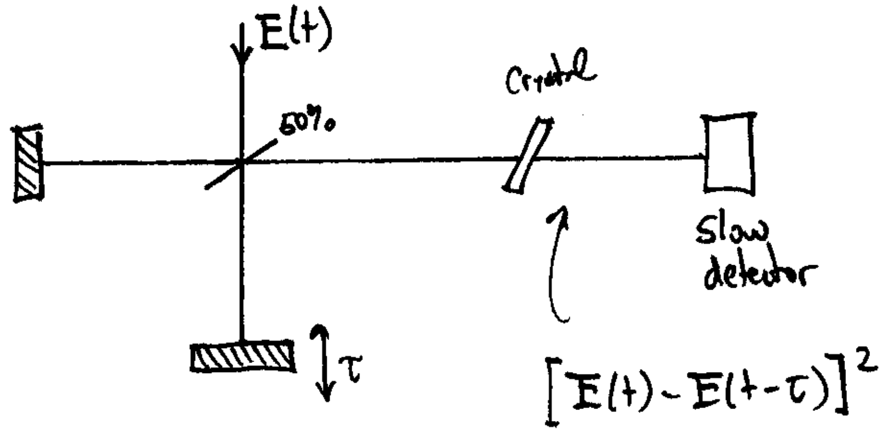
$$I_{AC}(t) = \frac{3}{\sinh^2(t/T)}\left(\left(t/T\right)\coth(t/T) - 1\right)$$

What is wrong with this?



Another method: Interferometric Autocorrelator

Michelson interferometer with SHG crystal



$$I_{IAC}(\tau) = \int_{-\infty}^{\infty} \left| [E(t) - E(t-\tau)]^2 \right|^2 dt$$

Notice the difference from the Michelson with + without the SHG crystal

without $\quad I_M(\tau) \sim \int_{-\infty}^{\infty} |E(t) - E(t-\tau)|^2 \, dt$

with $\quad I_{IAC}(\tau) \sim \int_{-\infty}^{\infty} \left| [E(t) - E(t-\tau)]^2 \right|^2 dt$

$$I_{IAC}(\tau) \equiv \int_{-\infty}^{\infty} \left| E^2(t) + E^2(t-\tau) - 2E(t)E(t-\tau) \right|^2 dt$$

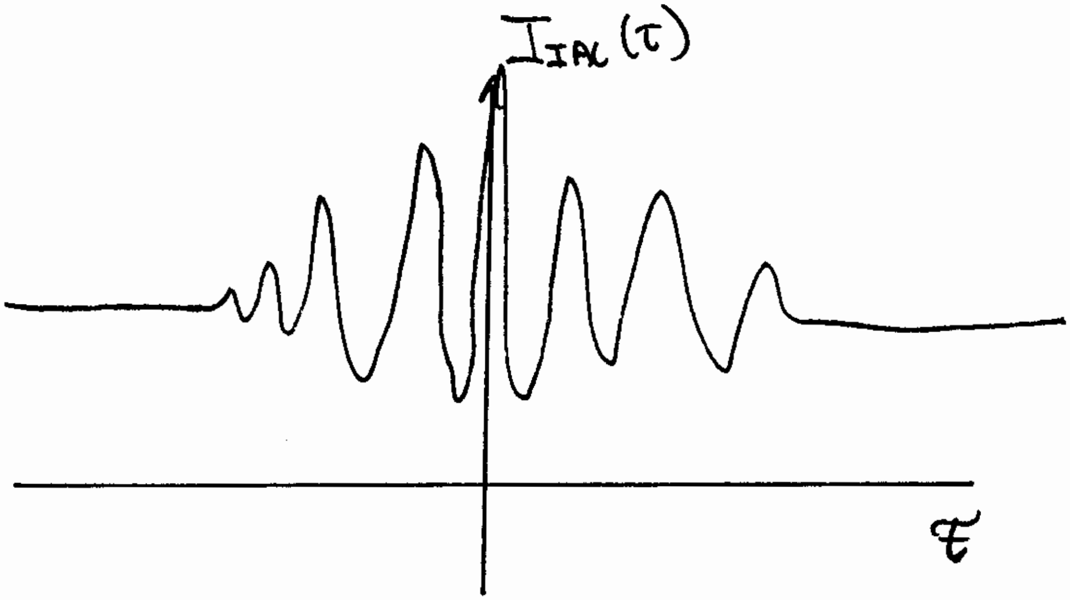
Expand this

$$I_{IAC}(\tau) = \int_{-\infty}^{\infty} \left(I^2(t) + I^2(t-\tau)\right) dt \qquad \text{(constant)}$$

$$+ 4\int_{-\infty}^{\infty} I(t)\, I(t-\tau)\, dt \qquad \text{(Intensity AC)}$$

$$+ 2\int_{-\infty}^{\infty} \left[I(t) + I(t-\tau)\right] E(t) E^*(t-\tau)\, dt + c.c. \qquad \left(\begin{array}{c}\text{Sum of intensities} \\ \text{weighted interferogram} \\ (\omega_0)\end{array}\right)$$

$$+ \int_{-\infty}^{\infty} E^2(t) E^{2*}(t-\tau)\, dt + c.c. \qquad \left(\begin{array}{c}\text{Interferogram of second} \\ \text{harmonic} \\ 2\omega_0\end{array}\right)$$



Advantages:

- Measure pulse duration.
- Qualitative test of phase modulation, Quantitative for linear chirp
- Method to get complete $E(t)$ using pulse spectrum (PICASSO)

Lecture 16 More Applications of SHG : FROG

Frequency Resolved Optical Gating

- Pulse characterization What do we wish to measure?

Intensity + **Phase**

$I(t)$ $\phi(t)$

+

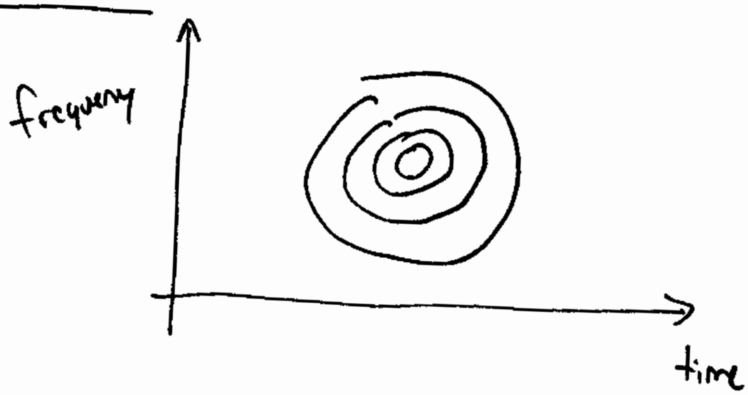
$I(\omega)$ $\phi(\omega)$

Intensity AC does not provide this information

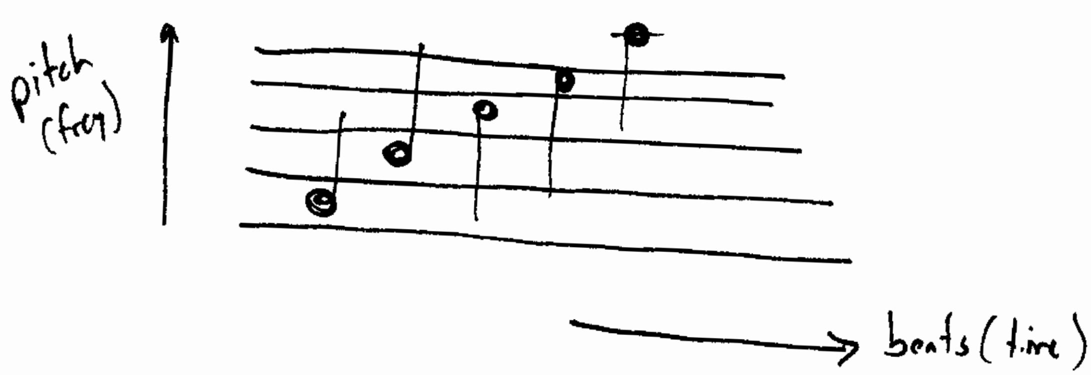
$$I_{AC}(\tau) \sim \int I(t)\, I(t-\tau)\, dt$$

Not enough data here to provide intensity + phase. Can we somehow get more data?

Time frequency Domain



Like a musical score



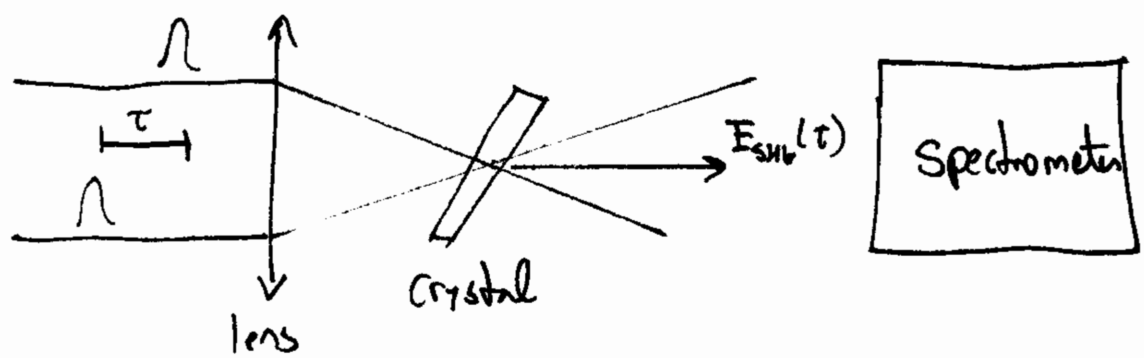
We can set up an experiment to measure the spectrogram or time-frequency representation of the pulse

$$I_{FROG}^{SHG}(\omega,\tau) = \left| \int_{-\infty}^{\infty} E(t)\,E(t-\tau)\,e^{-i\omega\tau}\,dt \right|^2$$

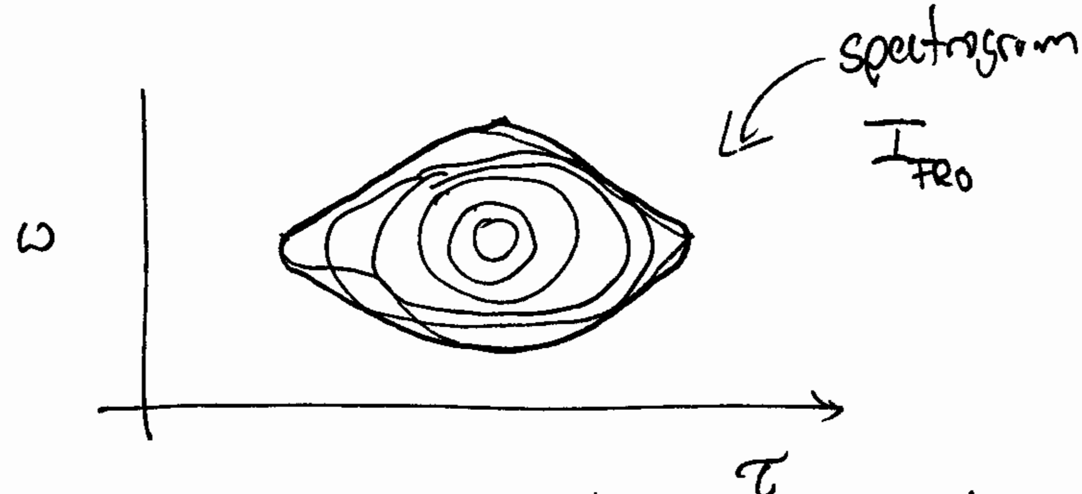
SHG FROG TRACE

Form of nonlinearity

How to do this? Use our intensity autocorrelator....



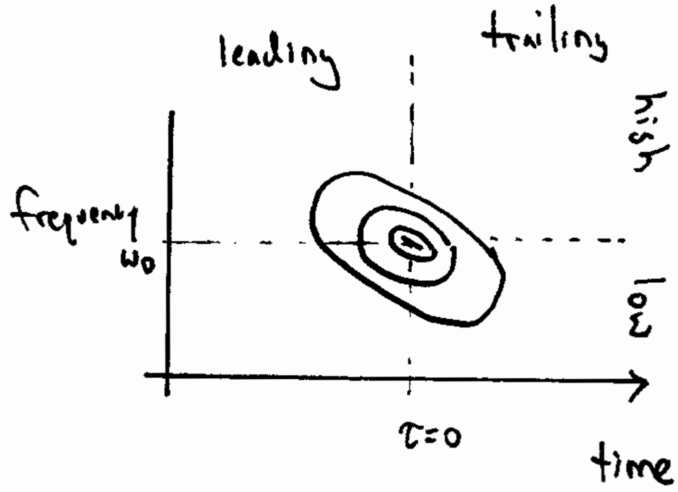
Here we spectrally resolve the SHG as a function of τ



Spectrally resolved intensity autocorrelation.

FROG traces are very intuitive

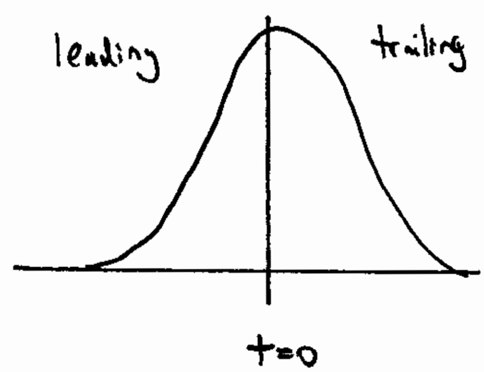
~ FROG

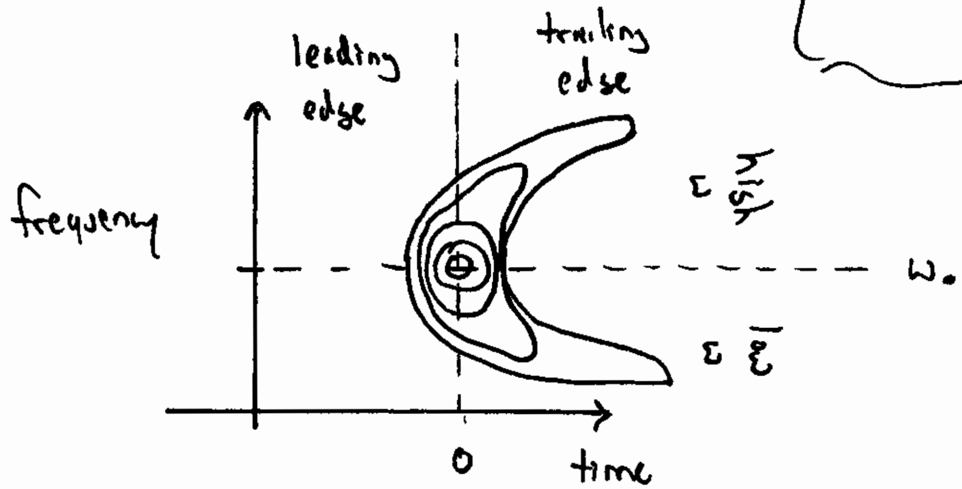


This spectrogram tells me that the higher frequency components of a pulse arrive before the lower frequency components

$\Rightarrow$ negative chirp

Aside: Pulse in time





This shape tells me that the center frequency arrives 1st the the ~~[illegible]~~ high + low

Except! SHG spectrograms are always symmetric about τ + ω!!

Not intuitive!!

SHG FROG Experimentally is a spectrally resolved intensity autocorrelation.

In general FROG can be used with other nonlinearities.

$$I_{FROG}(\tau,\omega) = \left| \int E_{sig}(t,\tau)\, \exp(-i\omega t)\, dt \right|^2$$

$$E_{sig}(t,\tau) \sim \begin{cases} E(t)\,|E(t-\tau)|^2 & \text{polarization gate} \\ E^2(t)\, E^*(t-\tau) & \text{self diffraction} \\ E(t)\, E(t-\tau) & \text{SHG} \\ E^2(t)\, E(t-\tau) & \text{Third harmonic generation} \end{cases}$$

FROG consists of two parts

1) Measurement apparatus

2) 2D Phase retreival algorithm
(FROG Algorithm)

FROG Algorithm : 2D Phase retrieval

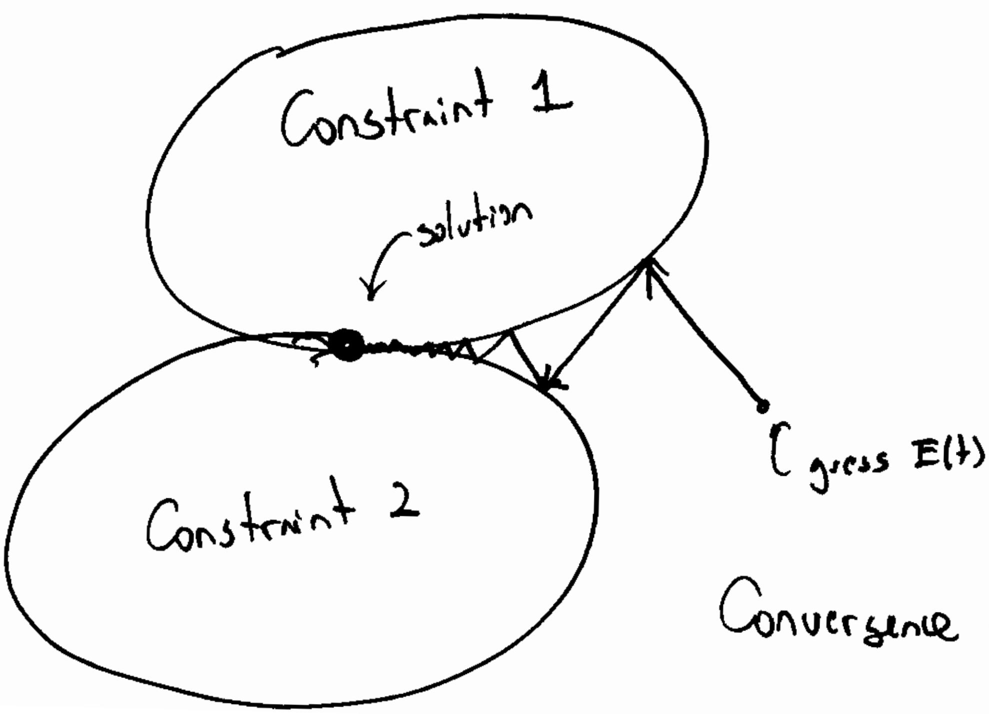
An iterative method to find Intensity + phase.
Solution must satisfy two constraints:

Constraint 1: Set of $E(t)$ that satisfy

$$E_{sig}(t,\tau) \sim E(t)\,E(t-\tau)$$

Constraint 2: Set of $E(t)$ that satisfy

$$I_{FROG}(\tau,\omega) = \left| \int E_{sig}(t,\tau)\exp(-i\omega t)\,dt \right|^2$$



Convergence is <u>not</u> guarenteed.

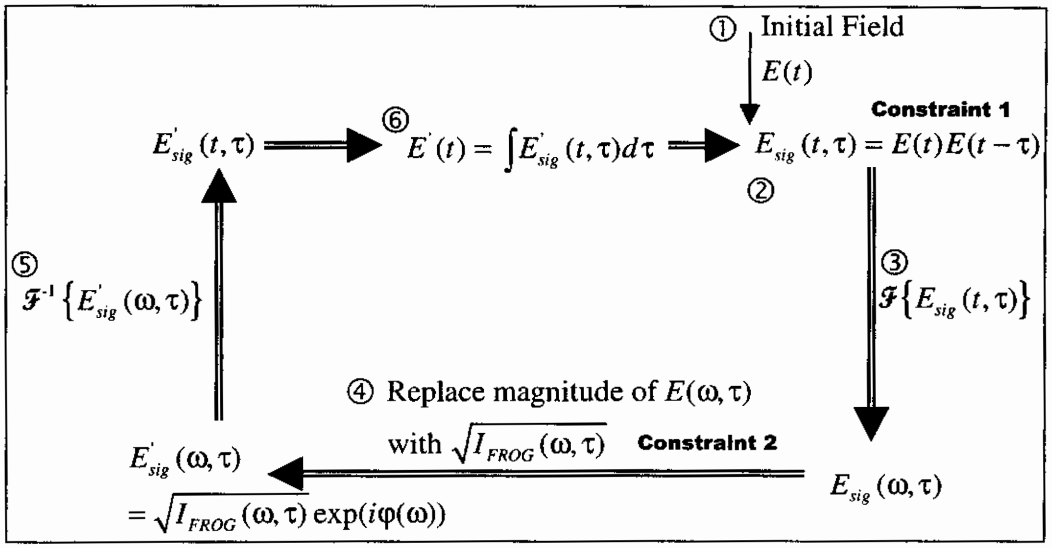


Figure 2.15 The FROG algorithm with generalized projections.

The steps of the FROG algorithm are:

① First, an initial guess electric field, $E(t)$, is generated, typically intensity noise or Gaussian profile.

② The quantity $E_{sig}(t,\tau)$ is calculated by Eq. (2.15), applying Constraint #1.

③ The quantity $E_{sig}(\omega,\tau)$ is determined using the 1D Fourier transform with respect to t.

④ In the frequency domain, the magnitude of $E_{sig}(\omega,\tau)$ is replaced by the experimental spectrogram $\sqrt{I_{FROG}(\omega,\tau)}$ while the phase is kept the same, applying Constraint #2.

⑤ The 1D inverse Fourier transform is performed to obtain $E'_{sig}(t,\tau)$.

⑥ Finally, a new $E'(t)$ is calculated from $E'_{sig}(t,\tau)$.

The new field $E'(t)$ is used as the new input to step ② and the process repeats. At the k^{th} iteration the FROG error G is calculated by

$$G = \sqrt{\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left[I_{FROG}^{(k)}(\omega_i,\tau_j) - I_{FROG}(\omega_i,\tau_j)\right]^2}\ . \tag{2.37}$$

FROG Advantages

- Provides information on intensity + phase (if you trust the algorithm)
- Has "self-checks" built in

  temporal + frequency marginals

FROG Disadvantages

- A bit of a complicated experiment
- Algorithm a "black box"
- Susceptible to systematic errors

  crystal thickness, misalignment, etc.

- SHG FROG does not give the sign of phase distortion.

Comparison of Ultrashort Pulse Functional Forms for the electric field: Gaussian and Sech

Brian Washburn version 1 9/21/07

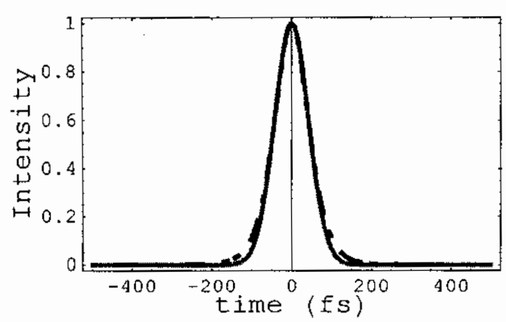
```
Off[General::spell];
<< Graphics`Graphics`
```

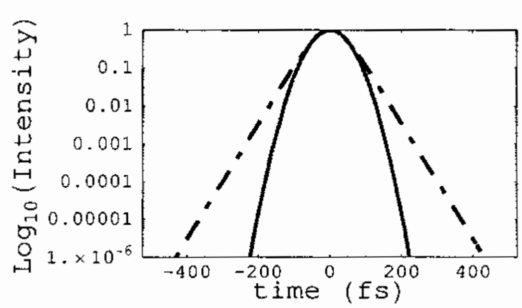
I wish to plot a sech^{2}pulse and a Gaussian pulse with the same intensity full width at half maximum and peak power.

```
Δt = 100;  Po = 1;

eg[t_] = √Po Exp[-2 Log[2] (t/Δt)^2];  es[t_] = √Po Sech[2 ArcSech[√0.5] t/Δt];
Ig[t_] = eg[t] * Conjugate[eg[t]];   Is[t_] = es[t] * Conjugate[es[t]];
```

Here I plot both pulse shapes. The Gaussian is the solid line and the sech^2 is the dotted line. The hyperbolic secant pulse has wider wings, which is quite pronounced on the Log plot.

```
p1 = Plot[{Ig[t], Is[t]}, {t, -500, 500},
   Frame -> True, PlotRange -> {All, All}, FrameLabel ->
    {StyleForm["time (fs)", FontSize → 14], StyleForm["Intensity", FontSize → 14] },
   PlotStyle → {{RGBColor[1, 0, 0], Thickness[0.01]}, {RGBColor[0, 0, 1],
      Dashing[{0.01, 0.05, 0.05, 0.05}], Thickness[0.01]}}, DisplayFunction → Identity];
p2 = LogPlot[{Ig[t], Is[t]}, {t, -500, 500}, Frame -> True, PlotRange -> {All, {10^-6, 1}},
   FrameLabel -> {StyleForm["time (fs)", FontSize → 14],
     StyleForm["Log10(Intensity)", FontSize → 14] },
   PlotStyle → {{RGBColor[1, 0, 0], Thickness[0.01]}, {RGBColor[0, 0, 1],
      Dashing[{0.01, 0.05, 0.05, 0.05}], Thickness[0.01]}}, DisplayFunction → Identity];

Show[GraphicsArray[{p1, p2}]];
```



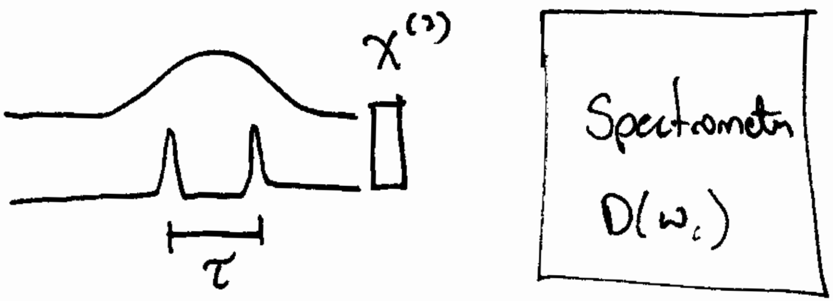


Other Pulse measurement techniques

Spectral phase interferometry for direct electric field reconstruction (SPIDER)

Idea:

Two replicas of the pulse are mixed with a highly chirped pulse in a nonlinear crystal

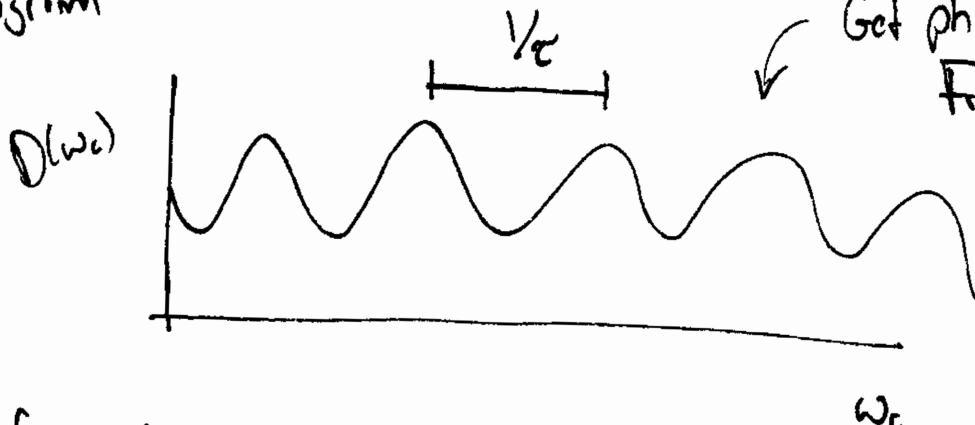


$D(\omega_c)$

$= |E_{1SHG}(\omega_c) + E_{2SHG}(\omega_c)|^2$

Get interferogram

Get phase from Fringes



Spectral interferometry + mixing in a nonlinear crystal

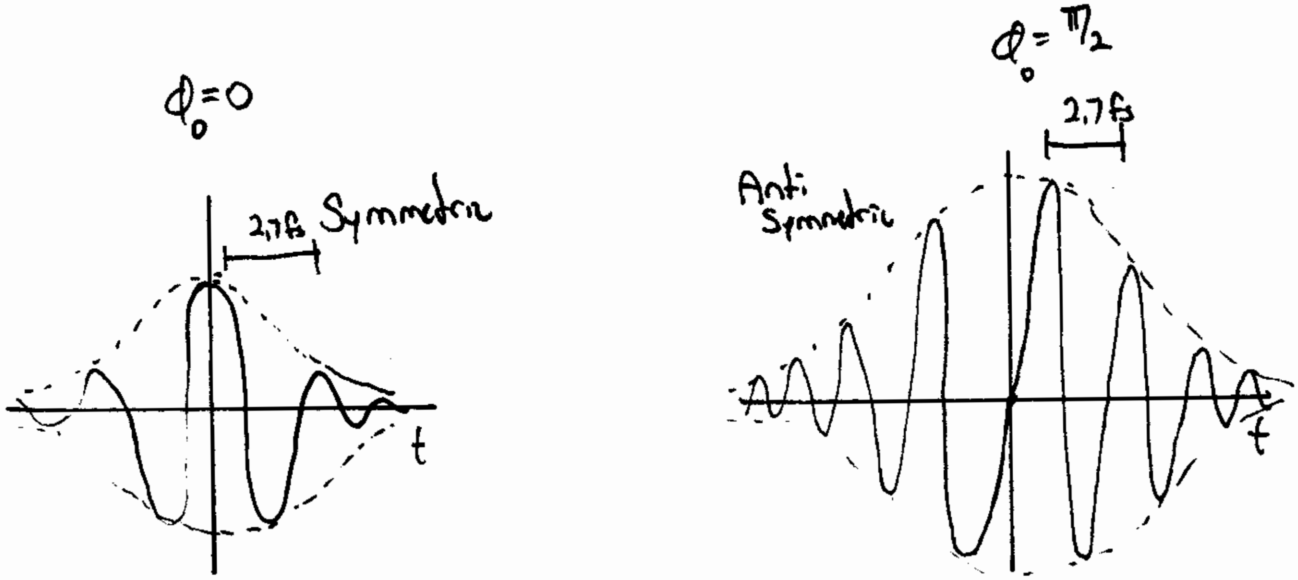
Lecture 17 Carrier Envelope Phase

$\Rightarrow$ The Frequency Comb

Consider the "real" electric field of a pulse

absolute phase

$$\mathcal{E}(t) = \mathcal{E}_0 \operatorname{sech}(t/\Delta t) \cos(\omega_0 t + \phi_0)$$

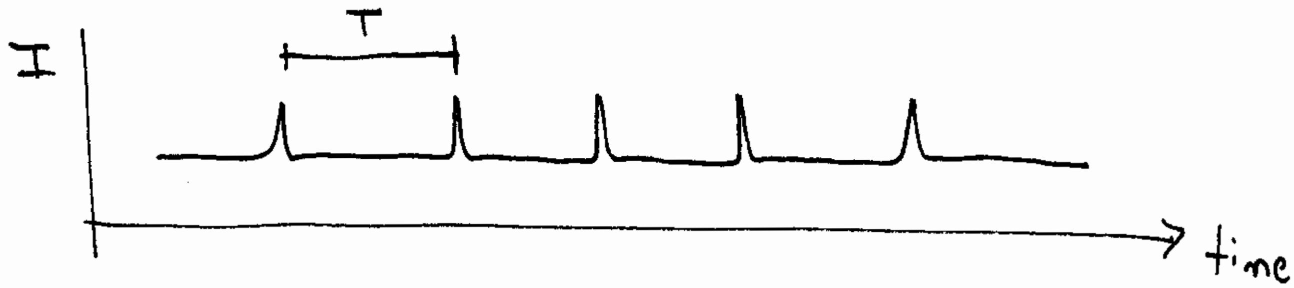


For a 800nm the period of a single cycle is 2.7 fs

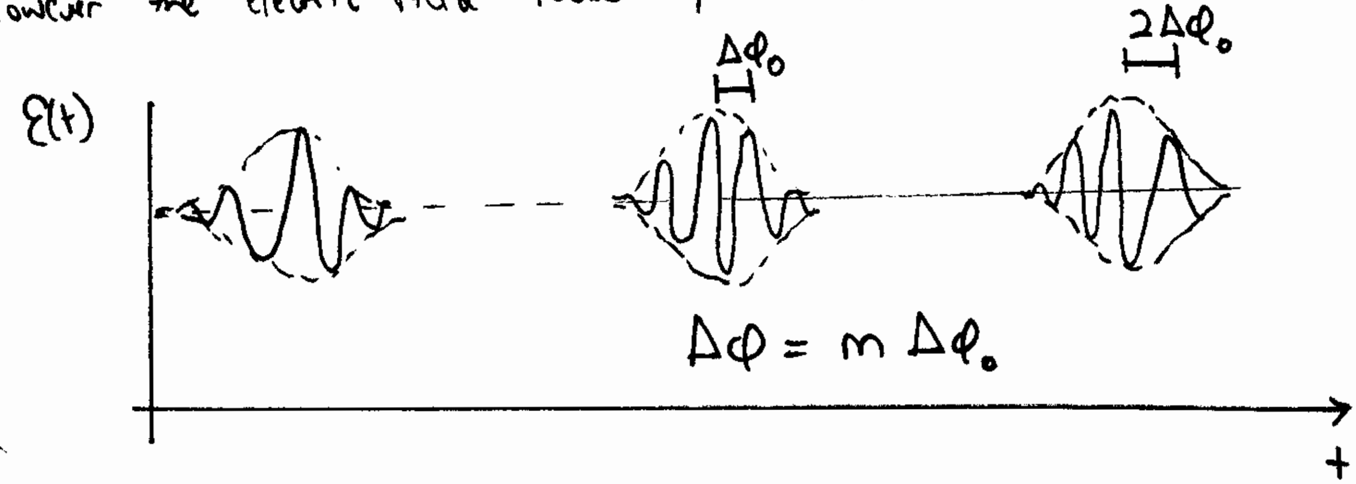
Perhaps we wish to control the absolute phase ϕ_0 of a pulse.

But we never have just <u>one</u> pulse. A modelocked laser produces many pulses, millions per second!!

We have a pulse train



However the electric field looks quite different



In the laser cavity the group + phase velocities are not the same and <u>not</u> constants over time.

$$\boxed{\Delta\phi_0 = \left(\frac{1}{v_g} - \frac{1}{v_p}\right) L\,\omega_0}$$

National Ignition Facility
1.8 MJ
3 times a day

$\Longrightarrow$ But $v_g(t)$ + $v_p(t)$ due to cavity flucuations

Each sucessive pulse will pick up an extra $\Delta\phi$ <u>even if</u> there are no cavity flucuations.

Derivation of Frequency Comb From a pulse Train

- Consider a pulsed electric field in the time domain, (sech^2)

$$\boxed{\begin{aligned} E(t) = E_0 \,\mathrm{sech}(\eta t/\Delta t)\, \exp(-i\,\varphi(t))\exp(-i(\omega_0 t + \varphi_0)) \\ \circledast \sum_{m=-\infty}^{\infty} \delta(t - m/F)\exp(-i m \Delta\varphi_0) \end{aligned}} \qquad (1)$$

Where $\Delta t \equiv$ FWHM, $\omega_0 \equiv$ carrier frequency, $\varphi_0 \equiv$ Absolute phase of 1st pulse, $\circledast \equiv$ convolution, $\Delta\varphi_0 \equiv$ CEO phase, $F \equiv$ Rep Rate, $\varphi(t) \equiv$ Temporal phase, $\delta(\,) \equiv$ Delta function, $\eta \simeq 1.763$

Fourier Transform this result

- $\mathcal{F}\{E(t)\} =$

$$\mathcal{F}\Big\{E_0\,\mathrm{sech}(\eta t/\Delta t)\exp(-i\,\varphi(t))\exp(-i(\omega_0 t + \varphi_0)) \circledast \sum \delta(t - m/F)\exp(-i m\Delta\varphi_0)\Big\}$$

Use convolution th$^{\underline{m}}$

$$\boxed{\begin{aligned} \mathcal{F}\{E(t)\} = \mathcal{F}\{E_0\,\mathrm{sech}(\eta t/\Delta t)\exp(-i\varphi(t))\exp(-i(\omega_0 t + \varphi_0))\} \\ \times\ \mathcal{F}\{\textstyle\sum \delta(t - m/F)\exp(-i m\Delta\varphi_0)\} \end{aligned}} \qquad (2)$$

To compute both FT, the shift th$^{\underline{m}}$ will be used multiple times

$$\mathcal{F}\{f(t-a)\} = \exp(i\omega a)\, f(\omega) \qquad \text{shift}$$

OR

$$\mathcal{F}\{f(t)\exp(+iat)\} = f(\omega - a) \qquad \text{Modulation}$$

(the converse th$^{\underline{m}}$ is also called the modulation th$^{\underline{m}}$)

– Derive the modulation thm

{ A phase shift in the time domain is a shift in absolute frequency in the frequency domain.

$$\mathcal{F}\{ f(t)\exp(+iat) \}$$
$$= \int f(t)\exp(+iat)\exp(-i\omega t)\, dt$$
$$= \int f(t)\exp(-i(\omega-a)t)\, dt$$
$$= f(\omega-a)$$

se the modulation thm on the 1st term of Eq 2.

$$\mathcal{F}\{ E_0 \operatorname{sech}(\eta t/\Delta t)\exp(-i\phi(t))\exp(-i\omega_0 t - i\phi_0) \}$$

$$= \exp(-i\phi_0)\, \mathcal{F}\{ \overbrace{E_0 \operatorname{sech}(\eta t/\Delta t)\exp(i\phi(t))}^{\text{complex envelope}} \overbrace{\exp(-i\omega_0 t)}^{\text{Carrier}} \}$$

$$= \boxed{\exp(-i\phi_0)\; E(\omega-\omega_0)} \quad \text{or} \quad \boxed{\exp(i\phi_0)\operatorname{sech}((\omega-\omega_0/\Delta\omega))\exp(i\phi(\omega))} \quad (3)$$

where $E(\omega-\omega_0)$ is the spectral representation of the electric field centered at ω_0

i.e. $E(\omega-\omega_0) = \sqrt{I(\omega-\omega_0)}\exp(i\phi(\omega))$

$$\underset{\text{spectrum}}{\uparrow} \qquad \underset{\text{spectral phase}}{\uparrow}$$

se Modulation thm on the 2nd term of Eq 2.

$$\mathcal{F}\{ \Sigma\, \delta(t - m/F)\exp(-im\Delta\phi_0) \}$$

– The FT of a comb function is another comb function, with different separation

$$\mathcal{F}\{ \sum_m \delta(t - m/F) \} = \sum_j \delta(\nu - jF) = \sum_j \delta(\omega - j2\pi F)$$

Using the modulation thm, the exponential term will shift the comb by $\Delta\phi_0/2\pi$, thus

$$\mathcal{F}\left\{ \sum_m \delta(t - m/F) \exp(-im\Delta\phi_0) \right\}$$

$$= \sum_j \delta\left(\omega - j2\pi F + \Delta\phi_0/2\pi\right)$$

$$= \boxed{\sum_j \delta\left(\omega - j2\pi F + \underset{\uparrow\ \text{CEO frequency}}{\delta_0}\right)} \qquad (4)$$

Combining (3) + (4)

$$\boxed{E(\omega) = \tilde{E}_0 \operatorname{sech}\left(\eta(\omega-\omega_0)/\Delta\omega\right) \exp(i\phi(\omega)) \left[\sum_j \delta(\omega - j2\pi F + \delta_0)\right] \exp(i\phi_0)}$$

$\Delta\omega \equiv$ spectral FWHM, $\delta_0 \equiv$ CEO frequency

$\phi(\omega) \equiv$ spectral phase, $\tilde{E}_0 \equiv$ spectral magnitude.

$\eta = 1.763$

The goal is to make every pulse in the pulse train to look identical $\Rightarrow$ Same form of $\mathcal{E}(t)$.

Why? $\Rightarrow$ Some experiments with (< 20 fs pulses @ 800nm) short pulses depend on the peak time of the peak of $E(t)$.

How to write the electric field of a pulse train?

$$\boxed{\begin{aligned}\underset{\sim}{E}(t) = \Big[& E_0 \operatorname{sech}(t/\Delta t \gamma) \exp(-i\phi(t)) \exp(-i(\omega_0 t + \phi_0)) \\ & \circledast \sum_{m=-\infty}^{\infty} \delta(t - m/F) \exp(-i m \Delta\phi_0) \Big] \, sq(T_N)\end{aligned}}$$

where

$F \equiv$ frep rate $= 1/T$

$\circledast \equiv$ convolution

$sq() =$ unit square function

$\gamma \simeq 2 \operatorname{sech}^{-1}\sqrt{1/2} \simeq 1.763$

$\omega_0 \equiv$ carrier frequency

$\delta \equiv$ delta function

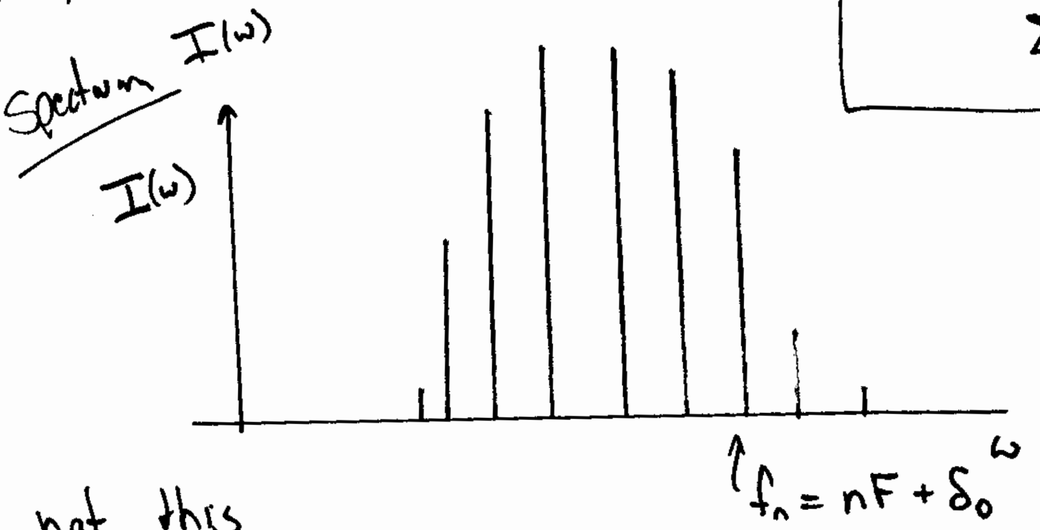
$T_N \equiv$ time the laser is on

The term $\boxed{\sum_{m=-\infty}^{\infty} \delta(t - m/F) \exp(-i m \Delta\phi_0)}$

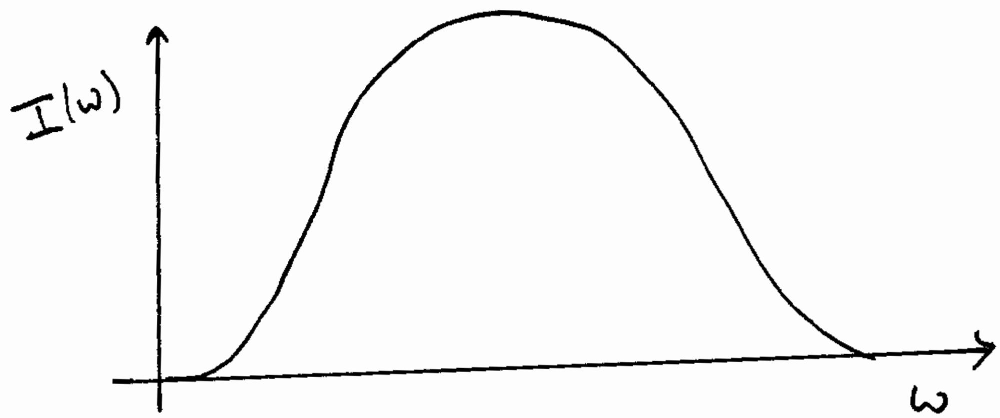
is an array of delta functions each with phase $m\Delta\phi_0$

The end result is the Fourier transform is the frequency comb.

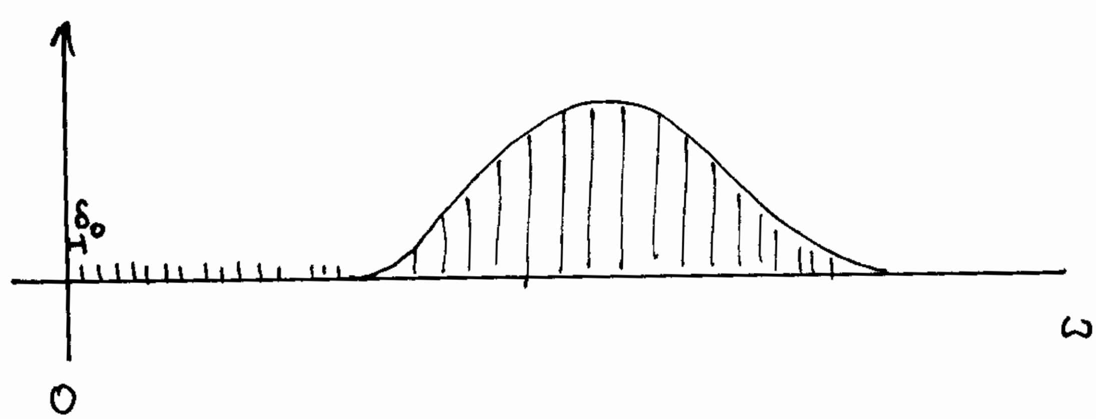
$$\boxed{E(\omega) \simeq A(\omega)\, e^{-i\phi_0} \sum_j \delta(\omega - j2\pi F + \delta_0)}$$



not this



The non zero $\Delta\phi$ lets the extended comb not to fall at $\omega = 0$



The rate of change of $\Delta\phi$ depends on δ_0.

If $\delta_0 = 0$ then $\partial_t \Delta\phi = 0$

To get every pulse in the train to have the same electric field we need to detect + fix δ_0

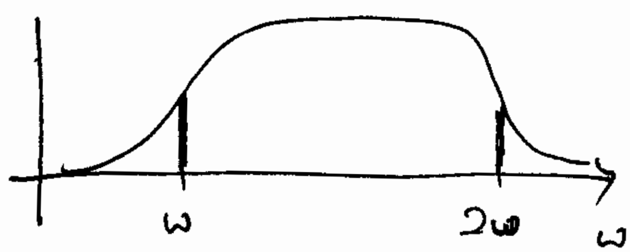
$$\boxed{\delta_0 = \frac{1}{2\pi} F \Delta\phi} \qquad (\delta_0, f_0, f_{ceo})$$

To do thing correct we need to detect + fix F as well.

How to detect F? $\Rightarrow$ Easy, fast photodiode

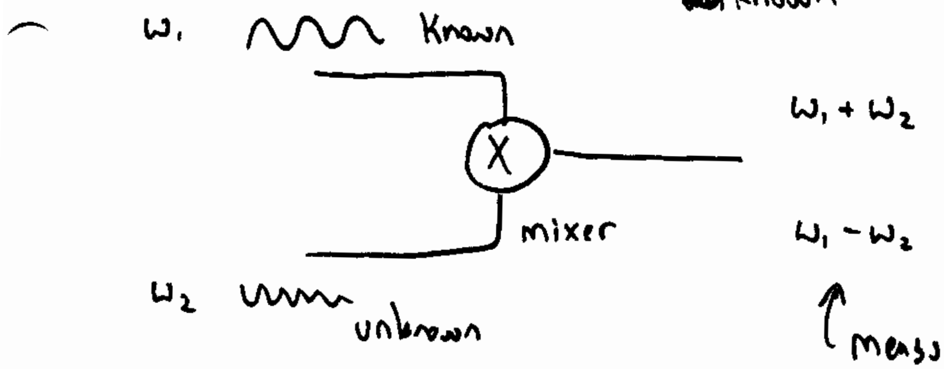
How to detect δ_0 $\Rightarrow$ Hard, use SHG with an octave spanning bandwidth.

Octave spanning $\Rightarrow$ spectrum covers ω_0 + $2\omega_0$

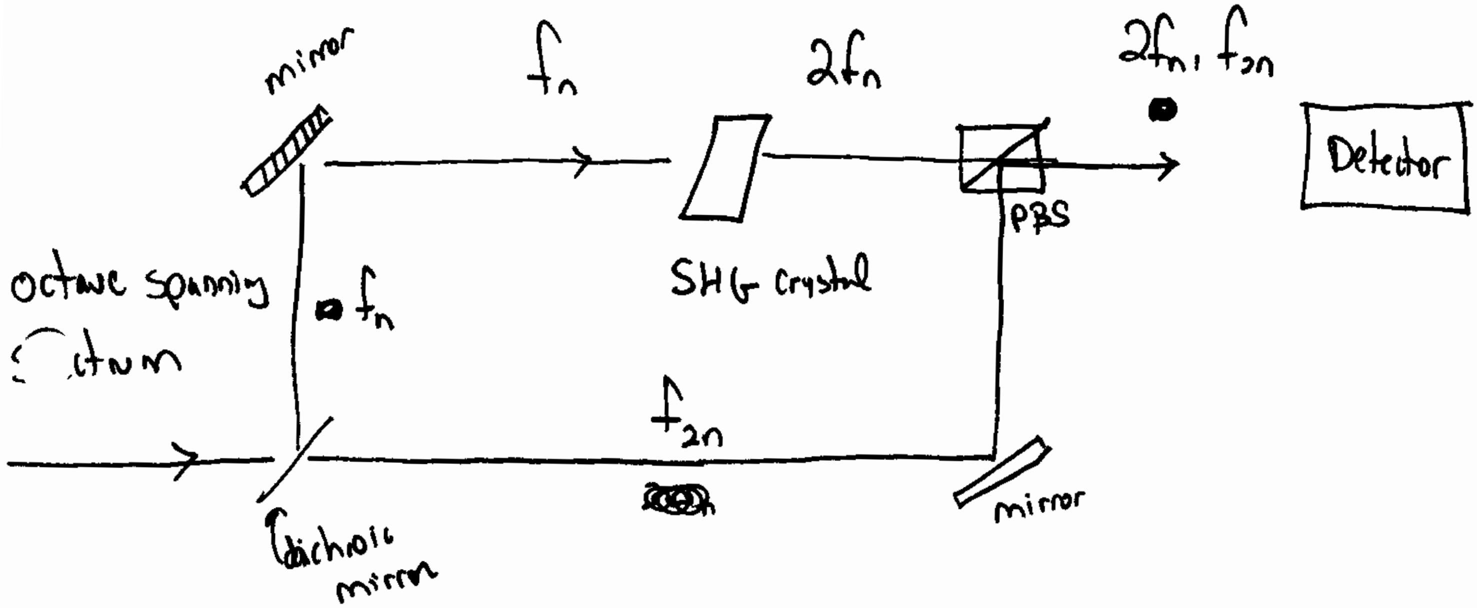


Use SHG generation of ω and beat against 2ω portion of the spectrum.

Hetrodyne detection : A method to measure a frequeny with an known frequenuy.



Beat notes



$$\boxed{f_n = nF + \delta_0}$$

Thru SHG we get $\Rightarrow$ $$\boxed{2f_n = 2nF + 2\delta_0}$$

But $\Rightarrow$ $$\boxed{f_{2n} = 2nF + \delta_0}$$

We hetrodyne $2f_n$ + f_{2n}

$$2f_n - f_{2n} = 2nF + 2\delta_0 - 2nF - \delta_0 = \boxed{\delta_0}\ !!$$

So we can detect both F + δ_0. Can we control them?

Control $F \Rightarrow$ Easy, just change the cavity length of the mode locked laser.

Control $\delta_0 \Rightarrow$ Difficult, we need to change the group velocity independent of the phase velocity

$$\boxed{\Delta\phi_0 = \left(\frac{1}{v_g} - \frac{1}{v_p}\right) L\, \omega_c}$$

Many ways to do this

1) Pump power modulation
2) Mirror tilt in the Fourier plane of a prism pair

Note that the detection of δ_0 depends on this octave spanning spectrum. Typically lasers do not produce an octave spanning spectrum, we need to do something to the pulse to broaden its spectrum. Thus we need another nonlinear effect to produce the octave spanning spectrum. That nonlinear effect will be a <u>third</u> <u>order</u> <u>nonlinearity</u>.

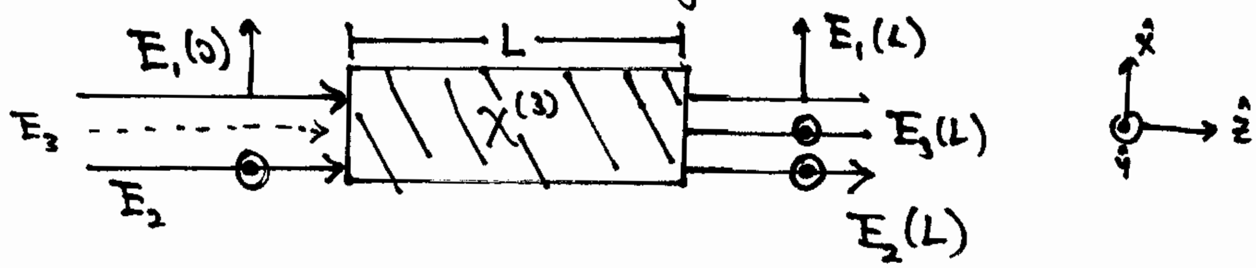
Lecture 18 Four wave mixing + the intensity-dependent index of refraction

Third order nonlinearities involve the interaction of three fields E to generate a nonlinear polarization

$$P_{NL} = \chi^{(3)} E_1(\omega_1) E_2(\omega_2) E_3(\omega_3)$$

We can, like for $\chi^{(2)}$ effects, write down the Maxwell's wave equation and solve for the new electric field. In general this process is known as four wave mixing (FWM) since <u>three</u> input fields induce a nonlinear polarization with induces a <u>fourth</u> field E.

To see how this works consider the case of third harmonic generation:

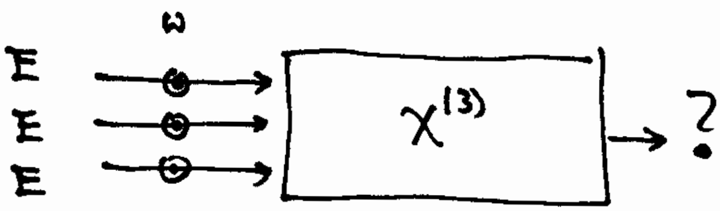


Inputs $E_1(0)$ + $E_2(0)$ which have orthogonal polarizations which both have frequency ω. The generated field $E_3(L)$ has frequency 3ω.

Write down the electric fields

Before we do this....

- Let's consider a simplier case of a FWM of three field with same polarization + frequency



$$\omega_1 = \omega_2 = \omega_3 = \omega$$
$$\bar{k}_1 = \bar{k}_2 = \bar{k}_3 = \bar{k}$$

The name for this process is completely degenerate four wave mixing. This will involve the tensor element

$$\chi^{(3)}_{xxxx}(\omega; \omega, -\omega, \omega)$$

Completely degenerate FWM leads to a change of the index of refraction that is dependent on the intensity I. This is also called single field degenerate FWM. The field changes the index of refraction it experiences!! This self modulation leads to two well known effects

- Self phase modulation (SPM)
- Self-Focusing

Let's derive an approximate equation that show the variation of the index by the intensity.

Completely degenerate FWM

$$\bar{k}_1 = \bar{k}_2 = \bar{k}_3 \qquad E_1 = E_2 = E_3$$

$$\omega_1 = \omega_2 = \omega_3$$

Remember:

$\mu = \cancel{K_m \mu_0} = K_m \mu_0$

$\varepsilon = \cancel{K_e \epsilon_0} = K_e \epsilon_0$

$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \quad v = \frac{1}{\sqrt{\mu \varepsilon}} = c/n$

$$P_{NL} \cong \epsilon_0 \chi^{(3)}_{xxxx} |E|^2 E$$

Into wave eq

$$P = P_L + P_{NL} = \epsilon_0 \left[\chi^{(1)} E + \chi^{(3)} |E|^2 E \right]$$

Assume $\partial_t^2 |E|^2 E \simeq |E|^2 \partial_t^2 E$ (SVEA)

$$\partial_z^2 E - \frac{1}{c^2} \partial_t^2 E = \mu_0 \epsilon_0 \left[\chi^{(1)} \cancel{E} + \chi^{(3)} |E|^2 \cancel{E} \right] \partial_t^2 E$$

OR

$$\boxed{\partial_z^2 E - \frac{\left[1 + \chi^{(1)} + \chi^{(3)} |E|^2 \right]}{c^2} \partial_t^2 E = 0}$$

Wave eq for a linear material

$$\partial_z^2 E - \frac{n^2}{c^2} \partial_t^2 E = 0$$

OR

$$\partial_z^2 E - \mu \varepsilon \partial_t^2 E = 0$$

Looks like

$$\partial_z^2 E - \frac{n^2}{c} \partial_t^2 E = 0 \quad \text{Where } n = \sqrt{1 + \chi^{(1)} + \chi^{(3)} |E|^2}$$

Remember $n = \sqrt{1 + \chi^{(1)}}$

Usually the linear index of refraction looks like

$$n_0 = \sqrt{1 + \chi^{(1)}}$$

$$n = n_0 \sqrt{1 + \chi^{(3)} |E|^2 / n_0^2}$$

assume the nonlinear term is $\ll n_0$

$$n \simeq n_0 \left(1 + \chi^{(3)} |E|^2 / 2 n_0^2 \right)$$

$$n \simeq n_0 + \chi^{(3)} |E|^2 / 2 n_0$$

$$\boxed{n \cong n_0 + n_2 I} \qquad I \sim |E|^2$$

nonlinear index of refraction

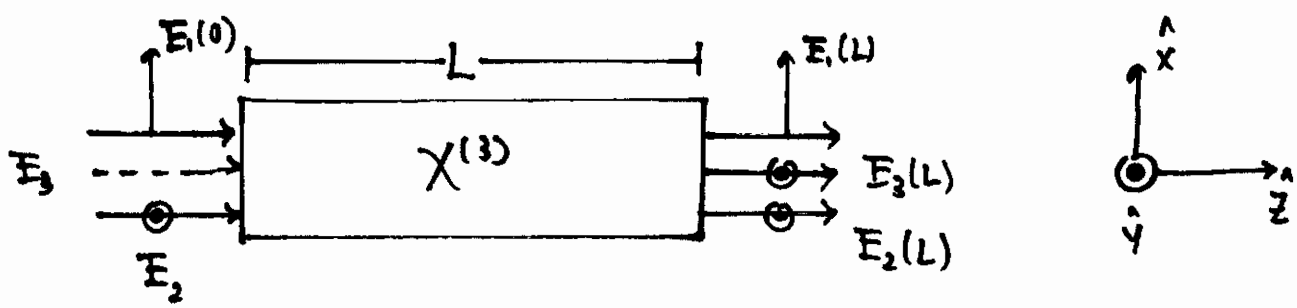
Really $\boxed{n_2 = \dfrac{3 \operatorname{Re}\{\chi^{(3)}_{xxxx}\}}{8 n_0}}$

Notice we never talked about <u>phase</u> <u>matching</u>.

Since this process is completely degenerate, it is automatically phase matched. In general FWM processes must be phase matched.

Third harmonic generation

We expect this to be a phase matched process. We will go back to the more general situation



Inputs $E_1(0)$ + $E_2(0)$ have frequency ω and are orthogonally polarized. The field E_3 has frequency 3ω.

Write down the field in real instantaneous forms

$$\vec{\mathcal{E}}_1 = \left[\tfrac{1}{2}E_{01}\ \exp(ik(\omega)z)\exp(-i\omega t) + c.c.\right]\hat{x}$$

$$\vec{\mathcal{E}}_2 = \left[\tfrac{1}{2}E_{02}\ \exp(ik(\omega)z)\exp(-i\omega t) + c.c.\right]\hat{y}$$

and

$$\vec{\mathcal{E}}_3 = \left[\tfrac{1}{2}E_{03}\ \exp(ik(3\omega)z)\exp(-i3\omega t) + c.c.\right]\hat{y}$$

where $\quad k(\omega) = \dfrac{n(\omega)\,\omega}{c} \quad + \quad k(3\omega) = \dfrac{n(3\omega)\,3\omega}{c}$

Make the assumption that $E_{02} < E_{01}$ + $E_{03} < E_{01} \Rightarrow$ Strong pump approximation

Find the nonlinear polarization

$$\vec{P}_{NL} = \epsilon_0 \chi^{(3)} (\vec{\mathcal{E}}_1 + \vec{\mathcal{E}}_2 + \vec{\mathcal{E}}_3)^3$$

$$= \epsilon_0 \chi^{(3)} \left[\mathcal{E}_1^3 + \mathcal{E}_2^3 + \mathcal{E}_3^3 + 6\,\mathcal{E}_1\mathcal{E}_2\mathcal{E}_3 + 3\,\mathcal{E}_1^2\mathcal{E}_2 + 3\,\mathcal{E}_1\mathcal{E}_2^2 \right.$$

$$\left. + 3\,\mathcal{E}_1^2\mathcal{E}_2 + 3\,\mathcal{E}_2^2\mathcal{E}_3 + 3\,\mathcal{E}_1\mathcal{E}_3^2 + 3\,\mathcal{E}_2\mathcal{E}_3^2 \right]$$

What a mess! To keep things simple since $E_{02} < E_{01}$ + $E_{03} < E_{01}$ we will only keep terms of the produce of $E_{01} \times E_{02}$ or $E_{01} * E_{03}$

The resulting P_{NL} terms will oscillate at ω + 3ω

$$P_{NL} = P_{NL}^{\omega} + P_{NL}^{3\omega} \left\langle \begin{array}{l} P_{NL}^{\omega}(\omega) = P_{NL}^{\omega}\, e^{-i\omega t} + P_{NL}^{\omega *}\, e^{+i\omega t} \\ P_{NL}^{3\omega}(\;\; = P_{NL}^{3\omega}\, e^{-i3\omega t} + P_{NL}^{3\omega}\, e^{+i\omega t} \end{array} \right.$$

Where

$$P_{NL}^{\omega} = \frac{3}{4}\epsilon_0 \chi_{yxxy}(\omega;\omega,-\omega,\omega)\left[2|E_{01}|^2E_{02} + E_{01}^2E_{02}^*\right]e^{ik(\omega)z}$$

$$+ \frac{3}{4}\epsilon_0 \chi_{yxxy}(\omega;-\omega-\omega,3\omega)(E_{01}^*)^2E_{03} \quad \exp(+i(k(3\omega)-2k(\omega))z)$$

$$P_{NL}^{3\omega} = \frac{6}{4}\epsilon_0 \chi_{yxxy}(3\omega;\omega,-\omega,3\omega)|E_{01}|^2E_{03}\,e^{ik(3\omega)z}$$

$$+ \frac{3}{4}\epsilon_0 \chi_{yxxy}(3\omega;\omega,\omega,\omega)E_{01}^2E_{02} \quad \exp(+i3k(\omega)z)$$

The wave eq gives

$$\boxed{\partial_z^2(\bar{\mathcal{E}}_1+\bar{\mathcal{E}}_2+\bar{\mathcal{E}}_3) - \mu_0\epsilon_0 n^2\partial_t^2(\bar{\mathcal{E}}_1+\bar{\mathcal{E}}_2+\bar{\mathcal{E}}_3) = \mu_0\partial_t^2\vec{\mathcal{P}}_{NL}}$$

Here we will invoke the slowly varying envelope approximation again

$$|k\,\partial_z E_{0i}| \gg |\partial_z^2 E_{0i}|$$

We will then get

Remember the slowly varying envelope approximation

$$|k\,\partial_z E| \gg |\partial_z^2 E|$$

It can be rewritten as

$$\left|\lambda\,\partial_z\left(\partial_z E_{0i}\right)\right| \ll \left|\partial_z E_{0i}\right|$$

The change in the slope of the field envelope over distance λ is much less than the magnitude of the slope itself.

This expression is valid for pulses except where

$$\boxed{\Delta t \simeq \frac{2\pi}{\omega_0}} \Longrightarrow \text{at } 800\text{ nm} \quad \frac{2\pi}{\omega_0} \simeq 2\pi\frac{\lambda_0}{2\pi c} = \frac{\lambda}{c} \simeq \frac{800\text{ nm}}{300\ \frac{\text{nm}}{\text{fs}}}$$

$$= 2.67\ \text{fs}$$

$$\left[(k(\omega))^2 (E_{01}+E_{02}) + i\,2k(\omega)\,\partial_z E_{02}\right] \exp(ik(\omega)z - i\omega t)$$

$$+ \left(k^2(3\omega) E_{03} + i\,2k(3\omega)\,\partial_z E_{03}\right) \exp(i\,k(3\omega)z - i3\omega t)$$

$$- \mu_0 \epsilon_0 n^2(\omega)\,\omega^2 (E_{01}+E_{02}) \exp(-i\omega t + k(\omega)z)$$

$$- \mu_0 \epsilon_0 n^2(3\omega)\,(3\omega)^2 (E_{03}) \exp(-i3\omega t + k(3\omega)z)$$

$$= \mu_0 \omega^2 P_{NL}^{\omega} \exp(-i\omega t) + \mu_0 (3\omega)^2 P_{NL}^{3\omega} \exp(-i3\omega t)$$

By separating the above eq in ω & 3ω, we get two coupled DEs

$$\frac{dE_{02}}{dz} = \frac{-i\,3\omega}{8\,n(\omega)c}\Big[\chi_{yxxy}(\omega;\omega,-\omega,\omega)\left[2|E_{01}|^2 E_{02} + E_{01}^2 E_{02}^*\right]$$
$$+ \chi_{yxxy}(\omega;-\omega,\omega,3\omega)\,E_{01}^{*2} E_{03}\, e^{-i\Delta k z}\Big]$$

$$\frac{dE_{03}}{dz} = \frac{-i\,3\omega}{8\,n(3\omega)c}\Big[6\chi_{yxxy}(3\omega;\omega,-\omega,3\omega)|E_{01}|^2 E_{03}$$
$$+ 3\chi_{yxxy}(3\omega;\omega,\omega,\omega)\,E_{01}^2 E_{02}\, e^{+i\Delta k z}\Big]$$

$$\frac{dE_{01}}{dz} = 0$$

where $\Delta k = 3k(\omega) - k(3\omega)$

If we also assume

$$\partial_z E_{o2} \simeq 0$$

look familiar

$$\frac{d E_{o3}}{dz} = \left(\qquad \right) \exp(+i \Delta k z)$$

or

$$E_{o3}(L) = \frac{-i\, 9\omega \chi_{yxxy}}{8\, n(3\omega) c} E_{o1}^2 E_{o2} L \frac{\sin(\Delta k L/2)}{(\Delta k L/2)} e^{+i\Delta k L/2}$$

$$I_{o3} = 2 \varepsilon_0 n_3 c |E_{o3}|^2 \sim \operatorname{sinc}^2(\Delta k L/2)\, L^2 \cancel{E_{o1}}$$

phase match process

where $\Delta k = 3k(\omega) - k(3\omega)$

Two important features

1) $I_{o3} \sim L^2$

2) $\operatorname{sinc}^2(\ )$

just like $\chi^{(2)}$!

Nonlinear Index of Refraction

All three inputs copolarized $\Rightarrow \chi^{(3)}_{xxxx}$

$$P_{NL}^{\omega} = \frac{3}{4}\epsilon_0 \chi_{xxxx}(\omega; \omega, -\omega, \omega)|E_0|^2 E_{01}\, e^{i k(\omega) z}$$

Put into wave eq.

$$-i\, 2k \frac{dE_{01}}{dz} - k^2 E_{01} + \omega^2 \mu_0 \epsilon_0 n_0^2 E_{01} = -\frac{3}{4}\omega^2 \mu_0 \epsilon_0 \chi_{xxxx} |E_{01}|^2 E_{01}$$

assume $\frac{dE_{01}}{dz} = 0$ Solve for k^2

$$k^2 = \omega^2 \mu_0 \epsilon_0 \left(n_0^2 + \frac{3}{4}\chi_{xxxx} |E_{01}|^2 \right)$$

Rewrite as . . . .

$$n = n_0 \left(1 + \frac{3\chi_{xxxx}}{4 n_0^2} |E_{01}|^2 \right)^{1/2}$$

$$n \simeq n_0 + n_2 I_{01}$$

where $\boxed{n_2 = \frac{3\chi^{(3)}_{xxxx}}{8 n_0}}$

Lecture 19 More on Self phase modulation + $\chi^{(3)}$

Many materials give rise to third order effects

Only materials that have a strong electronic component will offer a fast response

Nonlinear Response $\Rightarrow$ $R(t) = \left(\underset{\substack{\uparrow \\ \text{fast}}}{\delta(t)} + \underset{\substack{\uparrow \\ \text{slow}}}{h_R(t)} \right)$

SPM, FWM Raman effects

"Fast" $\Rightarrow$ Electronic contributions : $\sim 10^{-15}$ s

"Slow" $\Rightarrow$ Thermal / vibrational contributions : $\sim 10^{-12}$ s

Third order materials

- Gases
  - Noble gases
  - CS_2
- Liquids
  - Tea
- Isotropic Solid
  - glasses

Leads to nonlinear index of Refraction

$$\boxed{n_2 = \frac{3}{8 n_0} \operatorname{Re}\left\{ \chi^{(3)}_{xxxx} \right\}}$$

FWM $\Rightarrow$ Due to fast response of the $\chi^{(3)}$ medium

Intensity dependent index of Refraction

From Last time we derived:

$$\boxed{n = n_0 + n_2 |E|^2}$$

where $n_2 = \frac{3}{8n_0} \mathrm{Re}\{\chi^{(3)}_{xxxx}\}$

n_2 has units of m^2/V^2

Rewrite this equation in terms of Intensity where

$$I = 2^{-1} \varepsilon_0 c n |E|^2$$

So

$$\boxed{n = n_0 + n_2^I I}$$

where $n_2^I = \frac{2n_2}{\varepsilon_0 c n_0}$ in units of m^2/W

common notation replaces $\underline{\underline{n_2^I}}$ with $\underline{\underline{n_2}}$ (Redefine)

For fused silica

$$\boxed{n_2 = 3 \times 10^{-20}\ m^2/W}$$

$$n_2 \equiv \left(\frac{2}{\varepsilon_0 c n_0}\right)\left(\frac{3}{8n_0} \chi^{(3)}_{xxxx}\right)$$

Self Phase Modulation $\Rightarrow$ Completely degenerate FWM

$$\omega = \omega - \omega + \omega$$

$$P_{NL} = \frac{3}{4}\epsilon_0 \chi^{(3)}_{xxxx}(\omega;\, \omega, -\omega, \omega)\, E E^* E$$

and

$$\mathcal{P}_{NL} = \frac{1}{2}\left(P_{NL}\, e^{-i\omega t}\right) + c.c.$$

$$\mathcal{E} = \frac{1}{2} E\, e^{-i\omega_0 t} + c.c.$$

Sub into wave eq to find $E(z,t)$ generated

$$\partial_z^2 \vec{\mathcal{E}} - \mu_0 \epsilon_0 \partial_t^2 \mathcal{E} = \mu_0 \partial_t^2 \mathcal{P}$$

If we use the slowly varying envelope approximation

$$|k\, \partial_z \mathcal{E}| >> |\partial_z^2 \mathcal{E}|$$

we can rewrite the wave eq as

$$\partial_z E = i\, \frac{3\mu_0 \epsilon_0 c\, \omega}{8 n_0}\, \chi^{(3)}_{xxxx} |E|^2 E$$

Define ~~$E(t,z) = \sqrt{\frac{P_0}{\pi r^2}}\, U(t,z) \sqrt{\frac{2}{\epsilon_0 c n}}$~~

$U(z,t) \equiv$ normalized function

$$\boxed{E(z,t) = \sqrt{\frac{P_0}{\pi r^2}} \sqrt{\frac{2}{\epsilon_0 c n_0}}\; U(z,t)}$$

So we have (Remember $\epsilon_0 \mu_0 = \frac{1}{c^2}$)

$$\cancel{\sqrt{\frac{P_0}{\pi r^2}}}\ \partial_z U = i\,\frac{2 \cdot 3\mu_0 \epsilon_0 c\, \omega}{c n_0\, 8 n_0 \epsilon_0}\, \chi^{(3)}_{xxxx}\, |U|^2 U\, \cancel{\sqrt{\frac{P_0}{\pi r^2}}} \left(\frac{P_0}{\pi r^2}\right)$$

So $$\partial_z U = \frac{2}{\epsilon_0 c n_0}\left[\left(\frac{3\chi^{(3)}}{8 n_0}\right)\frac{\omega}{\pi r^2 c}\right] P_0\, |U(t)|^2\, U(t)$$

But $n_2 = \left(\frac{3\chi}{8 n_0}\right)\left(\frac{2}{\epsilon_0 c n}\right)$ Define the effective nonlinearity γ

in m^2/N

$$\gamma \equiv \frac{n_2 \omega}{(\pi r^2) c}$$

Thus
$$\boxed{\partial_z U(t,z) = i \gamma P_0\, |U(t,z)|^2\, U(t,z)}$$

Solution
$$\boxed{\begin{aligned} U(z,t) &= U(0,t)\exp(i\phi_{NL}(z,t)) \\ \phi_{NL}(z,t) &= \gamma P_0\, z\, |U(0,t)|^2 \end{aligned}}$$

Define nonlinear length $L_{NL} = \frac{1}{\gamma P_0}$

So $$\boxed{\phi_{NL}(z,t) = z/L_{NL}\, |U(0,t)|^2}$$

- Nonlinear length

$$\boxed{L_{NL} = \frac{1}{\gamma P_0}}$$ in units of meters

distance to travel to experience 1 radian nonlinear phase shift.

- Units for n_2 : Technically m^2/V^2 $\boxed{\chi^{(3)} \rightarrow m^2/V^2}$ General units $\left(\chi^{(n)} \rightarrow \frac{m^{n-1}}{V^{n-1}}\right)$

Common to quote n_2 in m^2/W $\boxed{n_2 \rightarrow \frac{2n_2}{\varepsilon_0 c n_0}}$

For fused silica $n_2 = 3\times10^{-20}\ m^2/W$

- Units for the effective nonlinearity

$$\gamma \rightarrow \frac{1}{W\,m}$$

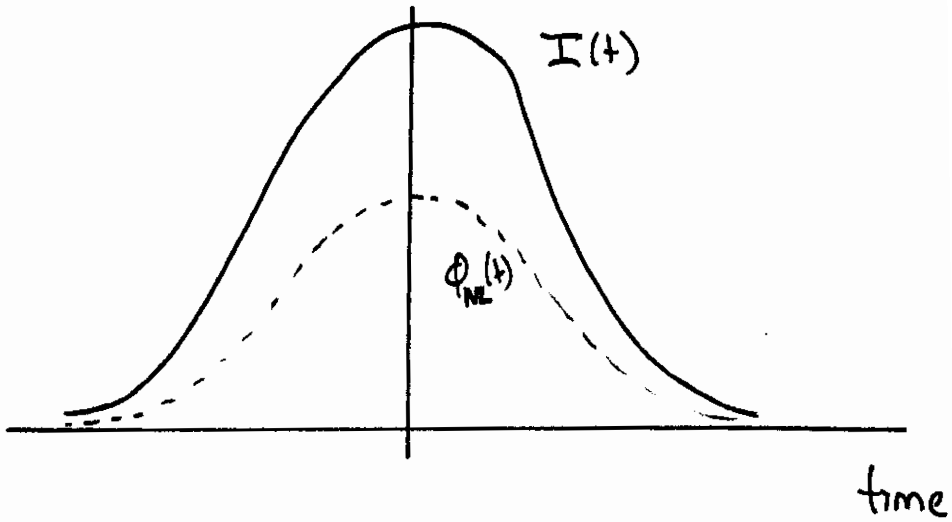
- Dispersion length

$$\boxed{L_D = \frac{T_0^2}{|\beta_2|}}$$ 1/e half width

where $T_0 = \frac{\Delta t}{2\ln(1+\sqrt{2})}$ FWHM

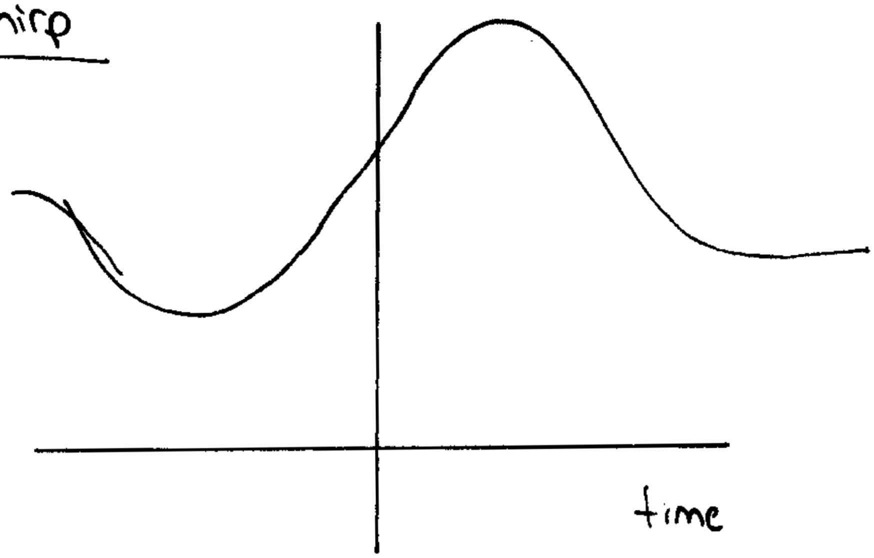
SPM For Pulses

$$E(t) = E_0 \operatorname{sech}(t/\Delta t)$$



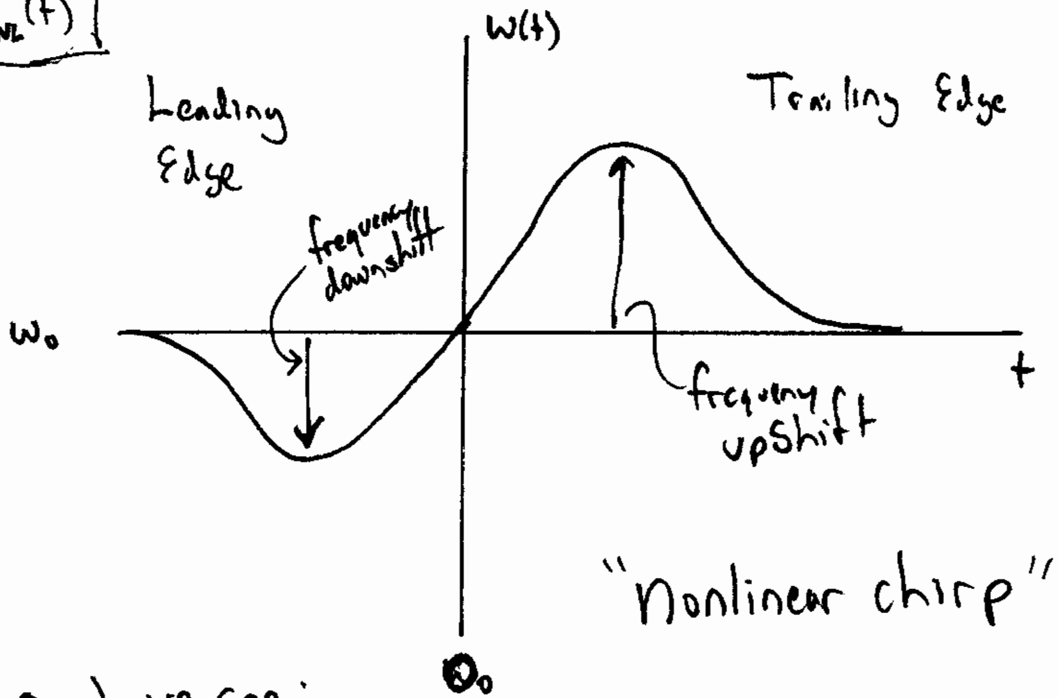
$$\begin{cases} I(t) \sim \operatorname{sech}^2(t) \\ \phi_{NL}(t) \sim \gamma P_0 \operatorname{sech}^2(t) \\ \text{Chirp} \sim \omega_0 - \partial_t \phi_{NL} \end{cases}$$

Chirp



How to think about this chirp!?

$\boxed{\omega_0 - \partial_t \phi_{NL}(t)}$



From the graph we see:

- Spectral components in leading edge experience frequency down shift (to lower ω) $\Delta\omega < 0$.
- Spectral components in trailing edge experience frequency up shift (to higher ω) $\Delta\omega > 0$.

So the effect of SPM Depends on how a pulse is initially chirped by say by dispersion. Consider what happens for negatively + positively chirped pulses.

- Negatively chirped pulses

  SPM Causes a frequeny downshift of the blue components, an upshift of the red components

  $\Rightarrow$ Spectral Compression

- Positively Chirped Pulses

  SPM Causes a frequeny downshift of the red components, an upshift of the blue components $\Rightarrow$ Spectral Broadening

Lecture 21 Pulse Propagation in Optical Fibers

Review We discussed how the geometry of an optical fiber alters the net dispersion.

$$\beta(\omega) = \frac{n_1(\omega)\omega}{c}\sqrt{1 + 2\Delta n\, b(\omega)}$$

$b(\omega) \equiv$ normalized propagation constant

$\Delta n \equiv$ index difference (~ 0.008)

Since Δn is small we can write

$$\beta(\omega) \simeq \frac{n_1(\omega)\omega}{c} + 2\Delta n\, b(\omega)$$

$$\beta_{total} \simeq \beta_{material} + \beta_{waveguide}$$

Role of Dispersion on Nonlinear effects

We wish to consider pulse propagation in a $\chi^{(3)}$ material.

In the absense of dispersion, the only effect is SPM

$$E(t,z) = E(t,0)\exp(i\,\phi_{NL}(t))$$

$$\phi_{NL}(t) = |U(t,0)|^2\, z/L_{NL} \qquad \left(L_{NL} = \frac{1}{\gamma P_0}\right)$$

This expression was derived assuming small (or zero) material dispersion. What do we set if we assume dispersion.

How to characterize dispersion : GVD

$$\beta_2 \equiv \text{GVD}$$

$\beta_2 > 0$ Normal

$\beta_2 < 0$ Anomalous

Dispersion Length (GVD)

$$\boxed{L_D \equiv \frac{T_0}{|\beta_2|}}$$

$T_0 \equiv 1/e$ halfwidth

$$\left(T_0 = \frac{\Delta t}{2\ln(1+\sqrt{2})} \quad \Delta t \equiv \text{FWHM for sech( )} \right)$$

A Gaussian pulse will increase its width by $\sqrt{2}$ by propagation L_D.

Compare L_D + L_{NL}

$L_{NL} >> L_D$ Nonlinear material / consider only SPM

$L_D >> L_{NL}$ Dispersive material / consider only GVD

The nonlinear length does not take into account any higher order nonlinearities.

Can we derive a more generic wave eq that is valid for both nonlinear + dispersive effects?

Start with wave eq

$$\nabla^2 \bar{E} - \frac{1}{c^2}\partial_t^2 \bar{E} = \mu_0 \partial_t^2 \vec{D}$$

Express this eq in Fourier Domain

$$\nabla^2 E(\omega) + \overbrace{\left(1 + \chi^{(1)} + \frac{3}{4}\chi^{(3)}_{xxxx}|E|^2\right)}^{\varepsilon(\omega)} \frac{\omega^2}{c^2} E(\omega) = 0$$

linear dispersion

$\chi^{(3)} \Rightarrow n_2$

Find solution of the form

$$E(r,\omega) = F(x,y)\, A(z,\omega) \exp(i\beta_0 z)$$

For a fiber this represents the fiber modes.

Subsitution gives two coupled Eqs assuming SVEA

call this $\eta(\omega)$

$$(1) \quad \partial_x^2 F + \partial_y^2 F - \left[\overbrace{\left(1 + \chi^{(1)} + \frac{3}{4}\chi^{(3)}_{xxxx} E\right)}^{\varepsilon(\omega)} \frac{\omega^2}{c^2} - \beta(\omega)\right] F(x,y) = 0$$

$$(2) \quad 2i\beta_0 \partial_z^2 E(z,\omega) + (\underbrace{\beta^2(\omega)}_{(\beta(\omega))} - \beta_0^2) E(z,\omega) = 0$$

How to solve this: Use a pertabative solution using terms of

$$\varepsilon(\omega) = 1 + \chi^{(1)} + \frac{3}{4}\chi^{(3)}_{xxxx}|E|^2$$

<u>Procedure</u> for solving $\beta(\omega)$

1. Zeroth order $\quad \varepsilon(\omega) = 1 + \chi^{(1)}$
   - use Eq (1) and solve for $\beta^{(0)}$ + $F(x,y)$
   - Assume $F(x,y)$ is a fiber mode (single mode)

2. 1st order $\quad \varepsilon(\omega) = 1 + \chi^{(1)} + \frac{3}{4}\chi^{(3)}|E|^2$
   - Sub $\varepsilon(\omega)$ into Eq (1) and solve for $\beta^{(1)}$
   - Sub $\beta^{(1)} + \beta^{(0)}$ into Eq (2)

From (2) we get

$$\partial_z E(z,\omega) = i\left[\beta^{(0)} + \beta^{(1)} - \beta_0\right] E(z,\omega)$$

effect of dispersion

Nonlinearity only $\Rightarrow$ $\beta^{(1)} = \gamma|E|^2$

Dispersion only (and absorption) $\Rightarrow$ $\beta^{0}_{\bullet}(\omega) = \beta_0 + \beta_1(\omega-\omega_0)^2 + \frac{1}{2}(\beta_2)(\omega-\omega_0)^3 = \beta(\omega)$

Solution

$$\underbrace{\underbrace{\partial_z E = -\left(\sum_{m=1} \beta_m \frac{i^{(m-1)}}{m!} \partial_m\right) E}_{\text{Dispersion}} + \underbrace{i|E|^2 E}_{\text{SPM}}}$$

The name of this equation is the nonlinear Schrödinger Equation (NLSE)

A more accurate form of this equation which takes into account higher order effects (self steepening + Raman effect)

$$\partial_z E = -\frac{\alpha}{2}E - \left(\sum_{m=2} \beta_m \frac{i^{m-1}}{m!} \partial_t^m\right) E + (1 - f_R)\left[i\gamma|E|^2 E - \frac{\partial\gamma}{\omega_0}\partial_t(|E|^2 E)\right]$$

$$+ i\gamma f_R\left(1 + \frac{i}{\omega_0}\partial_t\right)\left(E\int h_R(t)\,|E(z, t-t')|^2\,dt'\right)$$

Solitons

An analytic solution to the NLSE is of the form

$$\boxed{E(t) \sim \operatorname{sech}(t/\Delta t)\, e^{-i\omega_0 t} N\sqrt{P_1}}$$

where $\boxed{P_1 = \frac{1}{\gamma L_D}}$ $\boxed{N^2 = \frac{L_D}{L_{NL}} \equiv (\text{soliton order})^2}$

Lecture 22 — More on pulse propagation in fibers

- **Review** — Need to consider both dispersive (GVD) and nonlinear effects (SPM)

NLSE
$$\boxed{\partial_z E = -\frac{i}{2}\beta_2 \partial_t^2 E + i\gamma|E|^2 E}$$

$$\partial_z E = \underbrace{\left(-\frac{\alpha}{2}E\right)}_{\text{Absorption}} - \underbrace{\left(\beta_1 \partial_t E + \frac{i}{2}\beta_2 \partial_t^2 E\right)}_{\text{dispersion (GVD)}} + \underbrace{\left(i\gamma|E|^2 E\right)}_{\text{SPM}}$$

An analytic solution is given by

$$\boxed{E(t) \sim N\sqrt{P_1}\ \mathrm{Sech}(\eta\, t/\Delta t)}$$

Where $\frac{1}{\gamma L_D} \equiv P_1$ $\qquad N^2 = \frac{L_D}{L_{NL}}$ (Soliton order)

Optical Solitons (Self reinforcing solitary wave)

Balancing GVD + SPM

John Scott Russell (~183[illegible]) in water

Water wave ⇒ 2 miles (Union Canal)

See in plasma's + in optics

A 1^{st} order soliton occurs to the balanced effects of GVD + SPM

GVD ⇒ Temporal Broadening

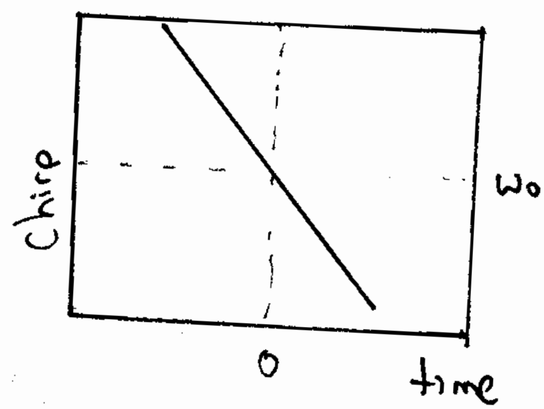
SPM ⇒ Spectral Broadening

For $N=1$ $\quad L_N = L_D$ $\quad$ look at chirp $\omega(t) = \omega_0 - \partial_t \phi(t)$

Compare the chirp due to GVD + SPM

Anomalous GVD

- $\beta_2 < 0$

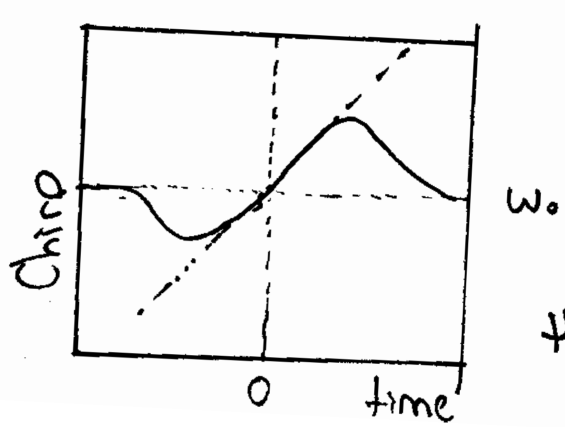


Negative chirp

SPM

$$\phi_{NL}(t) \sim \gamma \, |\operatorname{sech}(t/\Delta t)|^2 L$$

$$\omega(t) = \omega_0 - \partial_t \left(\gamma L \, |\operatorname{sech}(t/\Delta t)|^2 \right)$$



The slope of the chirp due to SPM across the width of the pulse is equal + opposite to that of anomalous GVD.

Heriot-Watt University

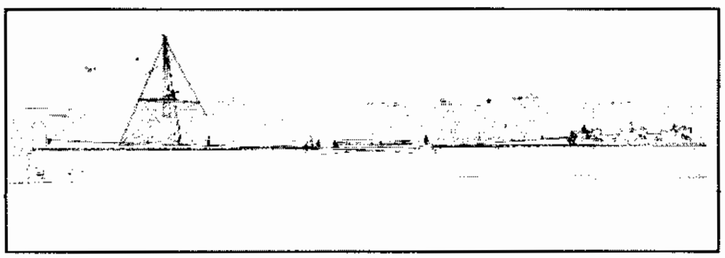
Department of Mathematics

John Scott Russell and the solitary wave

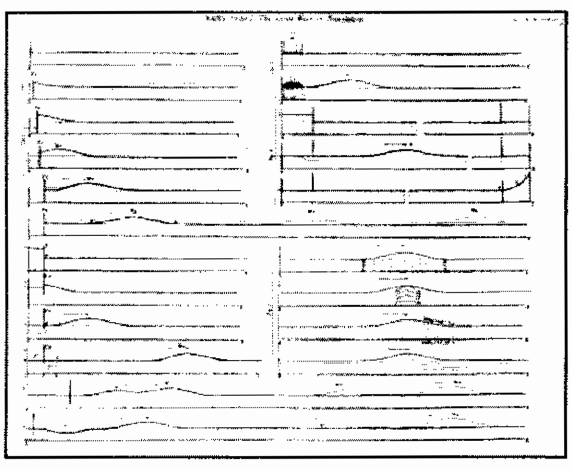
Over one hundred and fifty years ago, while conducting experiments to determine the most efficient design for canal boats, a young Scottish engineer named John Scott Russell (1808-1882) made a remarkable scientific discovery. As he described it in his "Report on Waves": (Report of the fourteenth meeting of the British Association for the Advancement of Science, York, September 1844 (London 1845), pp 311-390, Plates XLVII-LVII).

> ``I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation".
> (Cet passage en francais)

This event took place on the Union Canal at Hermiston, very close to the Riccarton campus of Heriot-Watt University, Edinburgh.



Following this discovery, Scott Russell built a 30' wave tank in his back garden and made further important observations of the properties of the solitary wave.



Throughout his life Russell remained convinced that his solitary wave (the ``Wave of Translation") was of fundamental importance, but ninteenth and early twentieth century scientists thought otherwise. His fame has rested on other achievements. To mention some of his many and varied activities, he developed the "wave line" system of hull construction which revolutionized ninteenth century naval architecture, and was awarded the gold medal of the Royal Society of Edinburgh in 1837. He began steam carriage service between Glasgow and Paisley in 1834, and made one of the first experimental observations of the "Doppler shift" of sound frequency as a train passes. He reorganized the Royal Society of Arts, founded the Institution of Naval Architects and in 1849 was elected Fellow of the Royal Society of London. He designed (with Brunel) the "Great Eastern" and built it; he designed the Vienna Rotunda and helped to design Britain's first armoured warship (the "Warrior"). He developed a curriculum for technical education in Britain, and it has recently become known that he attempted to negotiate peace during the American Civil War.

It was not until the mid 1960's when applied scientists began to use modern digital computers to study nonlinear wave propagation that the soundness of Russell's early ideas began to be appreciated. He viewed the solitary wave as a self-sufficient dynamic entity, a "thing" displaying many properties of a particle. From the modern perspective it is used as a constructive element to formulate the complex dynamical behaviour of wave systems throughout science: from hydrodynamics to nonlinear optics, from plasmas to shock waves, from tornados to the Great Red Spot of Jupiter, from the elementary particles of matter to the elementary particles of thought.

For a more detailed and technical account of the solitary wave, see for example R K Bullough, *"The Wave" "par excellence", the solitary, progressive great wave of equilibrium of the fluid - an early history of the solitary wave*, in **Solitons**, ed. M Lakshmanan, Springer Series in Nonlinear Dynamics, 1988, 150-281, or "The Spirited Horse, Three properties of solitons

Permanent form

Localized

Interact with other solitons, emerged unchanged.

Optical Solitons

Experimentally observed in 1973

Result from interplay of anomalous GVD + SPM

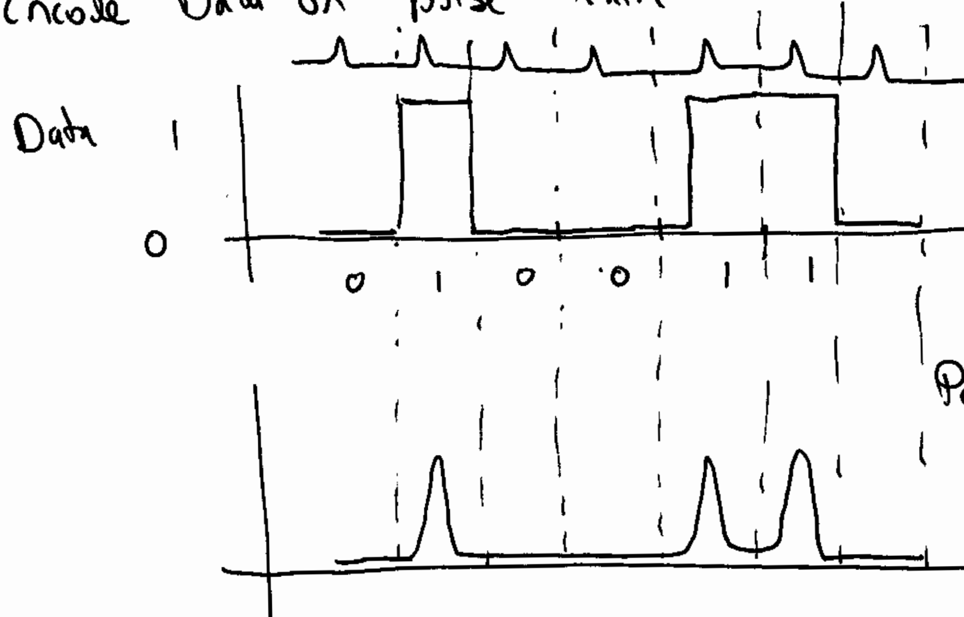
Fundamental Soliton $N = 1$ Constant Shape

Higher order solitons $N > 1$ Periodic Shape

Applications for solitons $\Rightarrow$ Optical Communications

(Time Division Multiplexing)

Encode Data on pulse train



Propagate this pulse train down many km of fiber. Do no want temporal broadening

Numbers: 20 Gb/s soliton train over 150000 km with 105 km amp spacing

How to Solve the NLSE: Split Step Fourier Method

Break the fiber in n steps of length h

Use operator method

Dispersion operator $\boxed{\hat{D} = -\frac{i}{2}\beta_2\,\partial_t^2}$

Nonlinearity operator $\boxed{\hat{N} = i\gamma|E|^2}$

Write NLSE

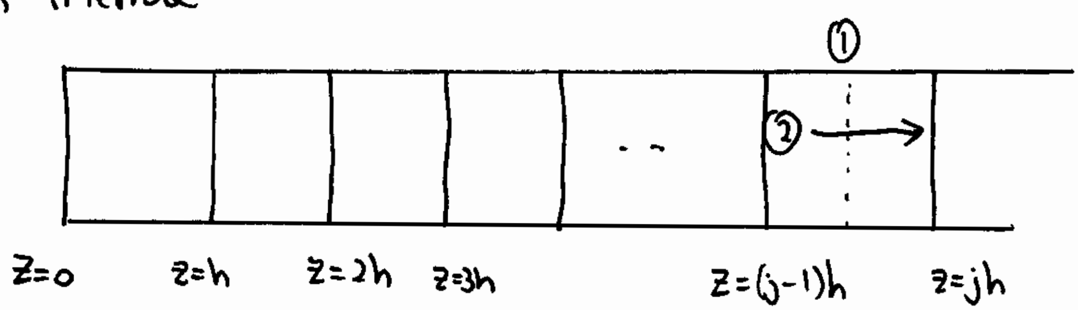
$$\boxed{\partial_z E = (\hat{D} + \hat{N})E}$$

So $$E(jh,t) = \exp(\hat{D} + \hat{N})\,E((j-1)h,t)$$

If dispersion acts independently of the nonlinearity (assumed for a small step size) then

$$\exp((\hat{D} + \hat{N})h) \simeq \exp(\hat{D}h)\exp(\hat{N}h)$$

Solution Method



In general

$$\boxed{\mathcal{F}\left\{ \partial_t^n f(t) \right\} = (i\omega)^n \, \mathcal{F}\left\{ f(t) \right\}}$$

1. Calculate nonlinearity at midpoint of step

$$\exp(h\hat{N})\, E((j-1)h, t)$$

2. Calculate Dispersion in frequency domain

$$\exp(h\hat{D}(\omega))\, \mathcal{F}\{\exp(h\hat{N})\, E((j-1)h, t)\}$$

Why? The operator $\hat{D}$ is a differential operator

$$\hat{D} \sim \partial_t^2$$

However

$$\mathcal{F}\{\partial_t^2\} = (i\omega)^2 \quad \left(\cancel{\text{In general } \mathcal{F}\{\partial_t^n\} \neq}\right)$$

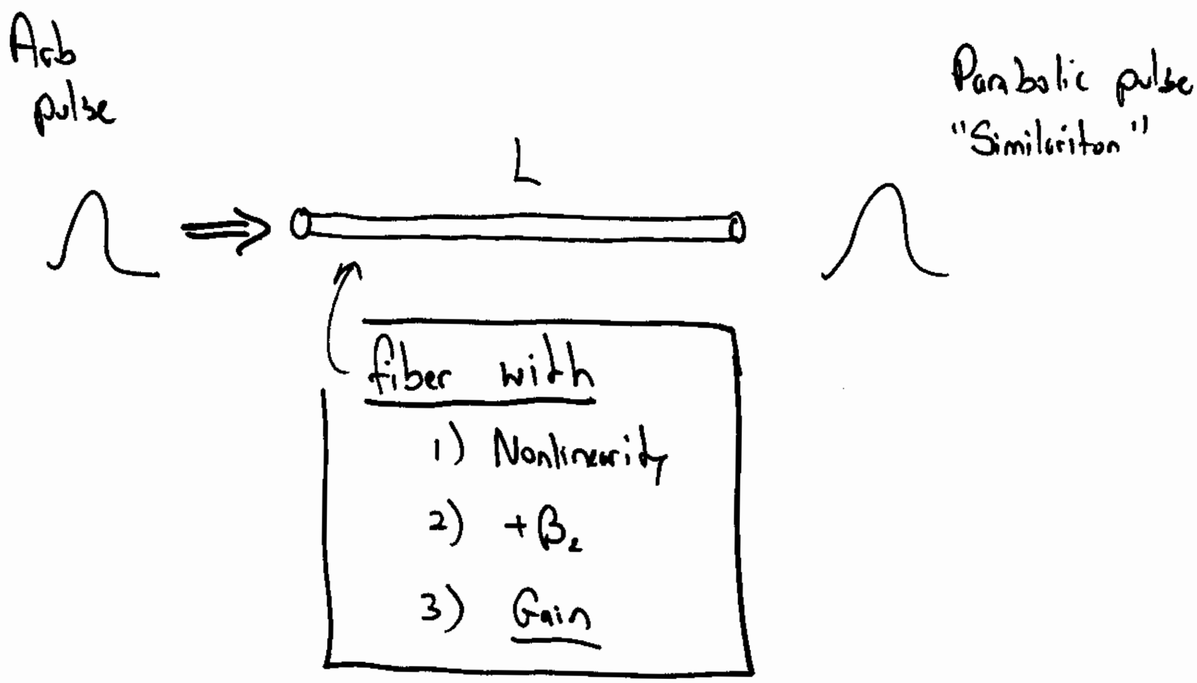
So $\hat{D}$ in the Fourier Domain is a <u>multiplicative</u> <u>operator</u>

3. The solution after the step is

$$\boxed{E(jh, t) = \mathcal{F}^{-1}\{\exp(\hat{D}h)\, \mathcal{F}\{\exp(\hat{N}h)\, E((j-1)h, t)\}\}}$$

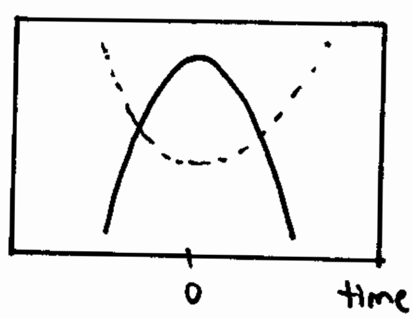
Repeat procedure over all steps to L.

One other "special" Pulse : Similariton



In the limit of infinite length the electric field profile is a parabola! It also has a parabolic phase distortion.

$$\boxed{E(\omega,z) \sim A_0 \sqrt{1 - \frac{\omega^2}{\omega_p^2}} \exp(i\,\omega^2/c)}$$



Intensity + Phase

Why are similarton's important

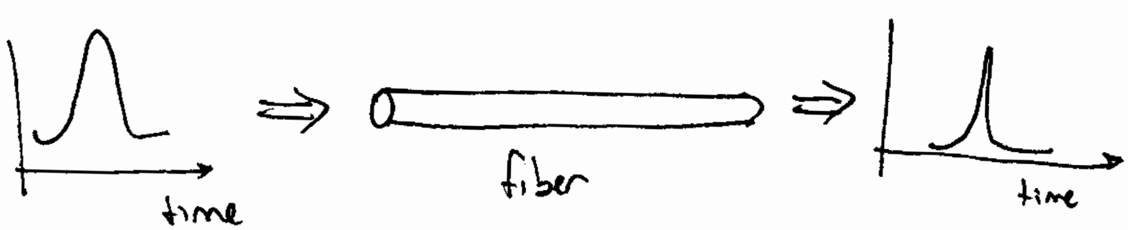
The pulse only has a quadratic phase distortion which can be compressed with GVD only!

Lecture 23 Applications of $\chi^{(3)}$ Effects

- Ultrashort pulse compression
- Self Focusing and Self Filamentation
- Supercontinuum Generation.
- Nonlinear Switching

Ultrashort pulse compression in optical Fibers

If a fiber exhibits a large nonlinearity with small β_2 GVD then the nonlinear spectral broadening can be used for pulse compression.

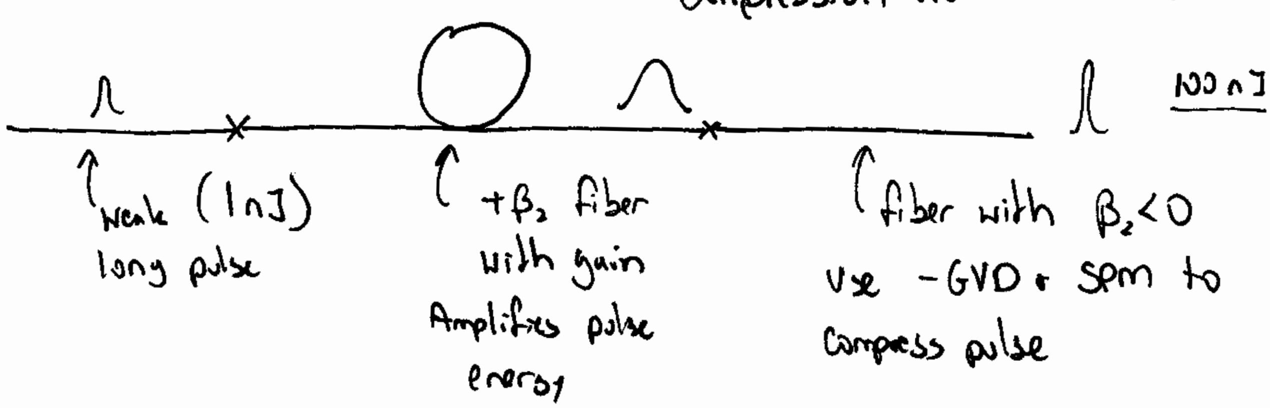


The process works better when the group velocity dispersion is near zero and anomalous.

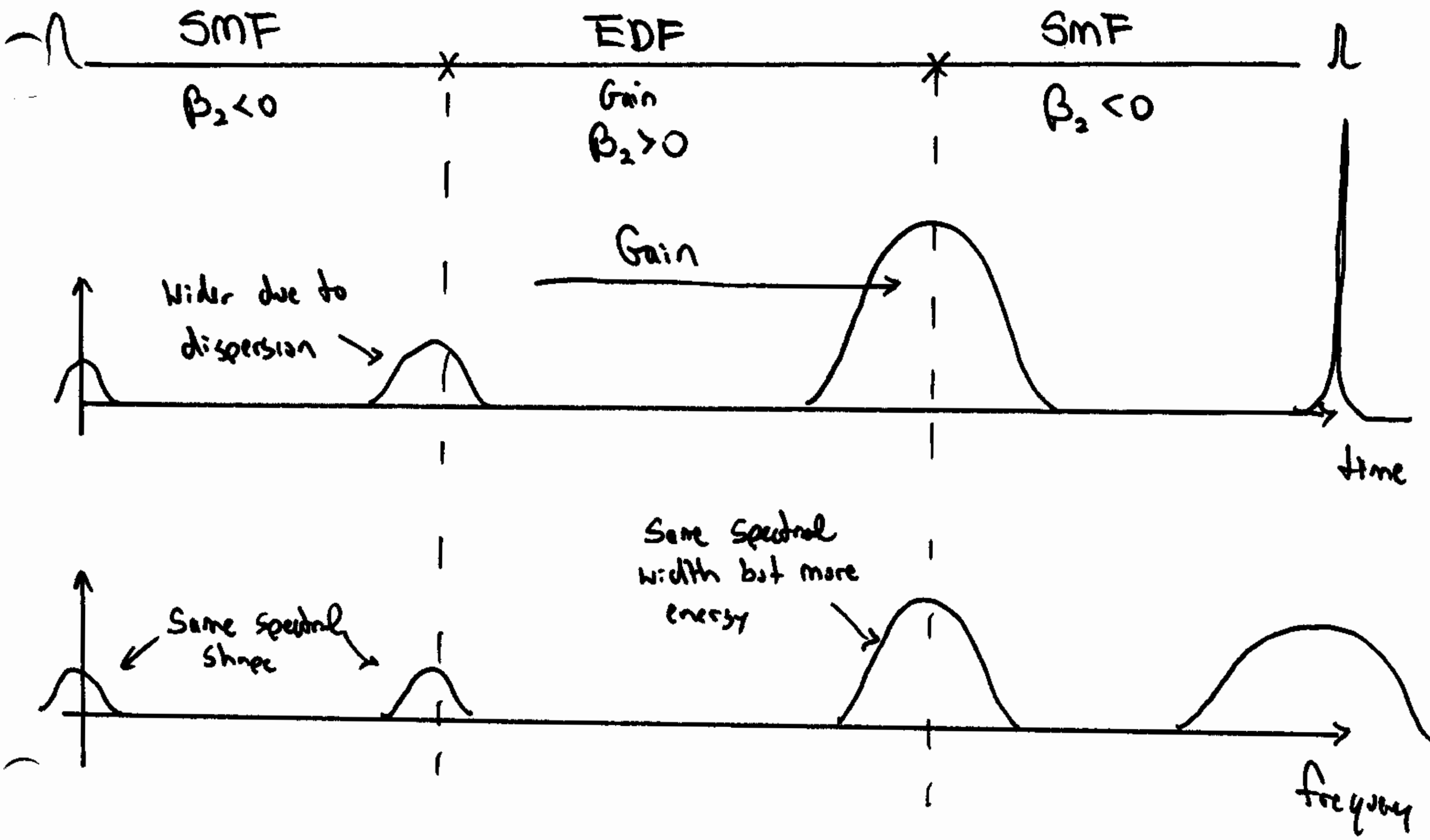
Anomalous Dispersion Compression Scheme in optical fibers (Compression + Amplification)

Compression from 200 fs to 50 fs

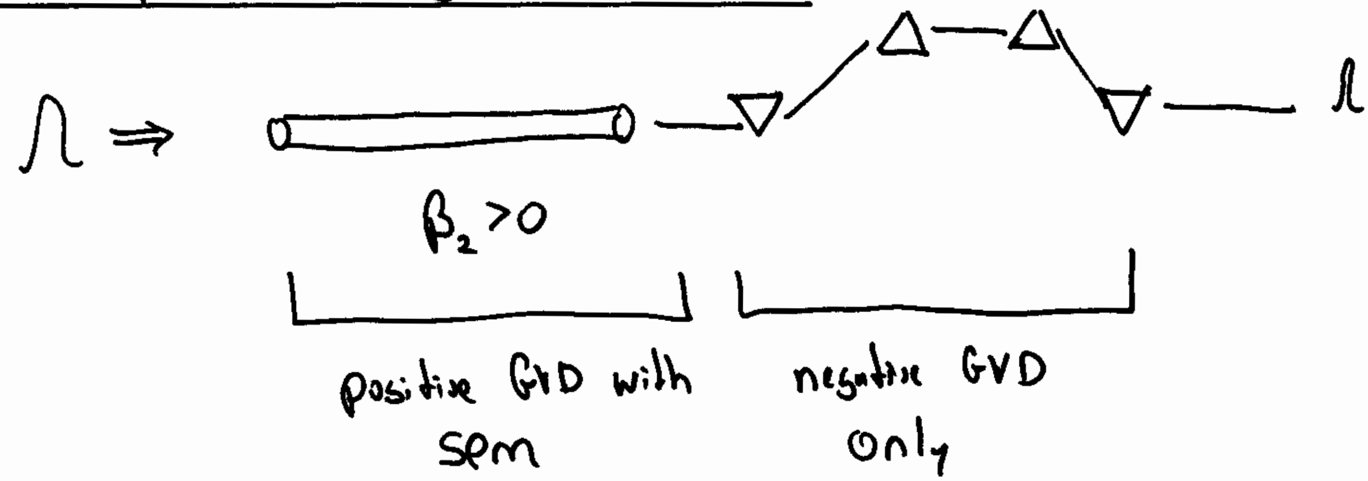
$(L_{NL} \simeq L_D)$



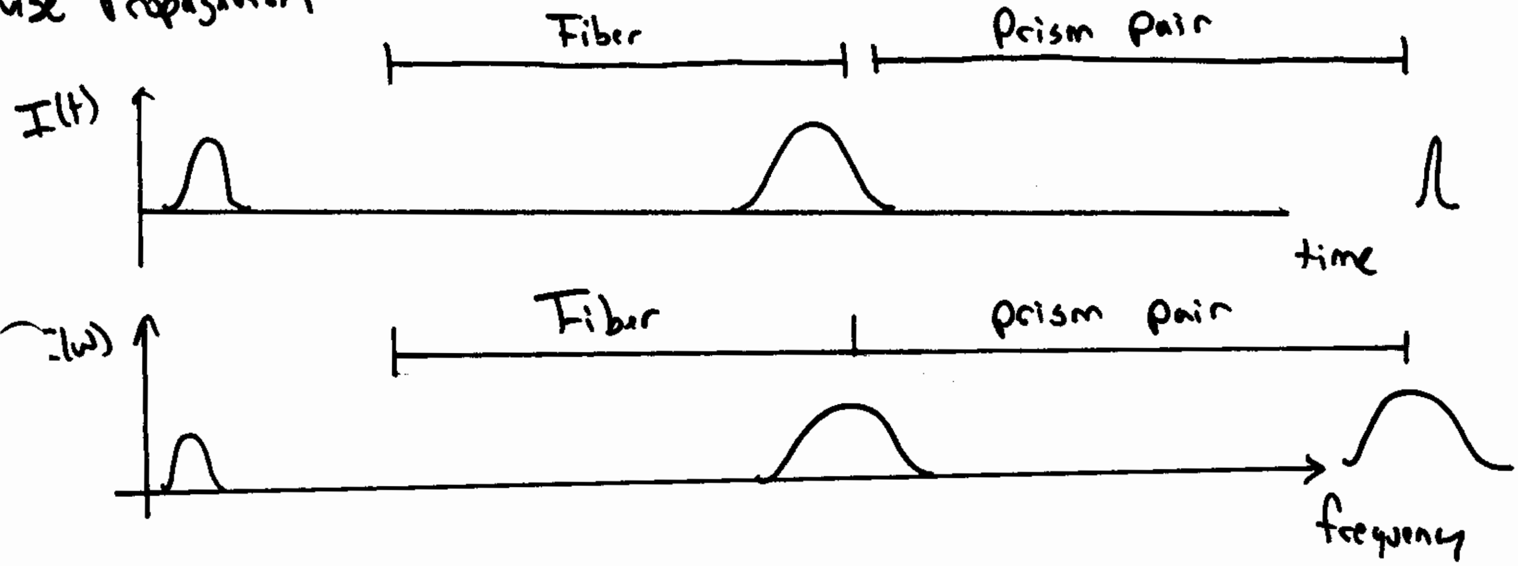
Pulse compression using negative GVD



Pulse Compression using positive GVD



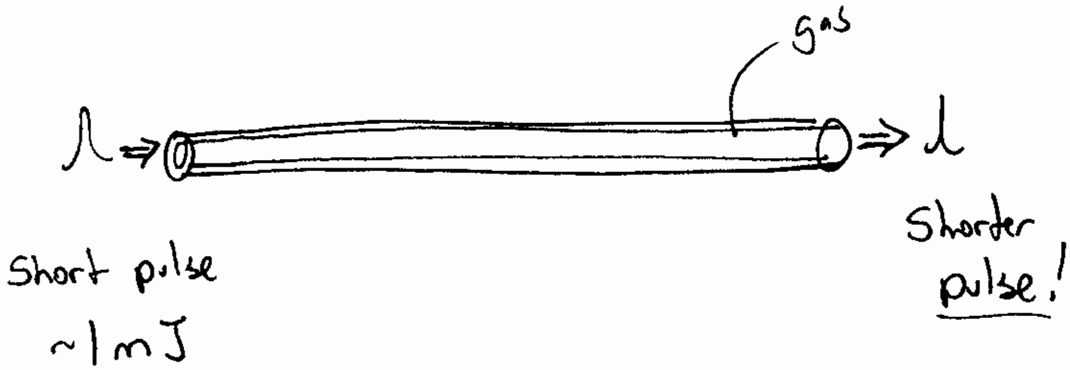
Pulse Propagation



Ultrashort pulse compression in Noble Gases.

- Specific Noble gases (Ne, etc) exhibit a large n_2 with small dispersion. The SPM due to the gas in the presence of small GVD will cause spectral broadening / temporal compression.

$L_{NL} \ll L_D$



- Compression from ~20fs to < 5fs

Using the interaction of SPM + GVD for pulse compression

~~Self Focusing~~

~~spatial analog to self~~

Supercontinuum Generation

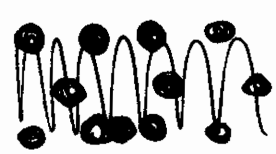
Generation of extremely broadband light ($\Delta\omega \simeq \omega_0$)

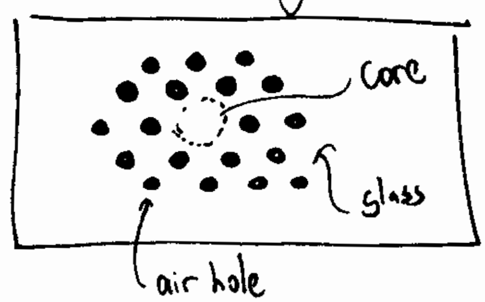
How? Typically two methods

1) ~ 1 mJ pulses in quartz or sapphire plate
2) ~ 1 nJ pulses in a "special" optical fiber

"special fiber"
- microstructured optical fiber
- photonic crystal fiber
- photonic bandgap fiber
- holey fibers

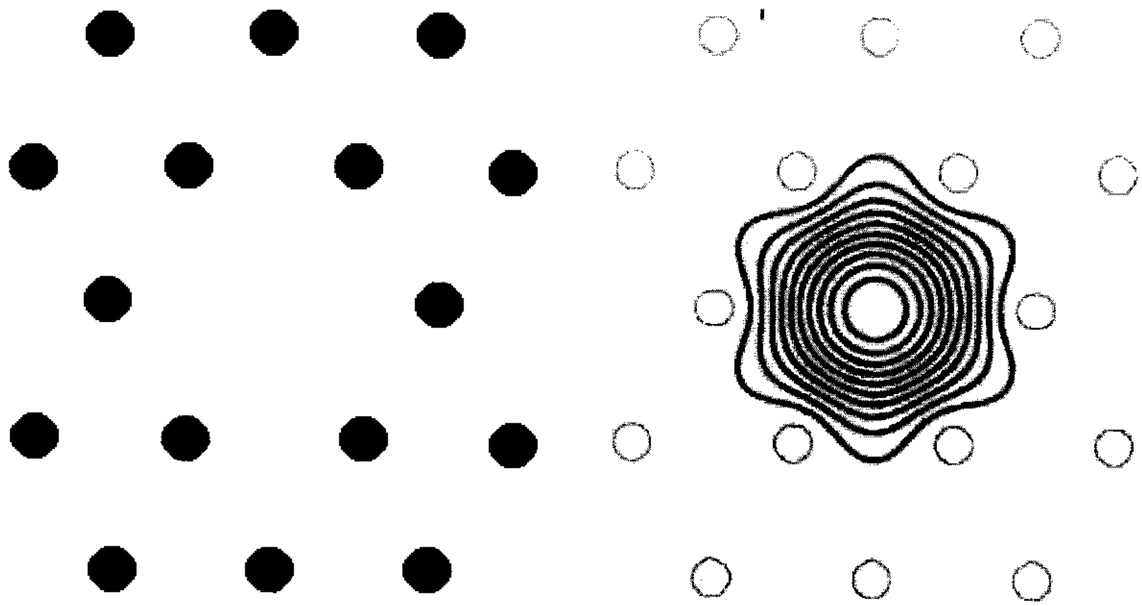
These are optical fibers that have a very small core surrounded by a cladding of air holes + glass.

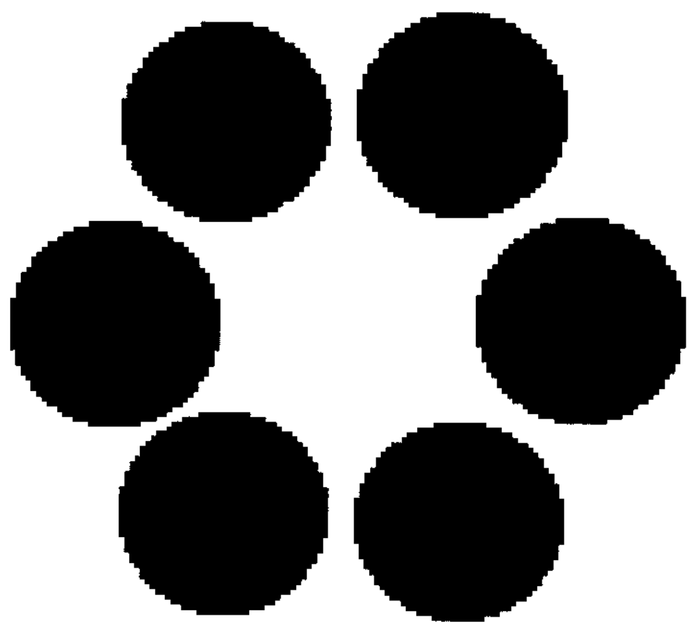




The presence of the air holes in the cladding reduces the effective cladding index thus increasing Δn. This makes the mode field diameter smaller thus a larger effective nonlinearity γ.

$$\gamma = \frac{n_2 \omega}{c(\pi r_0)^2}$$





microstructured optical fibers offer two advantages for supercontinuum generation.

1) Large effective nonlinearity $\left(\begin{array}{l} \gamma \simeq 110 \frac{1}{W km} \\ \gamma = 1 \frac{1}{W km} \end{array} \right)$

2) Small core moves zero GVD ~~[illegible]~~ wavelength to shorter λ.

Conventional fiber $\{ \lambda_{ZGVD} = 1300nm$, diameter $10\mu m$

microstructure fiber $\left\{ \begin{array}{l} \lambda_{ZGVD} = 900nm, \text{ diameter } 4\mu m \\ \lambda_{ZGVD} = 700 nm, \text{ diameter } 2\mu m \end{array} \right.$

Also microstructure fibers offer negative GVD at shorter λ.

For supercontinuum generation to occur

1. Large effective nonlinearity
2. Low loss
3. Low or zero GVD at center wavelength

Unfortunately associated with the spectral broadening is temporal ~~[illegible]~~ breakup of the pulse

5 nJ into 15cm at 800nm $\Rightarrow$ $\left\{ \begin{array}{l} \text{Broadening from 500–1200nm} \\ \text{Temporal Breakup over 3ps!} \end{array} \right.$

Why is Supercontinuum generation important?

1) Remember f_0 detection? Use supercontinuum generation to get f_n and f_{2n}

2) Wide spectral shape that is coherent.

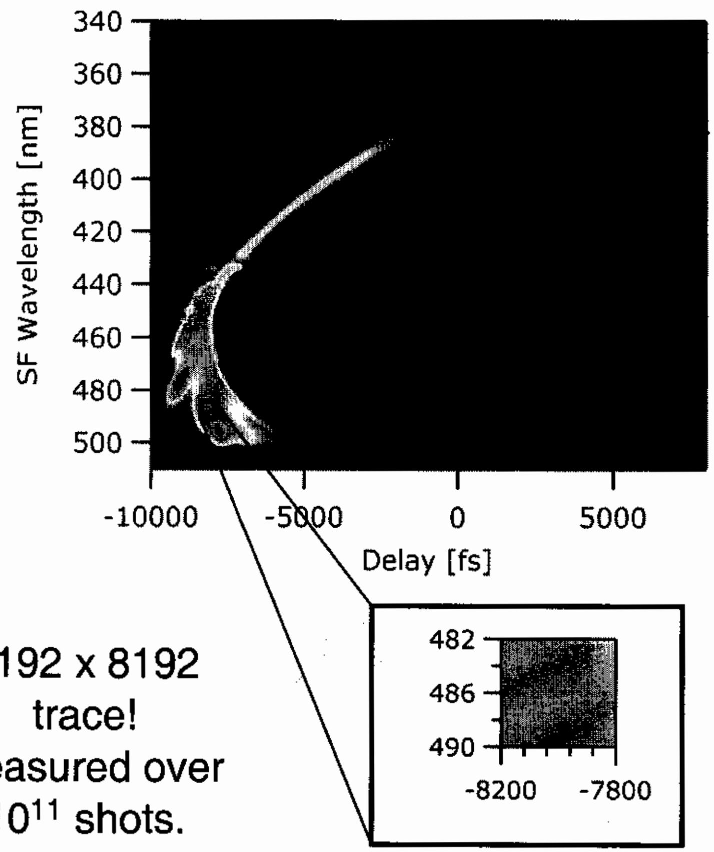
How to explain Supercontinuum generation

- Complicated nonlinear process: (Involves SPM, Stimulated Raman Scatt and Self steepening)

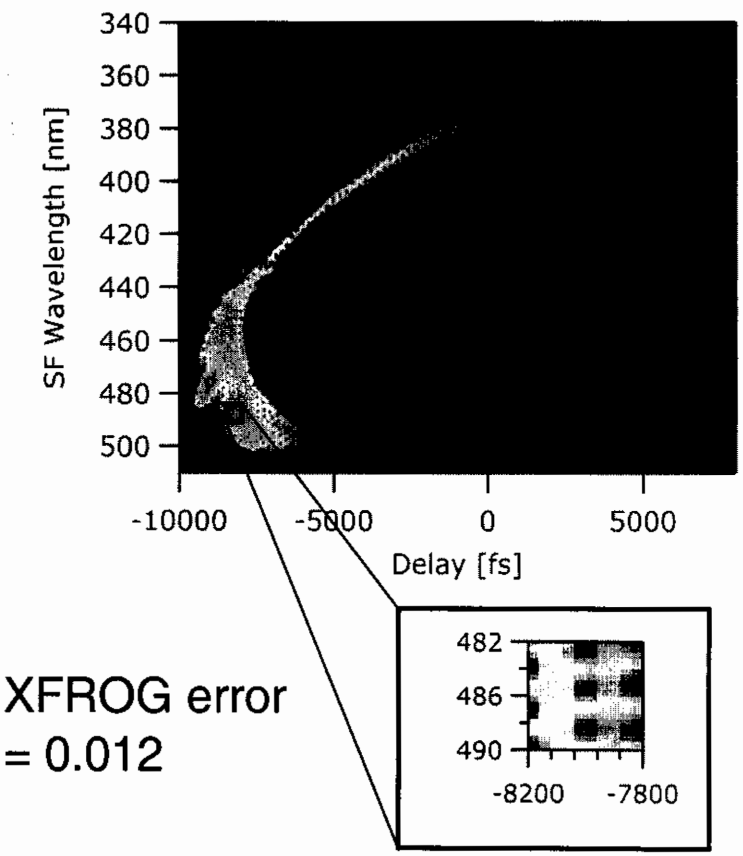
Solitons fission

{ A n^{th} order soliton will break up into n 1^{st} order solitons

XFROG measurement of the continuum



8192 x 8192 trace! Measured over 10^{11} shots.



XFROG error = 0.012

While the large-scale structure of each trace is identical, the measured trace lacks the fine-scale structure of the retrieved trace.

From Trebino's Lecture notes

Lucent/OFS Microstructure Fiber

— 92 μm —

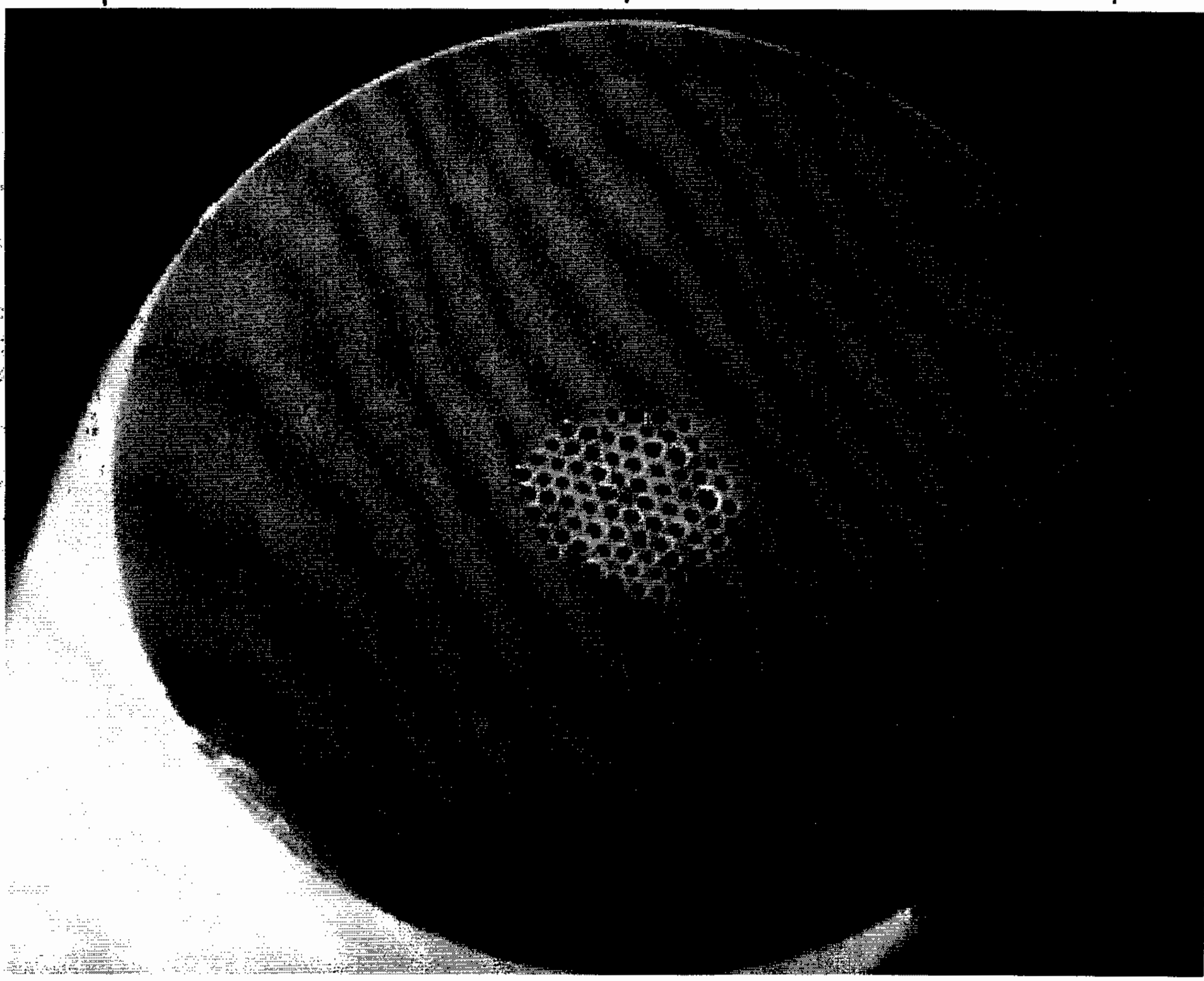
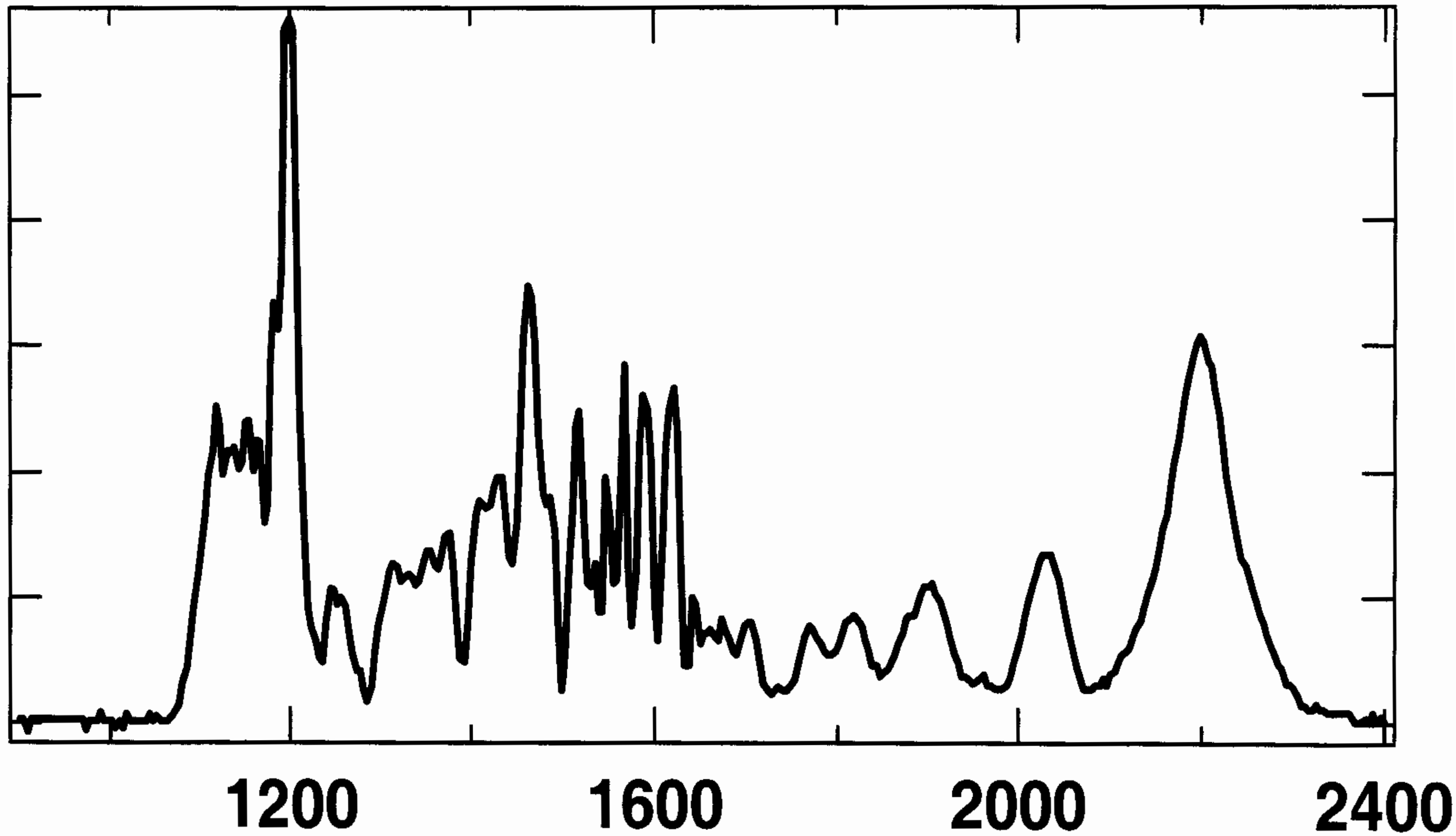
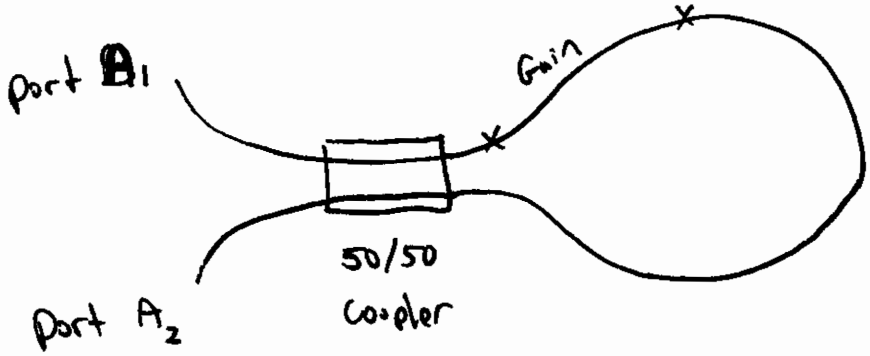


Figure courtesy of OFS


1200
1600
2000
2400

Nonlinear Switching Loop Mirrors

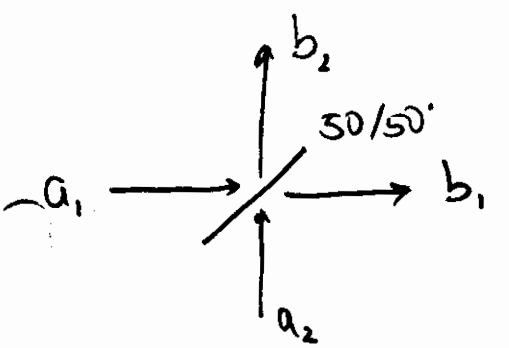


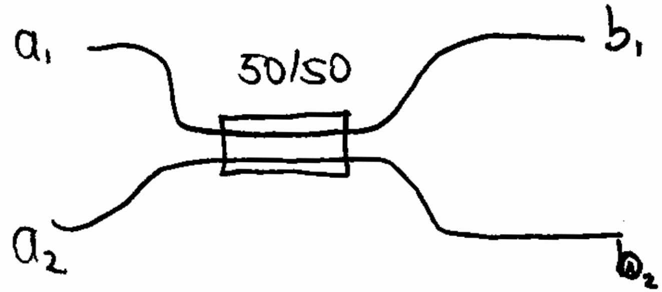
Loop mirror

Fast switch

Sagnac Interferometer

Difference between beam splitter + directional coupler





beam splitter

π phase shift between b_1 + b_2

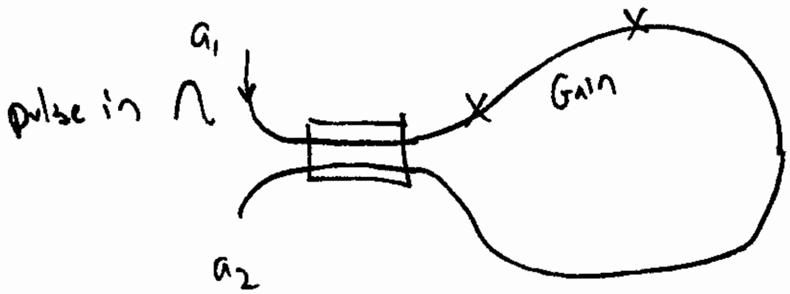
directional coupler

(uses evanescent wave coupling)

$\frac{\pi}{2}$ phase shift between b_1 + b_2 for 50/50.

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} t & e^{i\pi/2}\sqrt{1-t^2} \\ e^{i\pi/2}\sqrt{1-t^2} & t \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

The nonlinear phase shift is asymmetric



$G \equiv$ Gain

The directional coupler will split the input pulse into two:

1) Clockwise Pulse: Pulse that goes clockwise get first amplified and then experiences a large nonlinear phase shift

2) Counter Clockwise Pulse: Pulse that goes counter clockwise goes thru fiber and get a small nonlinear phase then goes thru amplifier

Difference in phase shift between two pulses.

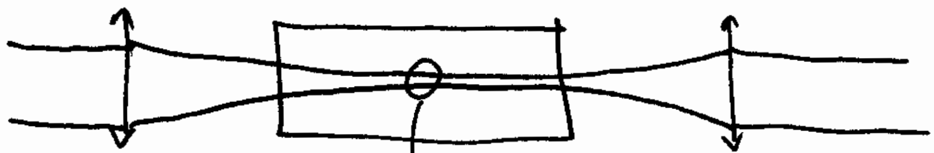
$$\Delta\phi \sim n_2 (G-1) I(t) L$$

If the pulses are weak then $\Delta\phi = 0$ and the pulse will exit the loop out port a_1

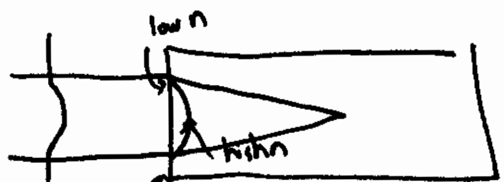
If the pulses are strong then $\Delta\phi \simeq \pi$ and the pulse will exit the loop out port a_2.

Lecture 24 Self focusing : Spatial $\chi^{(3)}$ Effects

- Spatial analog of Self phase modulation
- Intense beam of light modifies the medium (index) it experiences



Since $n_2 > 0$ the action of self focusing causes a larger index of refraction for high intensities. This ~~meant~~ creates another lens in the material.



Important ⇒ Critical Power not intensity

Spatial Solitons (self trapping)

Analog to temporal solitons

Balance of diffractive & nonlinear effects.

Critical Power

$$P_c = \frac{\pi (0.61)^2 \lambda^2}{8 n_0 n_2} = \frac{\lambda^2}{8\pi n_0 n_2} \quad \boxed{P_c > 1\,\mathrm{MW}}$$

Whole beam self focusing: Continuous wave

$$\left(\text{Phase distortion} \quad \boxed{\phi = \frac{n\omega}{c} L} \right)$$

$$n = n_0 + n_2 I$$

$$n = n_0 + n_2 I_0 \exp(-2r^2/w_0^2)$$

$$n \simeq n_0 + n_2 I_0 (1 - 2r^2/w_0^2)$$

Phase delay due to spatial nonlinearity

$$\phi(r) = n k_0 L = n_0 k_0 L + n_2 k_0 L I_0 (1 - 2r^2/w_0^2)$$

$$\phi(r) \sim -2 n_2 k_0 L I_0 \frac{r^2}{w^2} \Leftarrow \left(\begin{array}{l} \text{analog between} \\ \text{a lens and GVD} \end{array} \right)$$

quadratic spatial phase distortion.

This is what a lens does!

Critical Power is defined where $\Rightarrow \boxed{P_c = \lambda^2/8\pi n_0 n_2} \sim 1\,\text{MW}$

$$\boxed{\phi_{\text{self focus}}(r) = \phi_{\text{diffraction}}(r)}$$

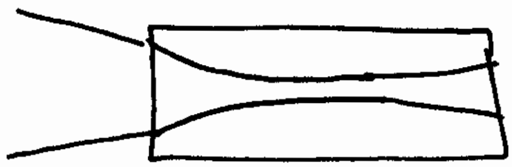
$$z_{sf} = \frac{1}{2} \left(\frac{\pi r_0^2 n_0}{\lambda} \right) \frac{1}{(P/P_c - 1)^{1/2}}$$

Self - defocusing

Higher order nonlinearities $\sim n_4 I^2$

Another way to look at this

Diffraction angle = self focusing angle.



Example of Self focusing ⇒ Kerr lens modelocking

Modelocking is a method to get short pulse formation in a laser cavity thru the coherent addition of cavity modes

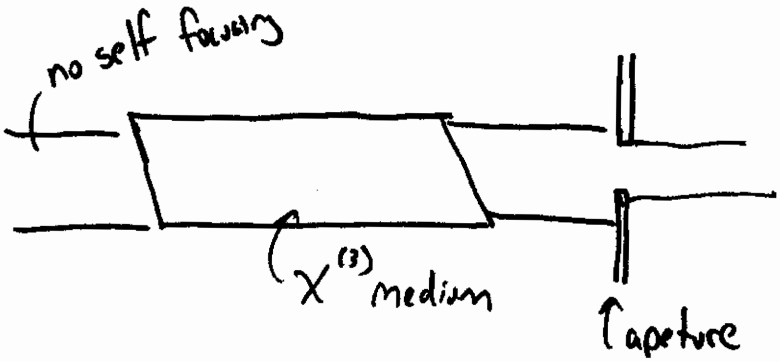
Add up modes to generate pulse train

This is typically done by setting up a condition in the cavity that favors high peak powers. (Self amplitude mod).

Kerr-lens ⇒ "lens" due to self focusing in a nonlinear crystal.

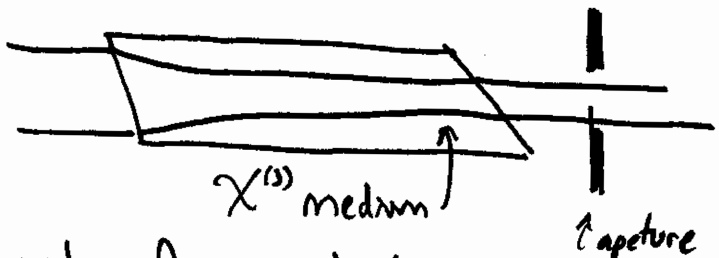
Mode-locked laser

How to use self focusing for mode locking?



(High loss for low P_0)

Set up a condition where there is low loss for high peak powers



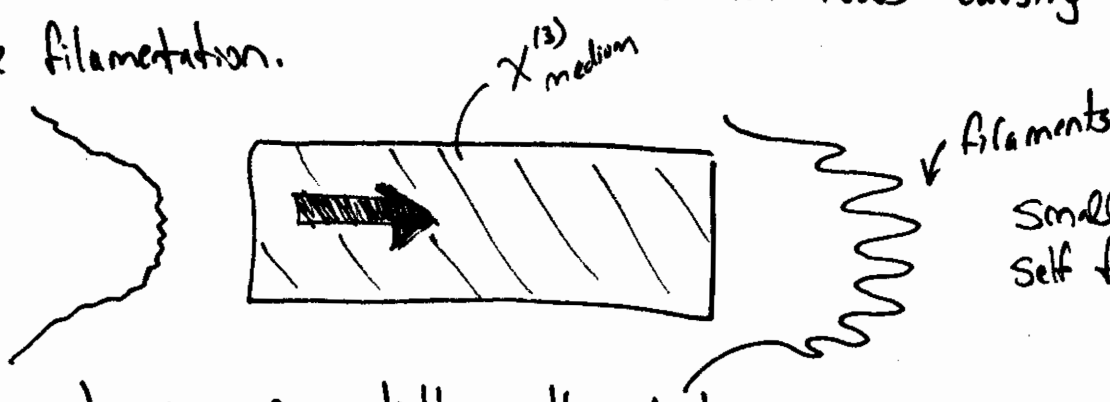
(Low loss for high P_0)

This condition ~~permits~~ favors high peak power pulses.

Self filamentation

During self focusing a single beam will break up into multiple small beams.

The spatial beam profile will have variations due to quantum noise. These variation each will focus causing the filamentation.



Small scale self focusing

Filaments grow exponentially with distance

Self filamentation leads to a beam with random intensity distributions

Unfortuntately, the power for self filamentation is on the same order of that of self focusing.

Light Bullets

$\Rightarrow$ 3D spatial solitons

Spatial / temporal coupling

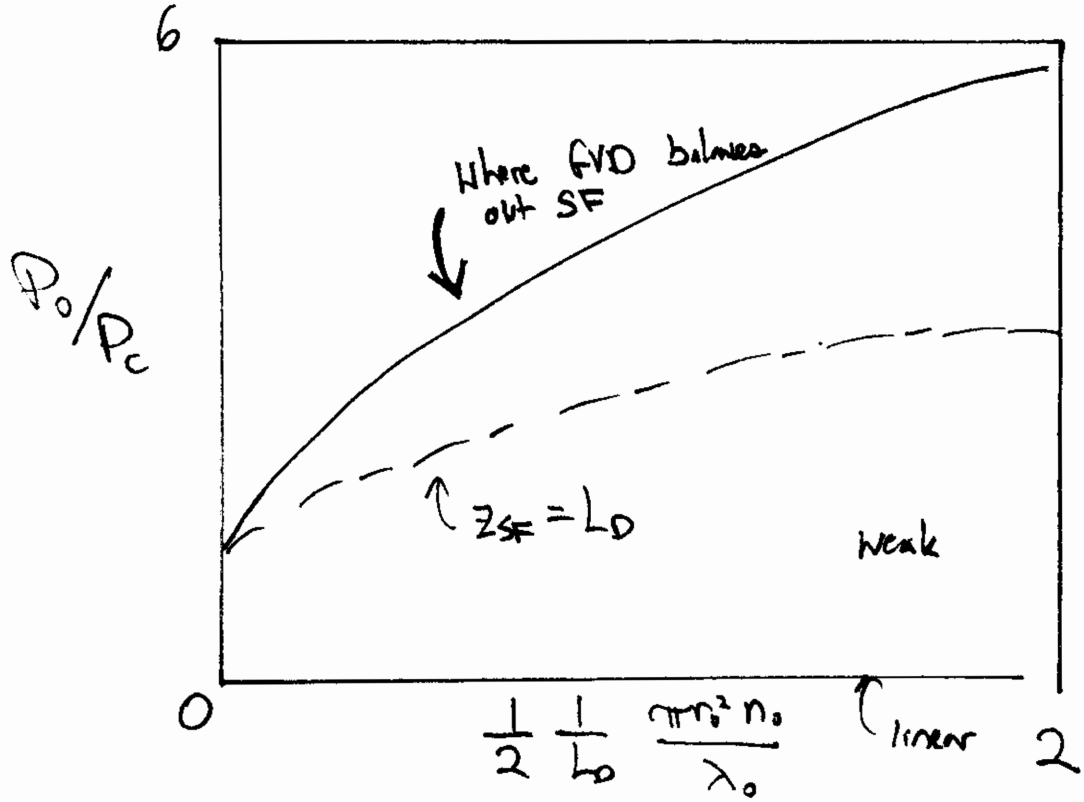
Self Focusing using pulses

- Need to include space - time coupling
  Material dispersion $\Rightarrow$ higher peak powers for self focusing compared to cw case

  Need to consider higher order dispersion + nonlinearities

$$\frac{1}{2}\frac{1}{L_D}\left(\frac{\pi r_0^2 n_0}{\lambda_0}\right) \equiv \text{measure of dispersion relative to diffraction}$$

- $$\frac{P_0}{P_{cr}} \equiv \text{Strength of pulse with respect to critical power}$$



Notes on SPM in gases

- SPM is more complex because it is coupled with self focusing
- Self focusing effects are detrimental for using SPM for temporal compression in gases.
- Ionization in gases modify the beam propagation + produces asymmetric SPM.

Saturation of the nonlinear response

Self Focusing in solids

Causes damage tracks (glass)

Fiber fuse effect $\Rightarrow$ Bad

Review of self focusing

The intensity dependent index of refraction creates a "lens" in the material

$$n = n_0 + n_2 I_0 \left(1 - 2r^2/w_0^2\right)$$

So $\phi(r) = \frac{n\omega}{c} L \simeq (n_0 + n_2 I)\frac{n_0\omega}{c} L - \frac{n_0\omega}{c} L\left(\frac{2r^2}{w_0^2}\right)$

Thru the results of Fraunhofer diffraction a lens induces the same phase shift to the wave

$$\boxed{\phi(r) \sim -r^2} \qquad \text{lens}$$

The electric field after the lens is

$$E(\vec{r},t)\, \exp(i\phi(r))$$

Domains for the electric field & Analogies

	Time / Frequency		position / k-space
Linear	Dispersion	$\Longleftrightarrow$	Diffraction
Linear	GVD	$\Longleftrightarrow$	Lens (quadratic phase distortion)
Nonlinear	SPM	$\Longleftrightarrow$	Self focusing

Lecture 25 Stimulated Raman Scattering

Spontaneous Raman effect (1928)

Scattering of light by vibrations of the medium

Scattering of photons by optical phonons

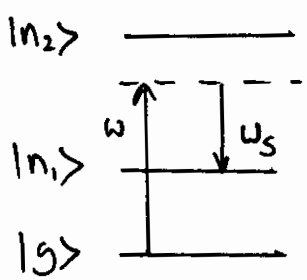
New spectral components

Stokes $\Rightarrow$ longer λ / smaller ~~larger~~ ω

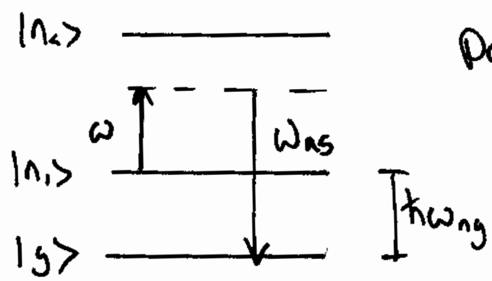
Anti Stokes $\Rightarrow$ shorter λ / larger ω

- Stokes components are an order of manitude larger than anti-Stokes.

Two photon process



Stokes



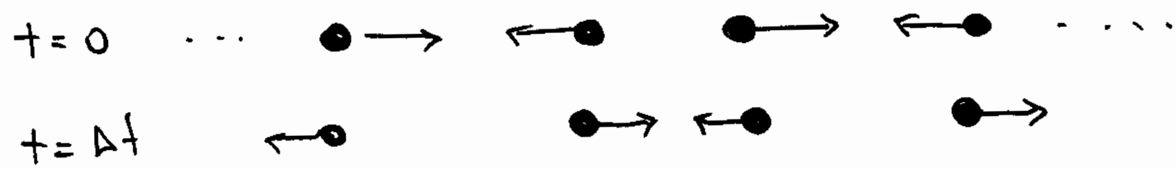
Anti Stokes

Anti-Stokes emission is smaller since at Room temp. mostly the $|g\rangle$ state is populated.

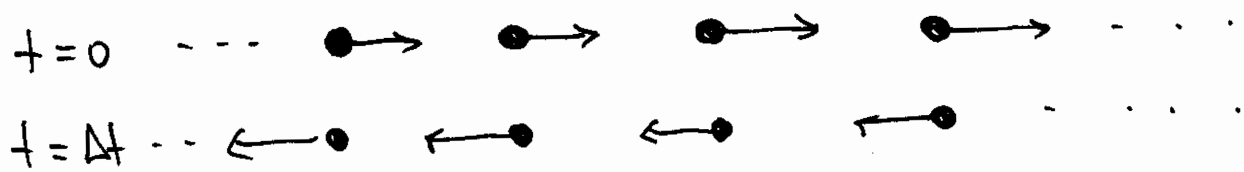
- $|n\rangle$ population smaller by $\exp(-\hbar\omega_{ng}/kT)$

— Optical Phonons

quantized lattice vibrations



Versus Acoustic Phonons



(Note: scattering of photons by acoustic phonons is called Brillouin scattering)

~~Stimulated Raman Scattering ⇒ Strong Pumps~~

~~Most stronger process~~

~~Forward scattering process~~

Properties of Spontaneous Raman Scattering

- Stokes intensity grows linearly with length of Raman material
- Process is weak scattering $\frac{\text{cross section}}{\text{Volume}}$ $\sigma \sim 10^{-6}\,\text{cm}^{-1}$

  (1 in 10^6 will be scattered)
- Two Photon Process

Stimulated Raman Scattering (SRS)

Four Photon Process

Slow resonance is excited by two optical fields at two frequencies that differ by the molecular resonant ~~frequency~~ frequency. These frequencies interact to produce sum + difference frequencies.

This is a nonlinear process since it depends on the product of fields.

Stokes generation dominates since

- upper levels are not previously excited
- phase matched for colinear propagation

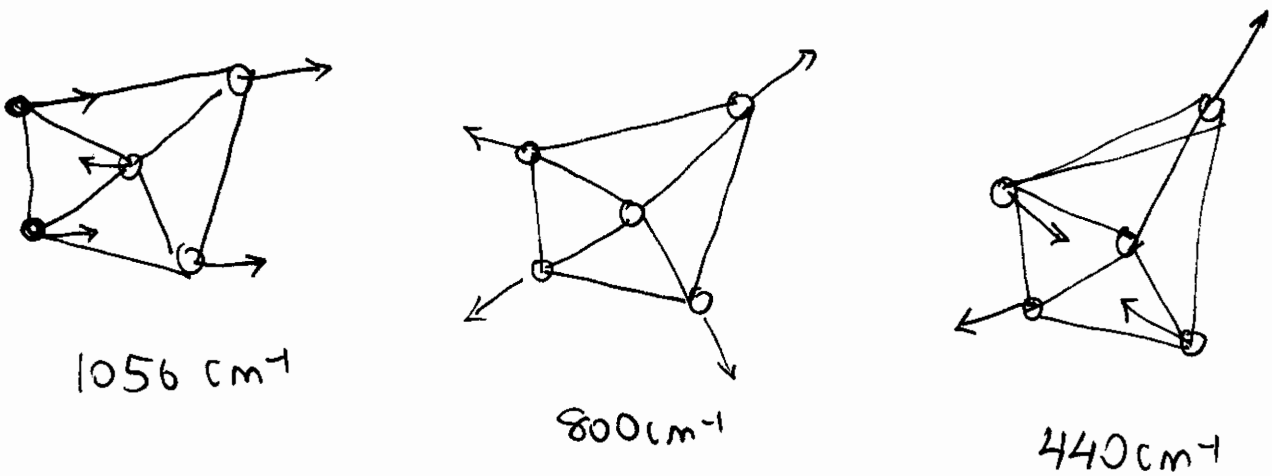
Unlike spontaneous Raman Scattering, SRS is a forward Scattering

Process of SRS

- Pump wave generates Stokes via spontaneous Raman Scattering
- The pump is constant so more Stokes photons are generated this also causes a slight excitation of the molecular resonance.
- As the scattered Stokes increases in intensity, the stimulated regime is reached.
- Now the Stokes wave interacts with the pump to further excite resonances, increasing the rate of frequency conversion.

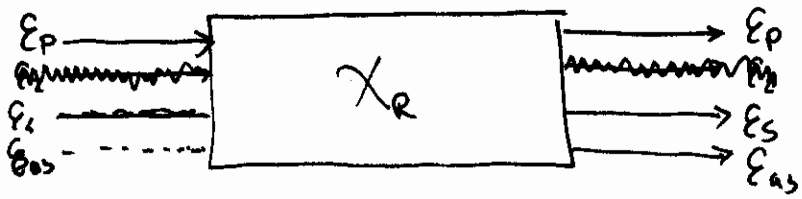
Raman Stretches in fused silica

SiO_2 tetrahedra



Classical Description of SRS

Treat medium as collections of atoms



(ignore anti stokes for this)

Atoms will obey the differential eq

$$m\ddot{q} + m\zeta_R\dot{q} + kq = e\mathcal{E} \qquad (1)$$

where $q \equiv$ atomic displacement $\quad k \equiv$ atomic spring const

$\zeta_R \equiv$ Raman damping $\quad \mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 = \mathcal{E}_p + \mathcal{E}_s$

Need to find the output electric fields

1) First find P + χ_R We can write

$$P_{NL} = \epsilon_0 N\alpha_p\, q(z,t)\, \mathcal{E}(z,t) \qquad (2)$$

$$P_{NL} = \epsilon_0 \chi_R \mathcal{E} \qquad (P = P_{NL} + P_L) \quad (3)$$

2) Solve for $q(z,t)$ using (1)

3) Find P using (2)

4) Find χ_R using (2) + (3)

$$\chi_R = -\frac{\epsilon_0 N\alpha_p^2}{8m\omega_0\zeta_R}\left(\frac{i + \delta_R}{1 + \delta_R^2}\right) \qquad \omega_0 \equiv \sqrt{k/m} \qquad \delta_R \equiv \frac{2}{\zeta_R}\left((\omega_2 - \omega_1) - \omega_0\right)$$

Threshold for SRS ⇒ Gain

$$\frac{dI_s}{dz} = g_R I_p I_s - \alpha_s I_s \qquad \left(g_R = \frac{\epsilon_0 N \chi_0 \omega_1 \alpha_p'^2}{4 m \omega_0 \zeta_0 c n^2} \right)$$

$$\frac{dI_p}{dz} = -\frac{\omega_p}{\omega_s} g_R I_p I_s - \alpha_p I_p$$

For small loss $\alpha_p \cong 0$ $\alpha_s \simeq 0$ we can rewrite

$$\frac{d}{dz}\left(I_s + \frac{\omega_s}{\omega_p} I_p \right) = 0$$

Ignoring pump depletion with loss

$$\frac{dI_p}{dt} = 0$$

$$\frac{dI_s}{dz} = g_R I_0 \exp(-\alpha_p z) I_s - \alpha_s I_s$$

Solution

$$\boxed{\begin{aligned} I_s(L) &= I_s(0) \exp(g_R I_0 L_{eff} - \alpha_s L) \\ L_{eff} &= \frac{1}{\alpha_p}\left(1 - \exp(-\alpha_p L)\right) \end{aligned}}$$

Where does $I_s(0)$ come from? ⇒ Spontaneous Raman Scattering

5) Find $\mathcal{E}_s$ + $\mathcal{E}_{as}$ by putting $P = P_{NL} + P_L$ into the wave equation.

Can rewrite wave eq as

$$\frac{dI_s}{dz} = g_R I_p I_s - \alpha_s I_s$$

$$\frac{dI_p}{dz} = -\frac{\omega_p}{\omega_s} g_R I_p I_s - \alpha_p I_p$$

where $$g_R \equiv \frac{\epsilon_0 N \omega_p \alpha_{pi}^2 \sqrt{\mu_0/\epsilon_0}}{4 m \omega_0 \zeta_R c n^2} \frac{1}{(1 + \delta_R^2)}$$

Define Raman threshold

Input pump power ~~at~~ which the output Stokes power becomes equal to the output pump power.

initial pump power

$$P_s(L) = P_p(L) = P_0 \exp(-\alpha_p L)$$

Approximation

$$\boxed{g_R P_0^{cr} L_{eff} / (\text{Area}) \simeq 16}$$

($g_R \simeq 10^{-13}$ m/W at 1μm for fused silica)

Raman Shift in fused silica

$$\Delta \nu = 440 \text{ cm}^{-1}$$

$$\Delta f = 13.2 \text{ THz}$$

at 1550nm

$$\Delta \lambda = \frac{\lambda_0^2}{c} \Delta f = \frac{(1550 \text{nm})^2}{(300 \text{ nm/fs})} \left(0.0132 \ \text{1/fs}\right)$$

$$\simeq \boxed{105.7 \text{ nm}}$$

at 800nm

$$\Delta \lambda = \frac{(800 \text{ nm})^2}{300 \text{ nm/fs}} (0.0132 \ \text{1/fs}) = \boxed{28 \text{ nm}}$$

Raman Active vibration

$\Rightarrow$ No change dipole moment due to atomic displacements.

Effect of SRS in Fibers

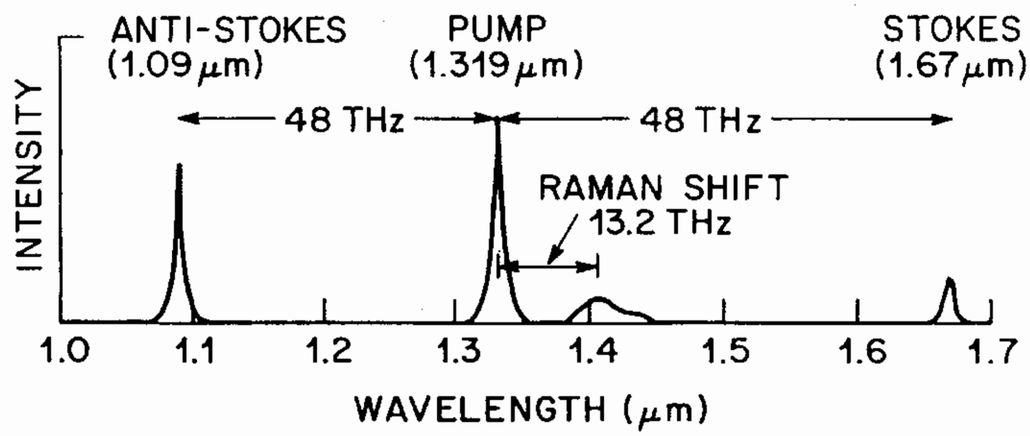
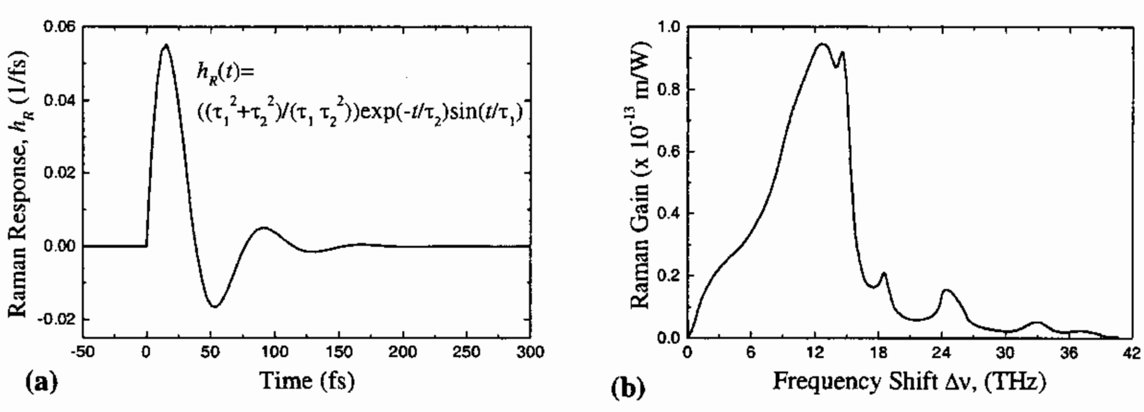


Figure 4-1 FWM Stokes and anti-Stokes components due to propagation in standard SMF near the zero dispersion wavelength (1319 nm). Stimulated Raman scattering also is present which produces spectral components near 1400 nm. Figure reproduced from Ref. [Lin, 1981 #32].

This plot shows Raman Scattering in a optical fiber
The vibrations of fused silica has a resonance at 13.2TH
Note that the stoke + anti stokes components <u>are</u> <u>not</u>
due to SRS but due to partially degenerate FWM.

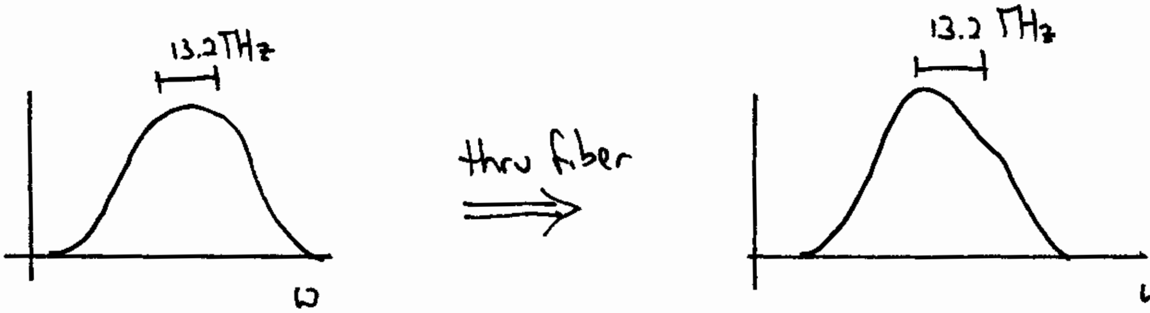


For optical fibers

$$P_0^{cr} \simeq 100\,W \text{ for } 10m$$

Interpulse Stimulated Raman Scattering

Short pulses may have a bandwith larger than 13.2 THz. As the pulse propagates, SRS will cause a shift of the spectrum to ~~shorter~~ longer wavelengths.



The process continues thru the length of the fiber. This effect is called the self frequency shift: you can change the center freq. of a pulse via SRS just by propagation in a Raman medium.

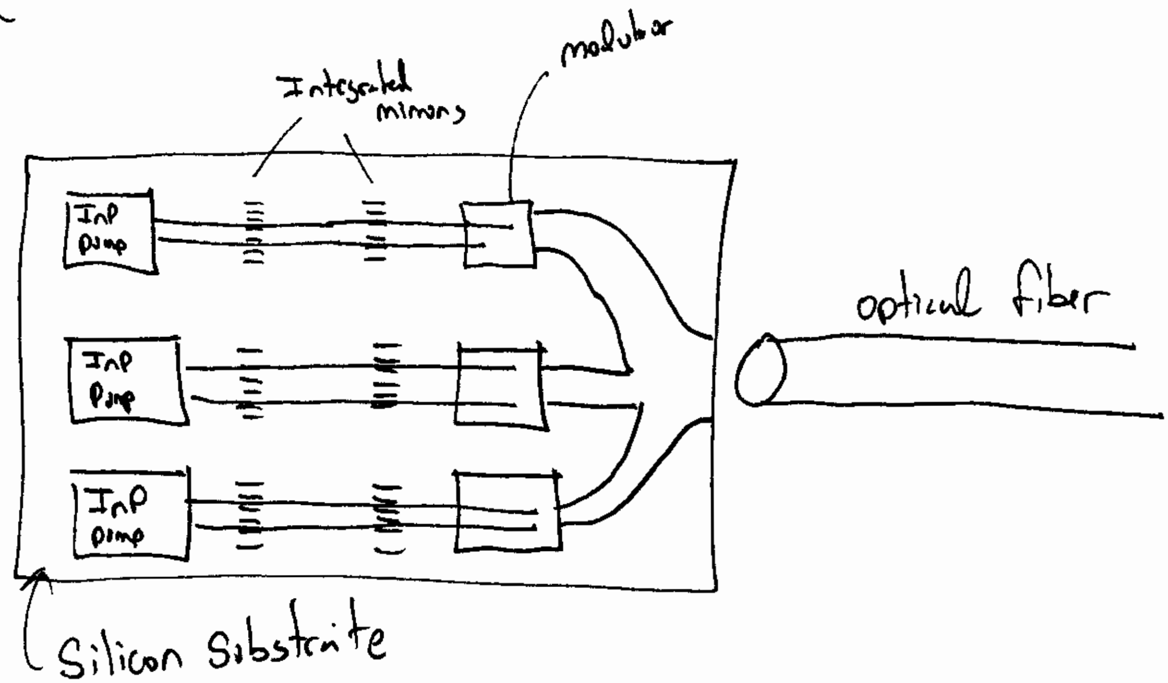
Intel + the Silicon "Laser"

A few years back, Intel announced the demonstration of a laser using silicon. This was a big deal since silicon does not directly lase (it has an indirect bandgap).
~~making a laser in a material that can lase~~

The silicon laser would allow a laser on your pentium chip, opening the door to computers that use light instead of electrons.

However, the problem here it isn't a laser, but a Raman amplifier. It uses a InP pump laser

Raman effect is 10000 times stronger in Silicon than fused silica

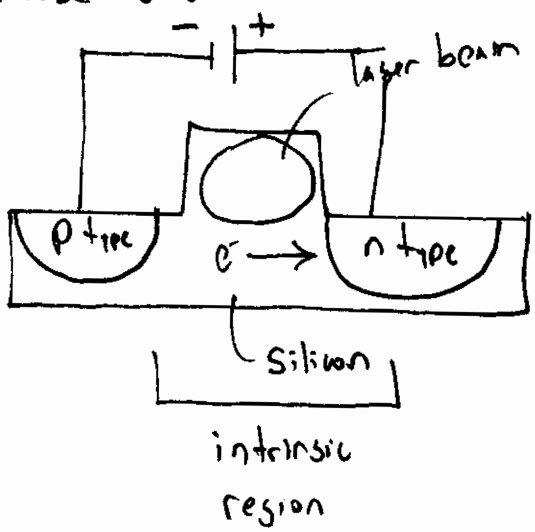


Two Photon absorption

- Silicon is transparent to IR light

For high powers two photons cause an atom to free its electron. ~~These free electrons~~ If the intensity is high enough the rate of generating free electrons will exceed the recombination rate. The free electrons will cause the material to have a higher absorption + prevent lasing + Raman Gain

Intel's solution was to use a p-i-n junction to "Sweep" those electrons out



p ≡ p type

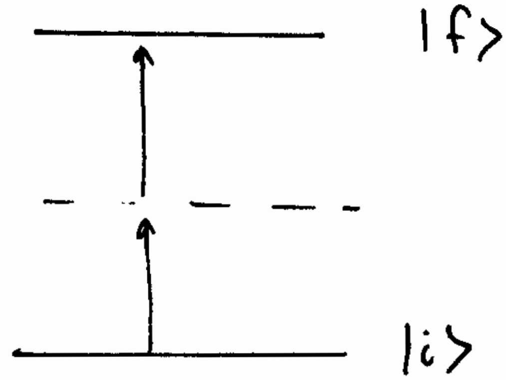
i ≡ intrinsic

n ≡ n. type

Two Photon Absorption

Nonlinear change to the absorption

Two photons simultaneously absorped to excite a state



absorption cross section is smaller than single photon process.

Corresponds to $\chi^{(3)}$ process

Stokes / Anti-Stokes Coupling

Equations using the SVEA

Extend the previous derivation: set susceptibility for stokes + Antistokes.

$$\frac{dA_p}{dz} = 0$$

$$\frac{dA_s}{dz} = -\alpha_s A_s + K_s A_2^* e^{i\Delta k z}$$

$$\frac{dA_{as}}{dz} = -\alpha_{As}^* A_{as}^* + K_{as}^* A_1 e^{-i\Delta k z}$$

Gaussian units (sorry)

$$\alpha_s = \frac{12\pi i \omega_s}{n_s c}\chi_R(\omega)|A_p|^2$$

$$K_s = \frac{6\pi i \omega_s}{n_s c}\chi_R A_p^2$$

$$\Delta k = 2\bar{k}_p - \bar{k}_s - \bar{k}_{as}$$

$\alpha_s \equiv$ Real part of Raman susceptibility

$K \equiv$

Solutions

$$A(z) = \left[(\quad)e^{s_+ z} + (\quad)e^{s_- z}\right]e^{+i\Delta k z/2}$$

$$A_2^*(z) = \left[(\quad)e^{s_+ z} + (\quad)e^{s_- z}\right]e^{-i\Delta k/2}$$

$$g_\pm = \pm\left[K_1 K_2 - (\Delta k/2)^2\right]^{1/2}$$ ⇐ Represents coupled gain

$$\simeq i\frac{\Delta k}{2}\left[1 - i\frac{4\chi_R}{\Delta k}\right]$$

(− ⇒ Stokes
 + ⇒ anti stokes)

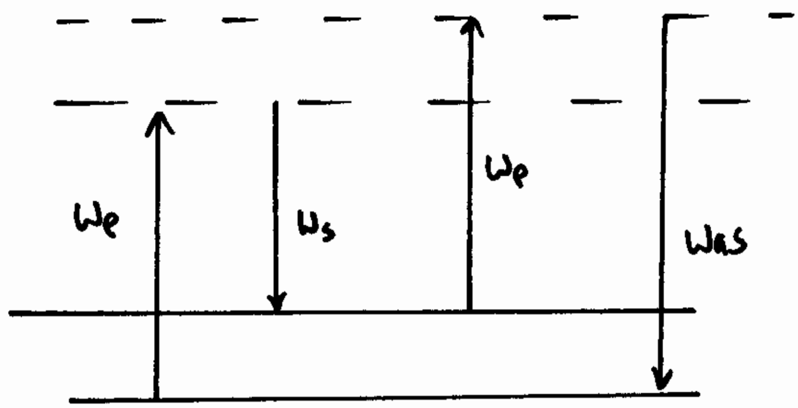
at $\Delta k = 0$ anti-stokes wave is strongly coupled to Stokes that prevents it to grow exponentially } Strongly matched coupled

$\Delta k > 0$ Strong stokes growth

$\Delta k < 0$ Strong anti-stokes growth

} Strongly mismatched Stokes + Antistokes are decoupled

Process for gain for Antistokes



Parametric process

Coherent Anti-Stokes Raman Scattering (CARS)

- Stimulated process that produces more photons than spontaneous Raman spectroscopy

- Developed in 1965 by the Ford Motor Co (no joke)

- Vibrationally sensitive nonlinear optical technique

Classical Description

Raman active medium of frequency ω_v.

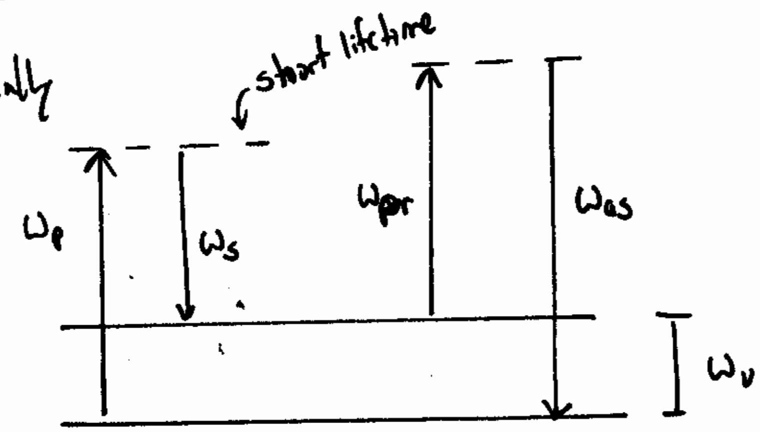
Medium driven by difference $\omega_p - \omega_s \Rightarrow$ tune ω_p

When $\omega_p - \omega_s$ is close to ω_v the medium responds with large amplitude.

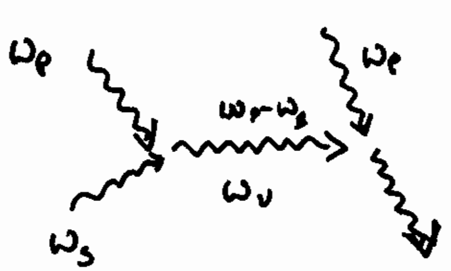
A third probe beam experiences a large change in index due to the large amplitude vibrations

Anti Stoke light is emitted

Quantum mechanically



$\omega_p = \omega_{pr}$



$\omega_{as} = 2\omega_p - \omega_s$

Quantum Mechanical

- joint action of pump + stokes establishes a coupling between the ground state + vibrationally excited state
- molecule is in coherent superposition of the two states
- The probe beam investigates the coherence between states
  It promotes to a virtual state
- The molecule falls to the ground state emitting as photon.

- Probe beam interrogates medium superposition of states —

Lecture 27 Quantum Mechanical Description of Nonlinear optical susceptibilities

So far, we have describe nonlinear optics in classical terms treating the medium as a collection of dipoles with a continous spread of energies.

The question is, do we lose "Something" treating the system classically? Does a Quantum description provide more or a better explaination? To answer these questions, we will need to develope a quantum treatment. More specifically, we will use a <u>semi-classical</u> <u>treatment</u>

- treat material quantum mechanically
- treat E/m field classically

We can do this since the number of photons are large. (High intensities)

Now we really have not discused what a photon is, this will come later in our discussion of quantum optics

Density Matrix Formalism

A single quantum mechanical state can be described by the state vector

$$|\psi\rangle = \sum_n c_n |\psi_n\rangle$$

This is a <u>pure state</u>. If I have a collection of N quantum systems I cannot use a state vector to describe the total system. This is called a mixed state. Here, we have an ensemble of N systems, n_i are in state $|\psi_i\rangle$

The ensemble is described by an occupancy number n_i

A way to assembly the information on an ensemble is the density matrix

$$\boxed{\hat{\rho} = \sum_i p_i \, |\psi_i\rangle\langle\psi_i|}$$

$$p_i = \frac{n_i}{N}$$

probability to be picked Randomly

pure state : $p_i = 0$ except one state

Ensemble average of property Ω

$$\boxed{\langle\bar{\Omega}\rangle = \sum_i p_i \langle\psi_i|\Omega|\psi_i\rangle = \mathrm{Tr}(\hat{\rho}\,\Omega)}$$

Here there a two averagings 1) Quantum average $\langle\psi_i|\Omega|\psi_i\rangle$ for each system i

2) Classical average over different states $|i\rangle$

Remember orthonormality

$$1 = \sum_j |j\rangle\langle j|$$

Consider

$$\mathrm{Tr}(\Omega\rho) = \sum_i \langle j|\Omega\rho|j\rangle$$

$$= \cancel{\sum_i \sum_j \langle j|\Omega|i\rangle\langle i|\rho|j\rangle}$$

$$= \sum_j \sum_i \langle j|\Omega|i\rangle\langle i|j\rangle p_i \quad \text{(use definition of } \rho\text{)}$$

$$= \sum_i \sum_j \langle i|j\rangle\langle j|\Omega|i\rangle p_i$$

But $\quad 1 = \sum_j |j\rangle\langle j|$

so

$$\mathrm{Tr}(\Omega\rho) = \sum_i \langle i|\Omega|i\rangle \rho_i$$

$$= \langle \bar{\Omega} \rangle \Rightarrow \text{Ensemble average of } \Omega$$

The density matrix contains all statistical information on the ensemble.

~~Define density matrix operator~~

$$\cancel{\hat{\rho} = |j\rangle\langle j|}$$

$$\cancel{\mathrm{Tr}(\hat{\rho}) = \sum_n \langle i|j\rangle\langle}$$

For a pure state

$$\hat{\rho} = |\psi_i\rangle\langle\psi_i|$$

$$\cancel{\mathrm{Tr}(\hat{\rho}) = \sum_j\sum_i \langle\psi_i|\psi_j\rangle\langle\psi_j|\psi_i\rangle\rho_i = \rho_i}$$

$$\mathrm{Tr}(\hat{\rho}) = \sum_n \langle n|\hat{\rho}|n\rangle = \sum_i\sum_n \langle n|i\rangle \rho_i \langle i|n\rangle$$

$$= \sum_i\sum_n \rho_i \langle i|n\rangle\langle n|i\rangle = \sum_i \rho_i \langle i|i\rangle = 1$$

Also $\quad \mathrm{Tr}(\hat{\rho}^2) = 1$

Important Result

$$\boxed{\begin{array}{l} Tr(\hat{\rho}^2) = 1 \\ Tr(\hat{\rho}^2) < 1 \end{array}}$$

pure state / ensemble

mixed state

The density operator describes a mixed state

More properties of the density operator

$$\rho^{\dagger} = \rho \qquad \rho^2 = \rho \text{ (pure ensemble)}$$

$$Tr\,\rho = 1 \qquad Tr(\rho^2) \leq 1$$

We can express the density matrix in matrix space

$$\rho_{nm} = \sum_i P_i\, C_m^* C_n$$

where $\psi = \sum_n C_n |n\rangle$

↑ energy eigensolutions to Schrödinger Eq

Physical ~~Int~~ Interpretation of density matrix

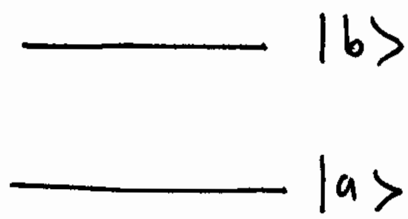
Diagonal Elements : probability of ~~state~~ system to be in eigenstate $n \Rightarrow \rho_{nn}$

Off diagonal elements : "Coherence" between levels n + m

$\rho_{nm} \neq 0$ if there is a coherent superposition of $|\psi_n\rangle$ + $|\psi_m\rangle$

The off diagonal terms are important since they are proportional to an induced dipole moment.

Let's look at an example: Two level ~~system~~ atom



Describe specific state s

$$|\psi_s\rangle = C_a^s\,|a\rangle + C_b^s\,|b\rangle$$

Density matrix

$$\hat{\rho} \Rightarrow \begin{pmatrix} \rho_{aa} & \rho_{ab} \\ \rho_{ba} & \rho_{bb} \end{pmatrix} \qquad \rho_{nm} = \sum_s P_s\, C_m^{s*}\, C_n^s$$

Dipole moment operator

$$\hat{\mu} \Rightarrow \begin{pmatrix} 0 & \mu_{ab} \\ \mu_{ba} & 0 \end{pmatrix} \qquad \mu_{ij} = -e\,\langle i|\hat{z}|j\rangle$$

Find expectation value of dipole $\langle\bar{\mu}\rangle = Tr(\hat{\rho}\hat{\mu})$

$$\hat{\rho}\hat{\mu} \Rightarrow \begin{pmatrix} \rho_{ab}\mu_{ab} & \rho_{aa}\mu_{ab} \\ \rho_{bb}\mu_{ba} & \rho_{ba}\mu_{ab} \end{pmatrix}$$

Then

$$\boxed{\langle\bar{\mu}\rangle = Tr(\hat{\rho}\hat{\mu}) = \rho_{ab}\mu_{ba} + \rho_{ba}\mu_{ab}}$$

Time dependences of Ensemble Systems

Expectation values are a function of time

$$\begin{pmatrix} \sum_s C_a^s(t)\, C_a^{s*}(t) & \sum_s C_a^s(t)\, C_b^{s*}(t) \\ \sum_s C_b^s(t)\, C_a^{s*}(t) & \sum_s C_b^s(t)\, C_b^{s*}(t) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_s |C_a^s(t)|^2 & \sum_s C_a(t)\, C_b^*(t) \\ \sum_s C_b^s(t)\, C_a^s(t) & \sum_s |C_b^s(t)|^2 \end{pmatrix}$$

on diagonal terms $\Rightarrow$ positive + Real

off diagonal terms $\Rightarrow$ negative or complex

Coherences decay due to dephasing : Rates

Cancellation of emitted light.

Ensemble average of atomic wavefunctions add to zero over time

Two timescales

$T_1 \Rightarrow$ relaxation time	ρ_{aa} or ρ_{bb}	(level lifetime)	
$T_2 \Rightarrow$ dephasing time	ρ_{ab} or ρ_{ba}	(coherence lifetime) collision rate	

Typically $T_2 << T_1$

Lecture 28 Nonlinear Optical Perturbation Theory

We wish to solve for the time dependence of $\hat{\rho}$. Use Liouville Eq.

$$\boxed{\dot{\rho}_{nm} = \frac{-i}{\hbar}[\hat{H}, \rho]_{nm}} \quad \begin{pmatrix}\text{interaction} \\ \text{picture}\end{pmatrix} \text{ or } \begin{pmatrix}\text{Dirac} \\ \text{picture}\end{pmatrix} \quad (1)$$

$$\boxed{\hat{H}(t) = \hat{V}(t) + \hat{H}_0}$$

However, we include a phenomenological damping term

$$\dot{\rho}_{nm} = \frac{-i}{\hbar}[\hat{H}, \hat{\rho}]_{nm} - \gamma_{nm}(\rho_{nm} - \rho_{nm}^{eq})$$

- ρ_{nm} relates to ρ_{nm}^{eq} at rate γ_{nm}

Specifically

$$\rho_{nm}^{eq} = 0 \quad \text{for} \quad n \neq m$$

$$\gamma_{nm} = \gamma_{mn} \equiv 1/T_2 \Rightarrow \text{dephasing rate}$$

So

$$\boxed{\dot{\rho}_{nm} = \frac{-i}{\hbar}[\hat{H}, \hat{\rho}]_{nm} - \frac{1}{T_2}(\rho_{nm} - \rho_{nm}^{eq})} \quad (2)$$

If we ignore the dephasing, the differential eq (1) can be solved

Quantum "Pictures"	Heisenberg	Interaction	Schrödinger
State Ket	No change	Evolution by $V(t)$	Evolution by $H(t)$
Observable	Evolution by H	Evolution by H_0	No change

Some notes on Quantum Pictures

	Heisenberg picture	Schrödinger Picture
State ket	Stationary	Moving
Obs.	Moving	Stationary
Base ket	moving oppositely	Stationary

Moving State

$$c(t) = \langle a' | \left(U | a, t=0 \rangle \right) \quad SP$$

$$c(t) = \left(\langle a' | U \right) | a, t=0 \rangle \quad HP$$

Schrödinger Picture

State ket has time dependence

$$|\alpha, t_0; t\rangle = U(t, t_0) |\alpha, t_0\rangle$$

$$U(t, t_0) = \exp\left(\frac{-i H (t - t_0)}{\hbar} \right)$$

Heisenberg Picture

Observables have time dependence

$$\boxed{\frac{d\Omega^{(H)}}{dt} = \frac{-i}{\hbar} [H, \Omega^{(H)}]}$$

or

$$\boxed{\frac{d\Omega^{(H)}}{dt} = \frac{1}{i\hbar} [\Omega^{(H)}, H]}$$

Relationship between Schrödinger + Heisenberg Pictures

$$\boxed{\Omega^{(H)} = U^{\dagger} \Omega^{(S)} U}$$

Use Perturbation theory to solve for $\rho_{nm}(t)$

Write Hamitonian

$$\hat{H} = \hat{H}_0 + \hat{V}(t)$$

Interaction $\rightarrow$ dipole $\quad V(t) = -\hat{\mu} \cdot \bar{E}(t) \quad \bar{\mu} = e\bar{r}$

classical E/m field

$$[\hat{H}, \hat{\rho}] = [\hat{H}_0, \hat{\rho}] + [\hat{V}(t), \hat{\rho}]$$

$\hat{H}_0$ satisfies time independent Schrodinger Eq

$$H_0 |\psi_n\rangle = E_n |\psi_n\rangle$$

or

$$H_{0,nm} = E_n \delta_{nm}$$

$|\psi_n\rangle$ eigensolutions to time-independent S.E.

So

$$[\hat{H}_0, \rho] = \hat{H}_0 \hat{\rho} - \hat{\rho} H_0 = \sum_v \left(H_{0,nv} \rho_{vm} - \rho_{nv} H_{0,vm}\right)$$

$$= \sum_v \left(E_n \delta_{nv} \rho_{vm} - \rho_{nv} \delta_{vm} E_m\right)$$

$$= E_n \rho_{nm} - E_m \rho_{nm} = (E_n - E_m) \rho_{nm}$$

Define $\quad \omega_{nm} \equiv \dfrac{E_n - E_m}{\hbar}$

Example Time dependence for two level System

$$\dot{\rho}_{ba} = -i\omega_{ba}\rho_{ba} - \frac{1}{T_2}\rho_{ba}$$

$$\dot{\rho}_{bb} = \frac{1}{T_1}(\rho_{11} - \rho_{11}^{e})$$ equilibrium levels

$$\rho_{ab}(t) = \rho_{ba}^{*}(t) \qquad \rho_{aa}(t) = 1 - \rho_{bb}(t)$$

Back to perturbation theory $\quad (U(t)U^{*}(t') = U(t-t'))$

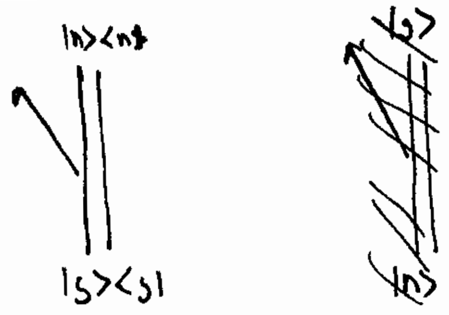
So in the interaction picture

$$V_I(t_1)\,\rho_I(t_0)\,V_I(t_2) = U^{*}(t)[-\mu\cdot E(t_1)]U(t_1)\,U^{*}(t_0)\rho^{(0)}(t_0)U(t_0)$$
$$U^{*}(t_2)[-\mu\cdot E(t_2)]\,U(t_2)$$
$$= U(t-t_1)(-\mu\cdot E)\,U(t_1-t_0)\,\rho^{0}(t_0)\,U(t_0-t_2)[-\mu\cdot E(t_2)]\,U(t_2-t)$$

For 2nd order

$$\left(\frac{1}{i\hbar}\right)\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\; U(t-t_1)[-\mu\cdot E(t_1)]\,U(t_1-t_0)\,\rho^{(0)}(t_0)$$
$$U(t_0-t_2)[-\mu\cdot E(t_2)]\,U(t_2-t)$$

For n^{th} order we have $\Rightarrow 2^n n!$ terms!



$$\frac{1}{\hbar}\left(\omega - \omega_{gn} + i\ 1/T_2\right)^{-1}$$

$$\hbar\left(\omega + \omega_{ng} + i\ 1/T_2\right)^{-1}$$

$$\text{Term} \Rightarrow -\frac{\rho_{nn}\ \mu_{ng}^{(1)}\ \mu_{gn}^{(1)}}{\hbar\left(\omega + \omega_{ng} + i\,1/T_2\right)}$$

But $\rho_{nn} = \rho_{gg} + 1$

So

$$\chi_{ij}^{(1)} = \rho_{gg}^{(0)}\frac{N}{\hbar}\left[\sum_{gn}\frac{\mu_{ng}^{(i)}\ \mu_{gn}^{(j)}}{\left(\omega + \omega_{ng} + i\ 1/T_{2ng}\right)} + \frac{\mu_{ng}^{(j)}\ \mu_{gn}^{(i)}}{\left(\omega - \omega_{ng} + 1/T_{2ng}\right)}\right]$$

To get classical result, define the oscillator strength

$$f_{ng} = \frac{2m\omega_{ng}\,|\mu_{ng}|^2}{3\hbar e^2} \qquad \rho_{gg}^{(0)} = 1$$

Dropping non resonant term

So

$$\chi_{ij}^{(1)} \simeq f_{na}\frac{Ne^2/m}{\left(\omega_{ng}^2 - \omega^2 - 2i\omega\ 1/T_2\right)}$$

Lorentzian

So

$$\boxed{\dot{\rho}_{nm} = -i\,\omega_{nm}\,\rho_{nm} - \frac{i}{\hbar}\left[\hat{V}, \hat{\rho}\right]_{nm} - \frac{1}{T_2}\left(\rho_{nm} - \rho_{nm}^{eq}\right)}$$

Solution $\Rightarrow$ Expand $\quad \rho(t) = \sum_n \rho^{(n)}(t)$

$$\rho^{(n)}(t) = \left(\frac{1}{i\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \left[V(t_1), \left[V(t_2), \cdots \left[V(t_n), \rho(t_0)\right] \cdots\right]\right]$$

(Dyson Series)

Note $\quad t_0 \le t_n \le t_{n-1} \cdots \le t_1 \le t \quad \Leftarrow$ time ordering

$\rho^{(n)}$ contains $2n$ terms

$$\left[V(t_1), \left[V(t_2), \rho(t_0)\right]\right] = V(t_1)V(t_2)\rho(t_0) - V(t_1)\rho(t_0)V(t_2) - V(t_2)\rho(t_0)V(t_1) + \rho(t_0)V(t_2)V(t_1)$$

Potential in interaction picture

$$V_I(t) = U^*(\mu \cdot E)\,U \qquad U \equiv \text{unitary matrix } \exp(-i H_0 t/\hbar)$$

Density matrix in interaction picture

$$\hat{\rho}_I = U^* \rho\, U$$

Expand terms from the Dyson series: <u>one</u> term

- In interaction picture

$$V_I(t) = U^\dagger(t)(-\bar{\mu}\cdot\bar{E})\,U(t) \qquad U(t) = \exp(-i\,tH_0/\hbar)$$

So $\quad V_I(t_1)\,\rho_I(t_0)\,V_I(t_2)$

$$= U^\dagger(t_1)\,[-\mu\cdot\bar{E}(t_1)]\,U(t_1)\,\overbrace{U^\dagger(t_0)\,\rho_I^{(0)}(t_0)\,U(t_0)}\,U^\dagger(t_2)(-\mu\cdot E(t_2))\,U(t_2)$$

But $\rho_I = U^\dagger(t)\rho(t)U(t)$

So $\Rightarrow$ $$= \underbrace{\left[U(t-t_1)\,[-\mu\cdot E(t_1)]\,U(t_1-t_0)\right]}_{\text{left}}\rho^0(t_0)\overbrace{\left[U(t_0-t_2)(-\bar{\mu}\cdot E(t_2))U(t_2-t)\right]}^{\text{Right}}$$

- <u>Left side</u> (~~Ket~~ Bra evolution)
propagation from

$$t_0 \rightarrow t_1 \rightarrow t$$

<u>Right side</u> (~~B~~ ket evolution)
propagation from

$$t_0 \rightarrow t_2 \rightarrow t$$

This is just <u>one</u> term, there will be many more.

- We need two sided diagram to handle both the ket + bra evolution

Time scales

Medium	T_1 (s)	T_2	σ (cm^2)
Solids doped with resonant atomic systems	$10^{-3} - 10^{-6}$	$10^{-11} - 10^{-14}$	$\sim 10^{-20}$
dye molecules	$10^{-8} - 10^{-12}$	$10^{-13} - 10^{-14}$	$\sim 10^{-16}$
semiconductors	$10^{-4} - 10^{-12}$	$10^{-12} - 10^{-14}$	

$T_1 \Rightarrow$ lifetime, longitudinal relaxation time

$T_2 \Rightarrow$ dephasing time, transverse relaxation time

Deriving the Polarization using perturbation theory $\Rightarrow \chi^{(n)}$

$$\langle \bar{P} \rangle = \langle \bar{P}^{(1)} \rangle + \langle \bar{P}^{(2)} \rangle + \langle \bar{P}^{(3)} \rangle \cdots$$

Where $\langle \bar{P}^{(n)} \rangle = T_r \langle \rho^{(n)} \bar{P} \rangle$

where

$$\begin{cases} V = e\vec{r}\cdot\vec{E} \\ E = E_1(\omega_1) + E_2(\omega_2) + \cdots\cdots \end{cases} \qquad \vec{P} = -Ne\vec{r} \qquad N \equiv \frac{\#\ \text{dipoles}}{\text{volume}}$$

50

$$\chi_{ij}^{(1)} = \frac{P_i^{(1)}(\omega)}{\epsilon_0 E_j(\omega)} \qquad \chi_{ijkl}^{(3)} = \frac{P_i^{(3)}(\omega)}{\epsilon_0 E_j(\omega_1) E_k(\omega_2) E_l(\omega_3)}$$

$$\chi_{ijk}^{(2)} = \frac{P_i^{(2)}(\omega)}{\epsilon_0 E_j(\omega_1) E_k(\omega_2)}$$

Lets write out the first ~~two~~ + 2nd order solutions for $\rho_{nm}^{(n)}$

$$\rho_{nm}^{(1)}(\omega_j) = \frac{[V(\omega_j)]_{nm}}{\hbar(\omega_j - \omega_{nm} + i\, 1/T_2)} \left(\rho_{mm}^{(0)} - \rho_{nn}^{(0)}\right)$$

$$\rho_{nm}^{(2)}(\omega_j + \omega_k) = \frac{[V(\omega_j), \rho^{(1)}(\omega_k)]_{nm} + [V(\omega_k), \rho^{(1)}(\omega_j)]_{nm}}{\hbar(\omega_j + \omega_k - \omega_{nm} + i\, 1/T_2)}$$

Now we use

$$\langle \vec{P}^{(n)} \rangle = \mathrm{Tr}\left(\rho^{(n)} \vec{P}\right)$$

+ find the polarization using the $\rho^{(n)}$ from the Dyson series

For $P^{(1)} \Rightarrow$ Two term $\quad \boxed{(2^n n!)} \Rightarrow$ Terms from Dyson Series

$P^{(2)} \Rightarrow$ Eight terms

$P^{(3)} \Rightarrow$ 48 terms

Then to find $\chi^{(n)}$ use the above expressions

$$\chi^{(2)} = \frac{P^{(2)}}{E(\,)\,E(\,)}$$

$$\chi^{(3)} = -\frac{Ne^2}{\hbar}\left[\sum_{s,n,n'} \left(\begin{array}{c}\text{terms from}\\ \text{perturbation}\\ \text{theory}\end{array}\right)\right]$$

Feynman diagrams for continuous wave case

Used to keep track of terms in perturbation ~~solutions~~ calculations. The density matrix involves products of two wavefunctions so Two diagrams are needed.

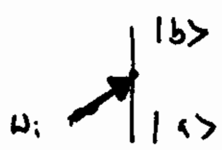
All diagrams give a simple picture of the corresponding physical process, allowing one to write down the corresponding mathematical expression.

1) Start system at $|g\rangle \rho_{gg}^{(0)} \langle g|$

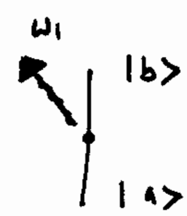
2) Draw ket on left, bra at right

3. A vertex bringing $|a\rangle$ to $|b\rangle$

On left



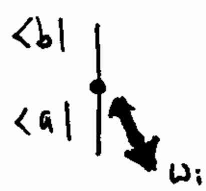
absorption



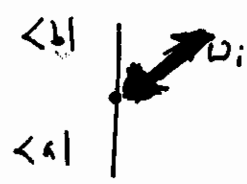
emission

$\Longrightarrow$ matrix elements $(1/i\hbar)\langle b|V|a\rangle$

On Right



emission



absorption

$\Longrightarrow$ matrix elements $-(1/i\hbar)\langle a|V|b\rangle$

4. Propagation from j^{th} vertex to $(j+1)$ along $|l\rangle\langle k|$ described by

$$\frac{1}{\hbar} \pm\left[\left(\sum_{i=1}^{j} \omega_i - \omega_{lk} + i\, 1/T_2\right)\right]^{-1}$$

$+$ ket side absorption
$-$ bra side absorption

5.

Sign of term	left	right
	+ if absorption	− if absorption
	− if emission	+ if emission

(OR)

	left	Right
Abs.	+	−
Emission	−	+

Term Sign $\pm [\quad]^{-1}$

6. Final State of System given by final ket + bra

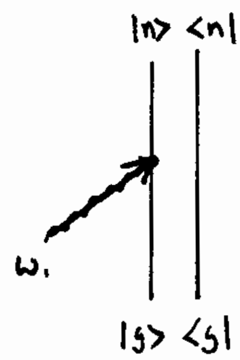
7. Product of all factors yields the susceptibility

Example : Linear optics — Absorption

Involves only one photon

No virtual processes

$|g\rangle$ to $|n\rangle$



Absorption

So

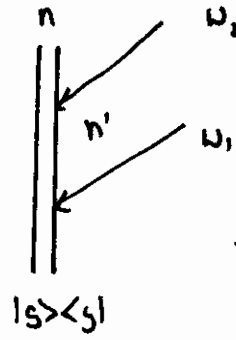
$$\boxed{+\frac{\rho_{gg}\,\mu^{(1)}_{gn}\,\mu^{(1)}_{ng}}{\hbar\left(-\omega_{ng}+\omega_1+i\,{}^1\!/\!(T_2)_{ng}\right)}}$$

To get classical result define oscillator strength

$$f_{ng} \equiv \frac{2m\omega_{ng}|\mu_{ng}|^2}{3\hbar e^2}$$

This is a complex Lorentzian function like we derived using our classical harmonic oscillator model.

Absorption on bra side



$$-\left(\omega_2+\omega_1-\omega_{n'g}+1/T_2\right)^{-1}$$

$$-\left(\omega_1-\omega_{ng}+1/T_2\right)^{-1}$$

Emission on ket side

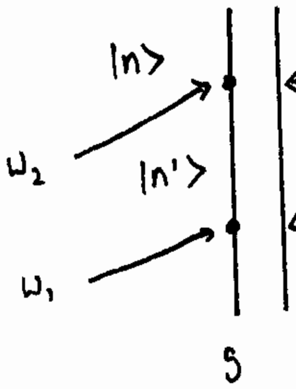
$$\left(\omega_1+\omega_{nn'}+1/T_2\right)^{-1}$$
$$=\left(\omega_1+\omega_{n'n}+1/T_2\right)^{-1}$$

Emission on bra Side

$$-\left(\omega_1+\omega_{n'n}+1/T_{2\,n'n}\right)^{-1}$$
$$=-\left(\omega_1+\omega_{nn'}+1/T_{2\,n'n}\right)^{-1}$$

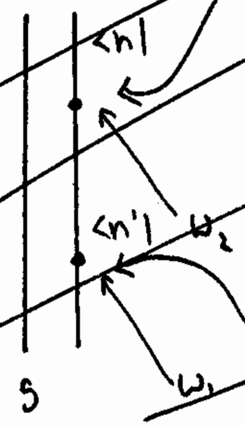
More Examples

Absorption



$$(\omega_1 - \omega_{n'g} + i\ 1/T_{ng})^{-1}$$

$$(\omega_2 + \omega_1 - \omega_{nn'} + i\ 1/T_{ng})^{-1}$$



$$\cancel{-(\omega_2 + \omega_1 - \omega_{n'n} + i\ 1/T_{nn'})^{-1}}$$

$$\cancel{-(+\omega_1 - \omega_{n'g} + i\ 1/T_{ng})^{-1}}$$

~~← here because on bra side due to complex conjugate~~

$$\cancel{= -(+\omega_1 + \omega_{gn'} + i\ 1/T_{ng})^{-1}}$$

$$\cancel{= +((\omega_1 + \omega_{gn'} + i\ (1/T_{ng})))^{-1}}$$

So

$$\boxed{\chi_{ij}^{(1)} = + \frac{Ne^2}{\hbar}\left(\frac{\rho_{gg}\,\mu_{gn}^{(i)}\,\mu_{ng}^{(j)}}{-\omega_{ng} + \omega_1 + i\left(1/T_2\right)_{ng}}\right)}$$

Relate to classical result by defining oscillator strength

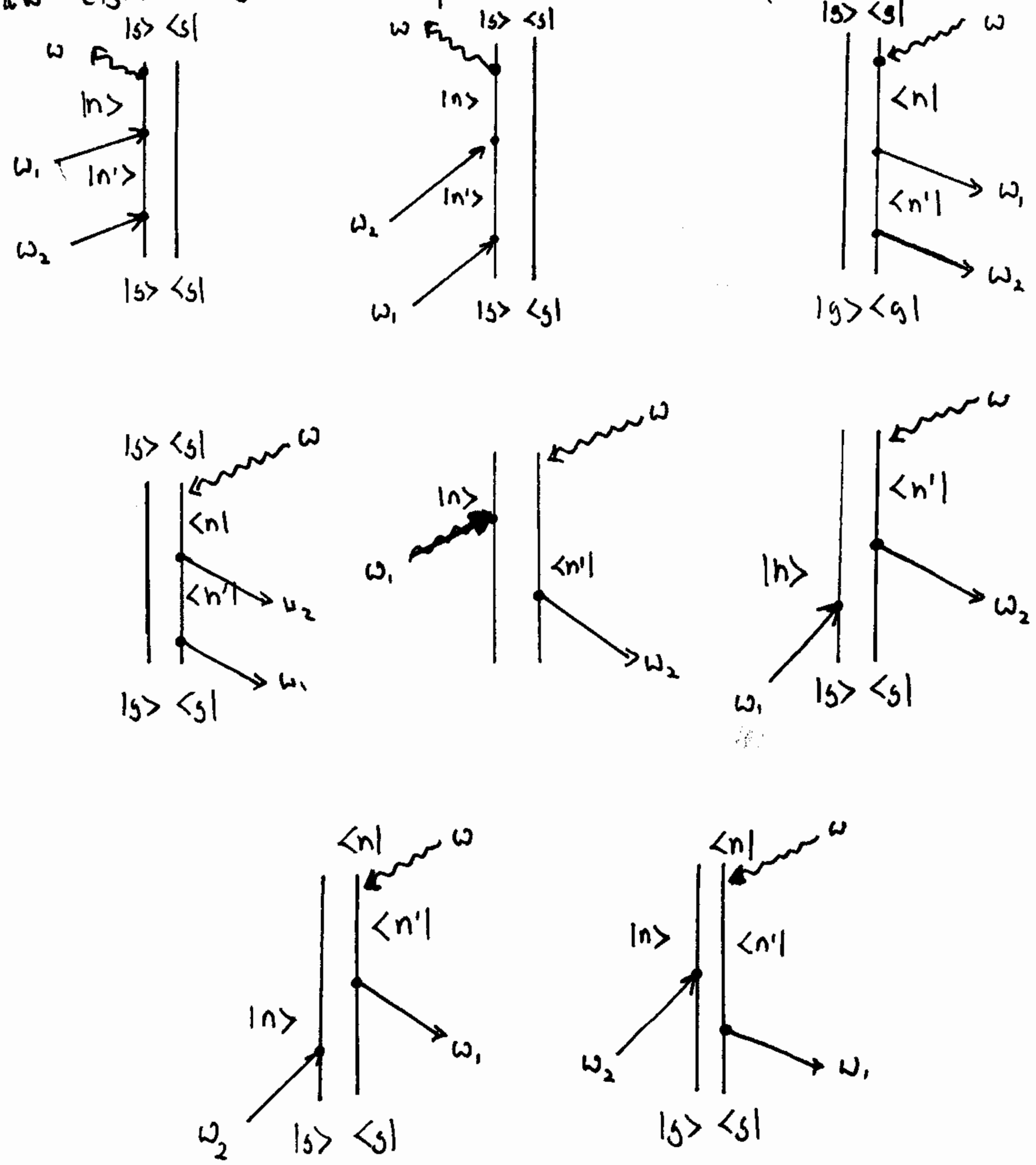
$$f_{ng} = \frac{2m\omega_{ng}\,|\mu_{ng}|^2}{3\hbar e^2}$$

2nd order Susceptibility

Draw Eight Diagrams for $\rho^{(2)}(\omega = \omega_1 + \omega_2)$

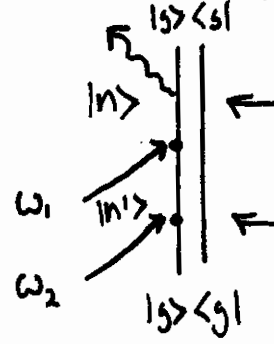
$\rho_{gg} \simeq 1$

(Three photon process)



Write down terms for $\chi^{(2)}$

We have a total of eight terms



$(\omega_1+\omega_2-\omega_{ng}+i\,1/T_{ng})^{-1}$

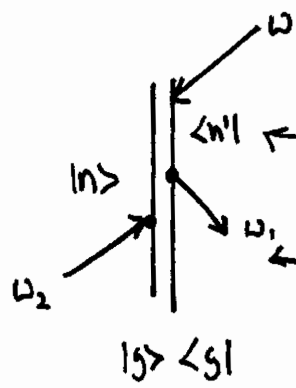
$(\omega_2-\omega_{n'g}+i\,1/T_{n'g})^{-1}$

(ω_1 + ω_2 are + since on right)

The term corresponding to this figure is

$$+\frac{\mu^{(2)}_{gn}\;\mu^{(1)}_{nn'}\;\mu^{(2)}_{n'g}\;\rho^{(0)}_{gg}}{\hbar^2(\omega_1+\omega_2-\omega_{ng}+i\,1/T_{ng})(\omega_2-\omega_{ng}+i\,1/T_{n'g})}$$

Another figure



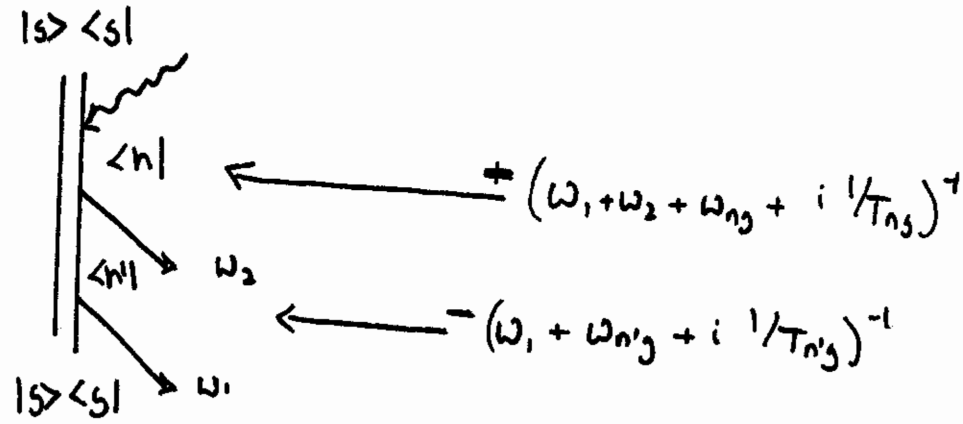
$-(\omega_1+\omega_2-\omega_{nn'}+i\,1/T_{nn'})^{2}$

$(\omega_2-\omega_{ng}+i\,1/T_{ng})^{-1}$

$$-\frac{\mu^{(2)}_{ng}\;\mu^{(1)}_{n'n}\;\mu^{(0)}_{gn'}\;\rho^{(0)}_{gg}}{\hbar^2(\omega_1+\omega_2-\omega_{nn'}+i\,1/T_{nn'})(\omega_2-\omega_{ng}+i\,1/T_{ng})}$$

($\mu^{(2)}_{n'n}$ ⇒ 2 ⇒ field 2, n' ⇒ final state, n ⇒ initial state)

Another figure



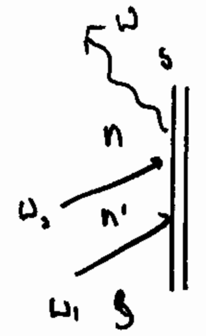
$$+\frac{\mu^{(1)}_{sn'}\,\mu^{(2)}_{n'n}\,\mu^{(0)}_{ng}\,\rho^{(0)}_{ss}}{\hbar^2\left(\omega_1+\omega_2+\omega_{ng}+i\,1/T_{ng}\right)\left(\omega_1+\omega_{n'g}+i\,1/T_{n'g}\right)}$$

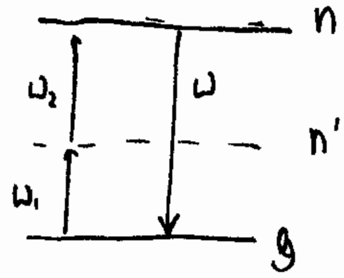
What are the other states $|n\rangle$ + $|n'\rangle$?

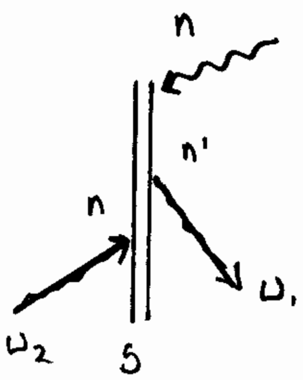
These are <u>virtual</u> <u>transitions</u>

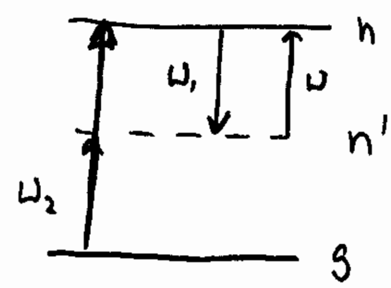
Transitions that occur + do not conserve energy.

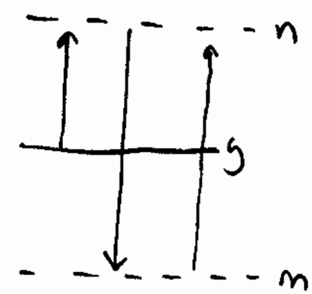
Physical Interpretation

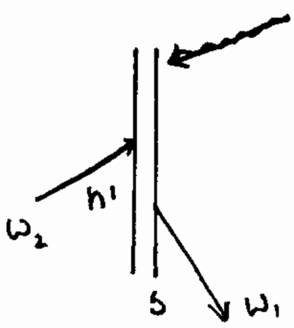


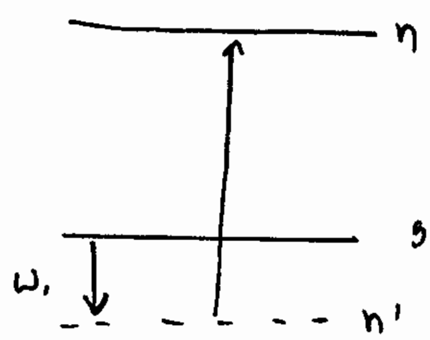












From Shen "The principles of nonlinear optics"

ies

$\chi^{(2)}_{ijk}$. The calculation can be
$_2$ + ...), which will have 48
in literature[5] and is not
, however, is discussed in
nstants in the denominators
otibility can then be reduced
terms in the expression for

ecules per unit volume, the
ate for gases or molecular
nn distribution. For solids
structure, the eigenstates are
distribution. The expression
Since the band states form
the resonant denominators
n with the photon wavevec-
the form[3]

(2.18)

' are the band indices, and
d arising from the induced
factor $L^{(n)}$ should then

appear as a multiplication factor in $\chi^{(n)}$. We discuss the local field correction in more detail in Section 2.4. For Bloch (band-state) electrons in solids with wavefunctions extended over many unit cells, the local field tends to get averaged out, and $L^{(n)}$ may approach 1.

2.3 DIAGRAMMATIC TECHNIQUE

Perturbation calculations can be facilitated with the help of diagrams. Feynman diagrams have been used in perturbation calculations on wavefunctions. Here, since the density matrices involve products of two wavefunctions, perturbation calculations require a kind of double-Feynman diagram. We introduce in this section a technique devised by Yee and Gustafson.[6] Only the steady-state response is considered here.

The important aspects of any diagrammatic technique are that the diagrams provide a simple picture to the corresponding physical process as well as allowing one to write down immediately the corresponding mathematical expression. It is essential to find the complete set of diagrams for a perturbation process of a given order. The scheme we adopt for calculating $\rho^{(n)}$ involves in each diagram a pair of Feynman diagrams with two lines of propagation, one for the $|\psi\rangle$ side of ρ and the other for the $\langle\psi|$ side. Figure 2.1 shows one of

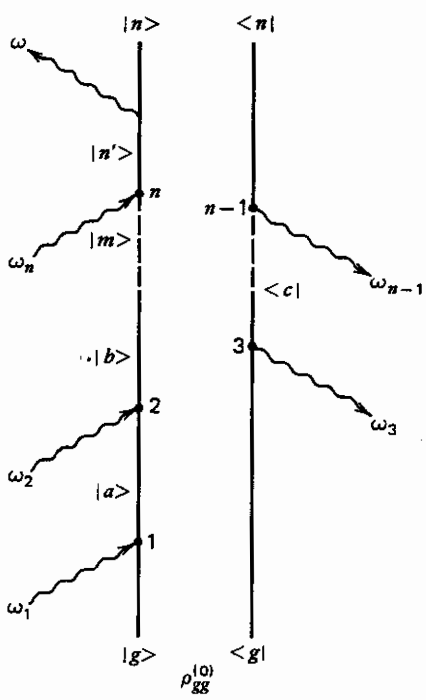


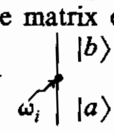
Fig. 2.1 A representative double-Feynman diagram describing one of the many terms in $\rho^{(n)}(\omega = \omega_1 + \omega_2 + \cdots + \omega_n)$.

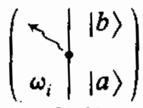
the many diagrams describing the various terms in $\rho^{(n)}(\omega = \omega_1 + \omega_2 + \cdots + \omega_n)$. The system starts initially from $|g\rangle\langle g|$ with a population $\rho_{gg}^{(0)}$. The ket state propagates from $|g\rangle$ to $|n'\rangle$ through interaction with the radiation field at $\omega_1, \omega_2, \ldots, \omega_n$, and the bra state propagates from $\langle g|$ to $\langle n|$ through interaction with the field at $\omega_3, \ldots, \omega_{n-1}$. Then, the final interaction with the output field at ω puts the system in $|n\rangle\langle n|$. Through permutation of the interaction vertices and rearrangement of the positions of the vertices on the lines of propagation, the other diagrams for $\rho^{(n)}$ can also be drawn.

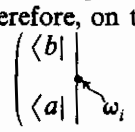
The microscopic expression for a given diagram can now be obtained using the following general rules describing the various multiplication factors:

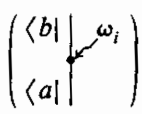
1 The system starts with $|g\rangle\rho_{gg}^{(0)}\langle g|$.

2 The propagation of the ket state appears as multiplication factors on the left, and that of the bra state on the right.

3 A vertex bringing $|a\rangle$ to $|b\rangle$ through absorption at ω_i on the left (ket) side of the diagram is described by the matrix element $(1/i\hbar)\langle b|\mathscr{H}_{\text{int}}(\omega_i)|a\rangle$ with $\mathscr{H}_{\text{int}}(\omega_i) \propto e^{-i\omega_i t}$ (denoted by 

in Fig. 2.1). If it is emission (

) instead of absorption, the vertex should be described by $(1/i\hbar)\langle b|\,\mathscr{H}_{\text{int}}^{\dagger}(\omega_i)|a\rangle$. Because of the adjoint nature between the bra and ket sides, an absorption process on the ket side appears as an emission process on the bra side, and vice versa.* Therefore, on the right (bra) side of the diagram, the vertices for emission (

) and absorption (

) are described by $-(1/i\hbar)\langle a|\mathscr{H}_{\text{int}}(\omega_i)|b\rangle$ and $-(1/i\hbar)\langle a|\,\mathscr{H}_{\text{int}}^{\dagger}(\omega_i)|b\rangle$, respectively.

4 Propagation from the jth vertex to the $(j+1)$th vertex along the $|l\rangle\langle k|$ double lines is described by the propagator $\Pi_j = \pm[i(\Sigma_{i=1}^{j}\omega_i - \omega_{lk} + i\Gamma_{lk})]^{-1}$ The frequency ω_i is taken as positive if absorption of ω_i at the ith vertex occurs on the left or emission of ω_i on the right; it is taken as negative if absorption of ω_i occurs on the right or emission on the left.

5 The final state of the system is described by the product of the final ket and bra states, for example, $|n'\rangle\langle n|$ after the nth vertex in Fig. 2.1 for $\rho^{(n)}$.

6 The product of all factors describes the propagation from $|g\rangle\langle g|$ to $|n'\rangle\langle n|$ through a particular set of states in the diagram. Summation of these

*If the field is also quantized, $\mathscr{H}_{\text{int}}(\omega_i)$ operating on a ket state will annihilate a photon at ω_i, while if operating on a bra state it will create a photon.

n $\rho^{(n)}(\omega = \omega_1 + \omega_2 + \cdots +$
population $\rho^{(0)}_{gg}$. The ket state
ı v the radiation field at
$\langle g|$ to $\langle n|$ through interaction
ıction with the output field at
ın of the interaction vertices
on the lines of propagation,

ı can now be obtained using
multiplication factors:

aultiplication factors on the

n at ω_i on the left (ket) side
ment $(1/i\hbar)\langle b|\mathscr{H}_{\text{int}}(\omega_i)|a\rangle$
ı Fig. 2.1. If it is emission

x should be described by

aature between the bra and
de appears as an emission
ore, on the right (bra) side
$\langle b|$ and absorption
$\langle a|$

$_{\text{nt}}(\omega_i)|b\rangle$ and $-(1/i\hbar)\langle a|$

th vertex along the $|l\rangle\langle k|$
$\Pi_j = \pm[i(\sum_{i=1}^{j}\omega_i - \omega_{lk} +$
absorption of ω_i at the ith
the right; it is taken as
r emission on the left.
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ion from $|g\rangle\langle g|$ to $|n'\rangle\langle n|$
am. Summation of these

annihilate a photon at ω_i, while

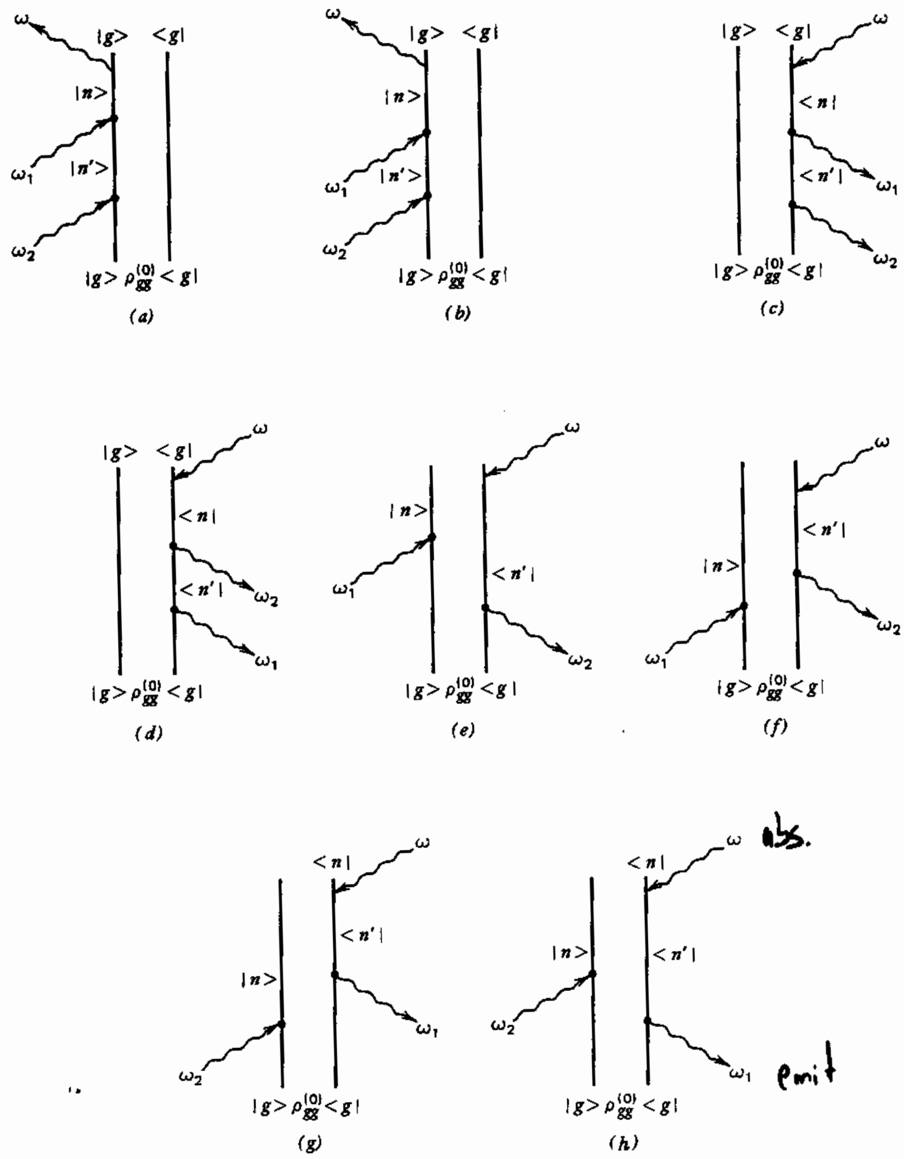


Fig. 2.2 The complete set of eight diagrams for the eight terms in $\rho^{(2)}(\omega = \omega_1 + \omega_2)$.

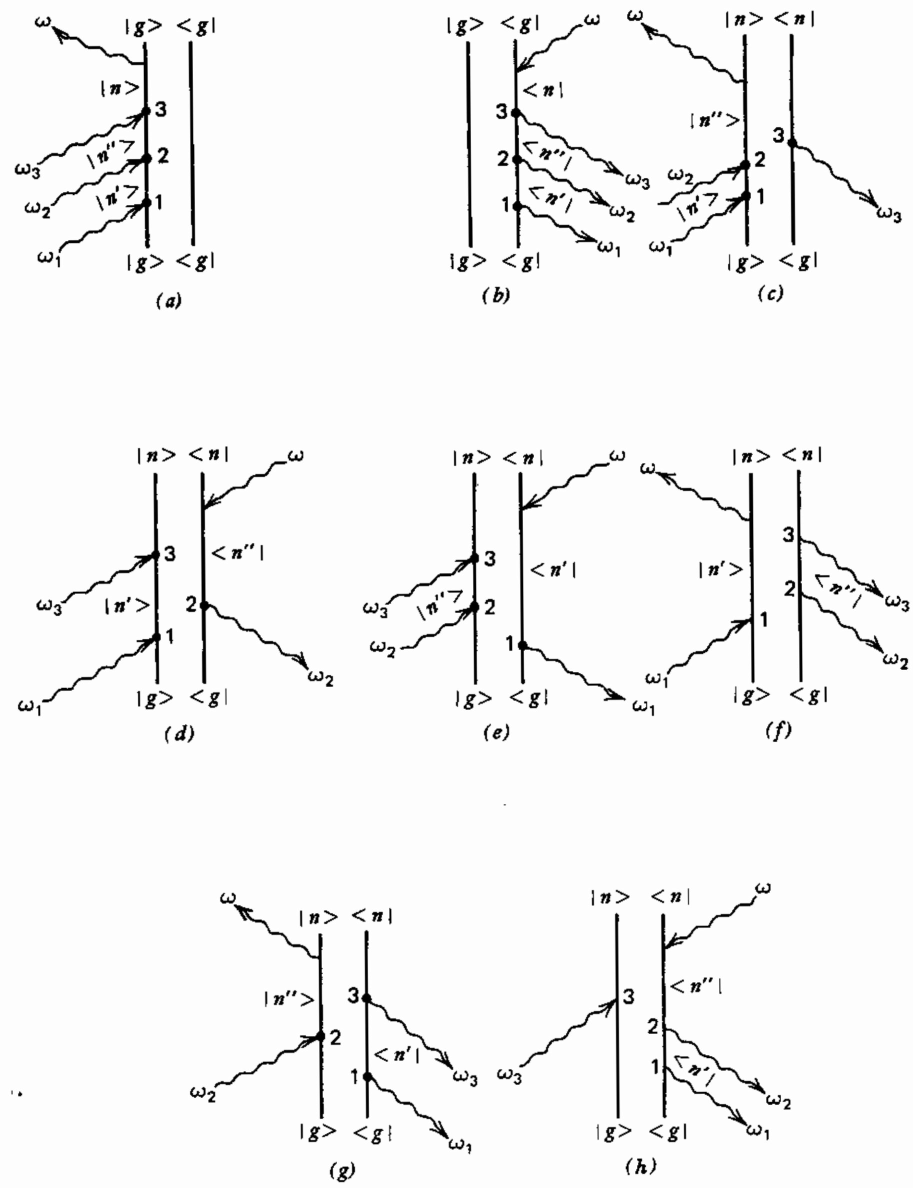


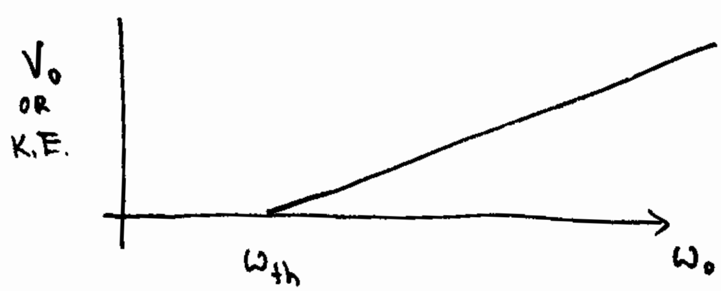
Fig. 2.3 The eight basic diagrams for $\rho^{(3)}(\omega = \omega_1 + \omega_2 + \omega_3)$.

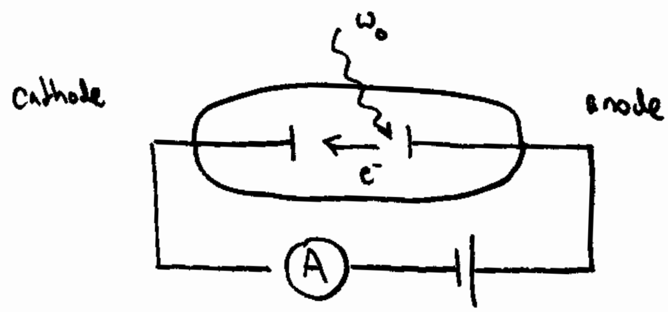
Lecture 29 What is a Photon? Part 1 : Photoelectric effect

Experiments of Lenard (1902) of light striking a metal

Observations

- Electrons are emitted a times very shortly after the onset of illumination ($\sim 10^{-9}$ s)
- Photo current rises linearly with light intensity
- Current to cathode decreases with increasing retarding potential, zero at stopping voltage V_0
- Stopping potential V_0 is linearly proportional to ω_0 and shows a threshold frequency ω_{th}





V ⇒ applied stopping voltage

It was thought that.....

Classical E/m does not explain observed behavior

(or does it!!)

(Sun Tsu
Hesel)

Einstein's Description (1906)

- Started with thermodynamical considerations
- Considered Blackbody radiation

"monochromatic radiation at low density behaves with respect to theory of heat as if it consisted of independent energy quanta of magnitude $h\nu$."

"if this is the case it is natural to investigate whether the laws of generation & transformation of light are such a kind as if light would consist of such energy quanta."

$$\hbar\omega_0 = KE + \phi_0$$

$$h\nu = \frac{1}{2}mv^2 + \phi_0$$

work function

The energy of a material oscillator with eigenvenfrequency ω_0 interacting with the radiation field can only take discrete values of $n\hbar\omega_0$

As einstein said

"our concept and the properties of the light electric field observed by Mr. Lenard, as far as I can see, are not in contradiction."

Einstein comment on nonlinear optics

"the # of energy quanta per unit volume being simultaneously converted is so large than an energy quantum of light generated can obtain its energy from several generating quanta.

1921 Nobel Prize "Services to Theoretical Physics, especially for his discovery of the law of photoelectric effect"

Photons (by name) 1926

Gilbert Lewis

" hypothetical new atom ... photon "

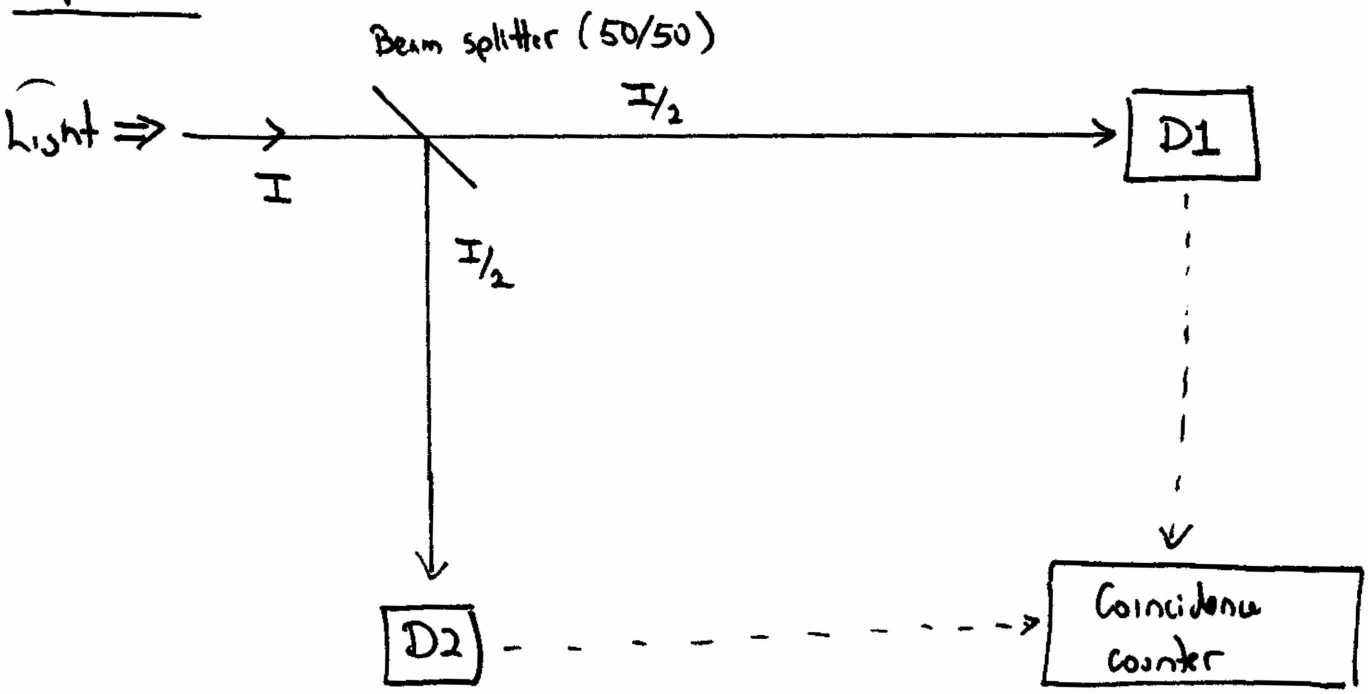
" hypothetical new entities as a particle of light ....
spends a minute fraction of its existance as a
carrier of radiant energy, the rest of the time as
an important structural element of the atom

Hanbury-Brown + Twiss Experiment (1956)

How to design an experiment to detect single photons?!

- photon ⇒ particle at one place
- Experiment to determine the position (here or there) of a photon
- single photon, two detectors
  Do these detectors "click" at the same time?

Experiment



Do detectors D1 + D2 "click" at the same time. Within the particle picture they should not.

- Coincidence ~~[illegible]~~ detection

— Records a count only if both detectors "click" at the same time

Experiment: Send light to beam splitter and measure the number of coincidences counts relative to the number of individual counts on the detector.

Anti correlation parameter

$$\boxed{A = \frac{P_c}{P_1 \; P_2}}$$

where
$P_c \equiv$ probability of coincidence
$P_1 \equiv$ measured prob. of detector 1 responding
$P_2 \equiv$ measured prob. of detector 2 responding

<u>Outcomes</u>

— Light as particles
~~[illegible] photons ([illegible])~~ $A = 0$

~~Light as waves $A \neq 0$~~

Light as a wave
— If detectors click randomly + independent $A = 1$ ← More specifically

(predicted by wave theory) $\left(P_{\text{both click}} = \left(P_{\substack{\text{one}\\\text{clicks}}}\right)\left(P_{\substack{\text{other}\\\text{clicks}}}\right) \right)$

— Independent of intensity

$P_c = P_1 P_2$ so $A = 1$

— If two detectors clicking together more often than random $A > 1$ "Clustering"
"~~[illegible]~~ ~~"Clustering"~~"

Write in terms of experimental results

– Probability of detection

$$P_{1 \text{ or } 2} = \frac{N_{1 or 2}}{\left(\frac{T}{\Delta t}\right)}$$

$\Delta t \equiv$ time resolution

$T \equiv$ experiment time

so

$$A = \frac{N_c}{N_1 N_2}\left(\frac{T}{\Delta t}\right)$$

Hanbury-Brown + Twiss

Quantum mechanics Prediction

for small intensities single photons are not split by the photodetector so $(A = 0)$ (anticorrelation)

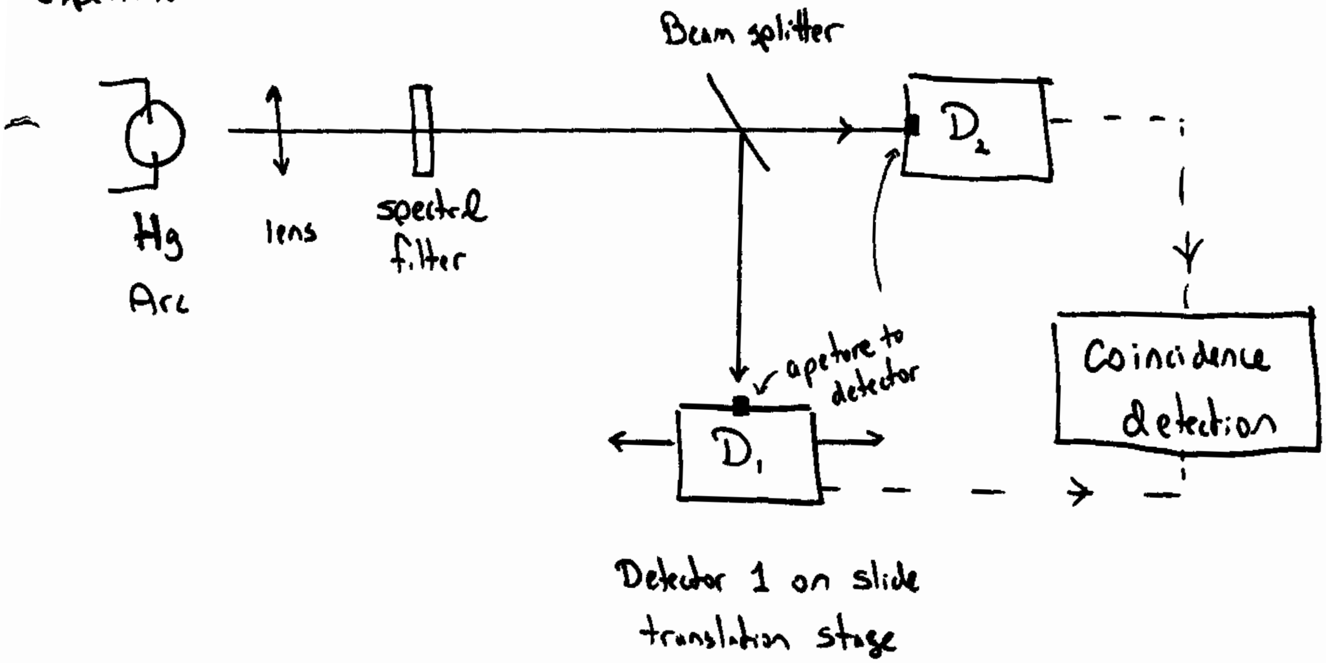
Wave theory Prediction

No matter the intensity of light at beam splitter it is halved in both directions, so random coincidences would be expected $(A = 1)$

⇒ They found the opposite result $A = 2$!!

When a click occured at one detector they found a higher than random probability that a click had simultaneously occurred at the other.

Experiment



This experiment failed to demonstrate the existence of photons and the indivisiblity of light. It showed that light travels thru space bunched up, you can divide the bunches in half but the bunches arrive at the same time

Origins of quantum optics $\Rightarrow$ understanding photon correlations

But is this classical or quantum description?

- Semi classical Description
  - Light classically
  - Detectors quantum mechanically

- Probability for transition

$$P_1 = \alpha_1 I \Delta t$$

$$P_2 = \alpha_2 I \Delta t$$

$$P_c = \alpha_1 \alpha_2 I^2 (\Delta t)^2$$

$$A = \frac{\alpha_1 \alpha_2 I^2 (\Delta t)^2}{(\alpha_1 I \Delta t)(\alpha_2 I \Delta t)} = 1 \Rightarrow \text{Not their result}$$

However if the light source produced a time varying intensity $\langle I \rangle$ produced by a collection of atoms:

$$P_1 = \alpha_1 \langle I \rangle \Delta t$$

$$P_2 = \alpha_2 \langle I \rangle \Delta t$$

$$P_c = \alpha_1 \alpha_2 \langle I^2 \rangle \Delta t$$

ave of intensity squared

Then $$A = \frac{\langle I^2 \rangle}{\langle I \rangle^2}$$

However $\langle I^2 \rangle \geq \langle I \rangle^2$ (Cauchy Swartz inequality)

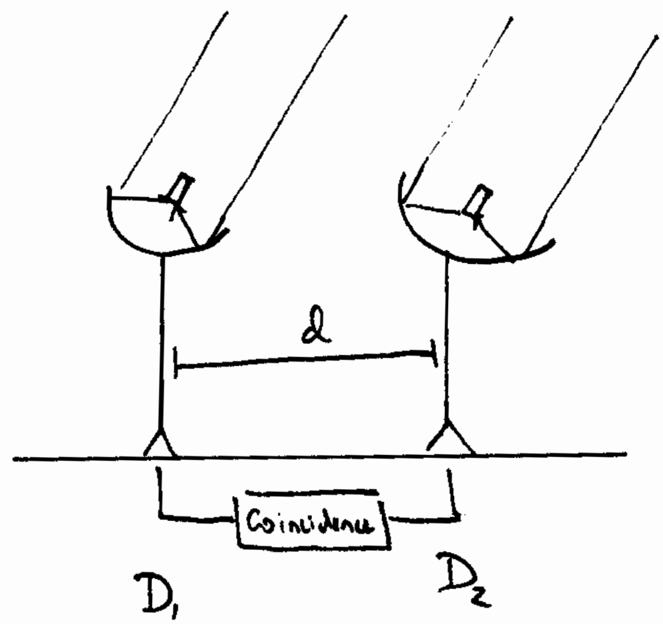
Thus $A \geq 1$

What happens when you use a laser?!

$A = 1$ since the intensity fluctuations are small

$$\langle I^2 \rangle = \langle I \rangle^2$$

Where did this experiment come from?: Radio astronomy



Measurement of spatial coherence of "light" from a star

- instantaneous phase of E changes slightly within coherence area.

- see similar time evolution between detectors if d is shorter than the transverse coherence length.

$$\Delta I_1 \simeq \Delta I_2 \qquad \text{for} \quad d < l_{coh}$$

$$\Delta I = I(t) - \bar{I}$$

Correlator gives $I_2(t)\, I_1(t)$

$$\overline{I_1(t)\, I_2(t)} = \bar{I}^2 + \overline{\Delta I_1}^2 \qquad d < l_{coh}$$

For $d > l_{coh}$

$$\overline{I_1(t) I_2(t)} = \bar{I}^2 \qquad \text{Decrease in intensity correlations.}$$

Instead of d we can use the star's angular diameter.

$d \simeq 188\text{m}$ $\qquad$ Δdiameter $\simeq$ 0.0005 arc sec.

from this experiment they noticed ...

for $d < l_{tran\ coh}$ when a ~~photon~~ click measured a D_1 the probability of detecting a "click" at D_2 was larger than the random case.

Does a photon "know" the outcome of the two detectors?!

Photoelectric effect revistited: Lamb + Scully (1969)
"The Photoelectric Effect without photons"

- Jaynes +
Lamb + Scully develope a semi-classical theory for the photoelectric effect ⇒ no more need for photons

Photoelectric effect is not a proof of the existance of photons.

Summary

This theory along with the failing of experiments to detect photons raised a few questions. What is the nature of a photon? Are there really photons? Do they exist?! Or are they artifacts of the tools we used to investigate light?!

The problem with the photoelectric effect + Hanbury-Brown Twiss Experiment was in the light source they used.

Anticorrelations are expected if the source produces light in an eigenstate of the photon number operator.

For both experiments ~~all large number of photons with~~, using a quantum description, a large # of photons were used.
- If another experiment is designed which uses one photon (that is an eigenstate of a photon number operator) then we would expect an anticorrelation $A = 0$.

Read: By Monday

1) Aspect el at Europhys Lett 1 (4) p 173 (1986)

2) Walther et al Phys Rev ~~[illegible]~~ A vol 35, 6, 1987

Survey $\Rightarrow$ <u>out</u> <u>tonight</u>

Questions: (About Hanbury-Brown + Twiss Experiment)

1. Why Did Hanbury Brown + Twiss measure coincidences in "cathodes aligned" positions + <u>no</u> coincidences in "cathodes not aligned" positions?

2. Why did Brannen el al <u>not</u> measure any coincidences?

3. Given Brannen el at experiment, what is the one thing they need in order to observe coincidences.

How would this "one thing" solve their problems?

(Brannen + Ferguson Nature 1956)

Notes $\left\{ \begin{array}{l} \tau_0 \sim \cancel{\ } 10^{-11}\text{s} \\ \text{resolving time} \quad 10\,\text{ns} \end{array} \right. \Rightarrow$ For Brannen et al so $T \simeq$ resolving time $\dfrac{\tau_0}{T} = \dfrac{10^{-11}}{10^{-8}} \simeq 10^{-3}$ $\Big\}$ Correlations 10^{-3}

What is a photon? Part 2

Lecture 30 Aspect Experiments in 1986

Here, we will discuss two experiments performed by A. Aspect et al in Europhys. Lett. 1 (4) pp 173-179 (1986)

Both experiments used an atomic cascade as a light source. and a triggered detection scheme. The source provided single photons unlike the experiment of Hanbury-Brown + Twiss.

Two Experiments

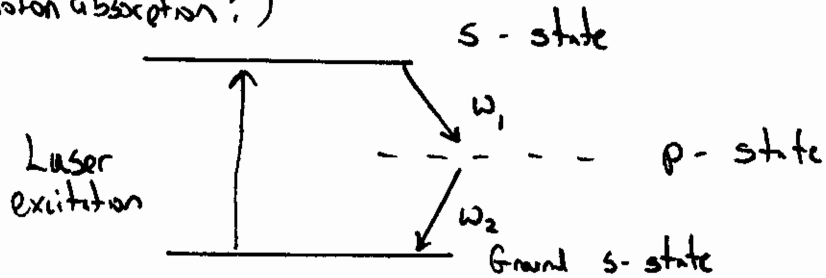
1. Test anticorrelation of the source (similar to Hanbury-Brown + Twiss)

2. Single photon interference experiments

The "single photon" Source.

Laser excitation of Ca atoms to a state that would decay by emitting two photons instead of one.

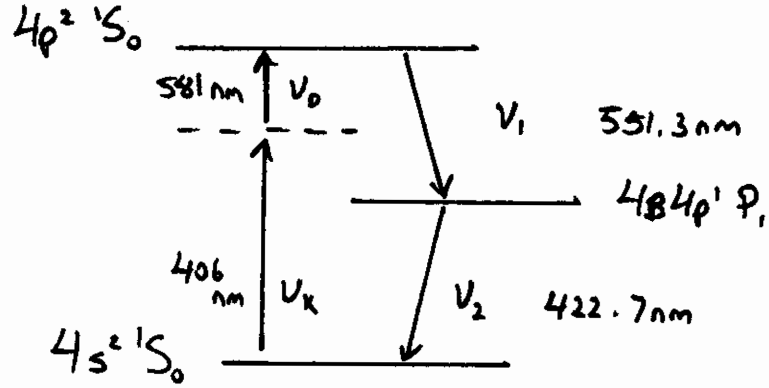
(Two photon absorption?)



How to detect these photon ω_2 from all other photons $\Rightarrow$ triggered detection

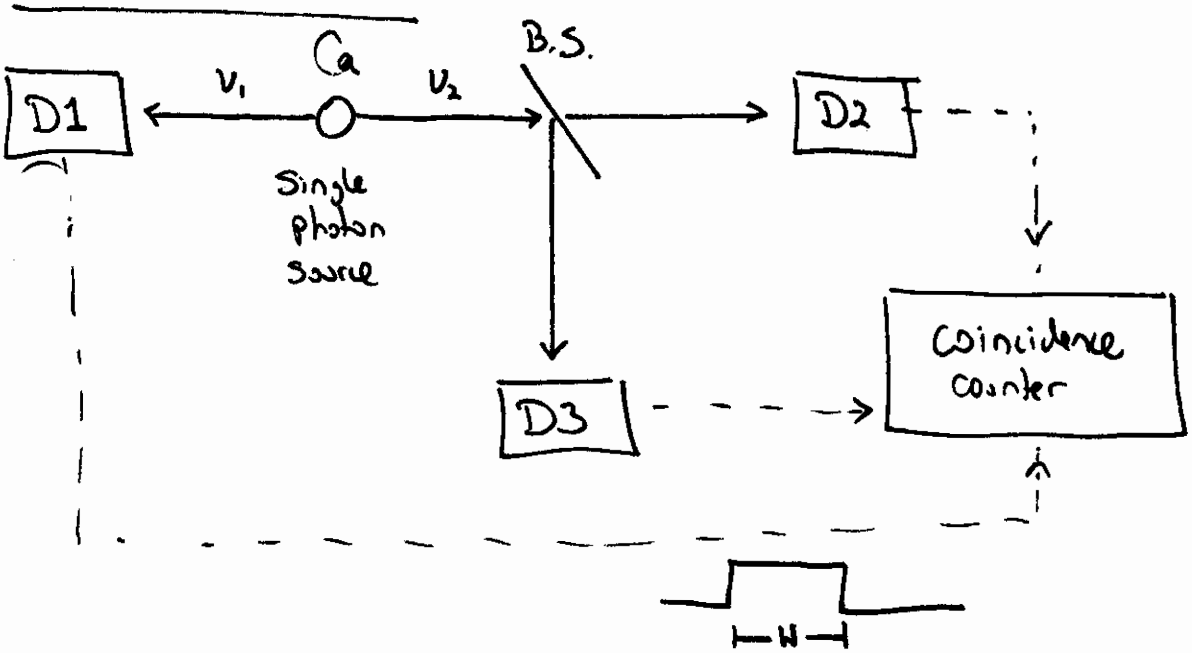
~~Experiment~~

Energy levels of Ca



Some numbers $\left\{ \begin{array}{l} \text{240 coincidences/s} \\ \text{90 accidental coincidences/s} \\ \text{100 s} \rightarrow \text{150 true coincidences} \end{array} \right.$

Detection Scheme



(gate time w

- Photon ν_1 triggers gate of duration w
- Look for coincidences during time w. Reject coincidences for times "outside" of w.

They measured $A = 0$! Single photons measured at last.

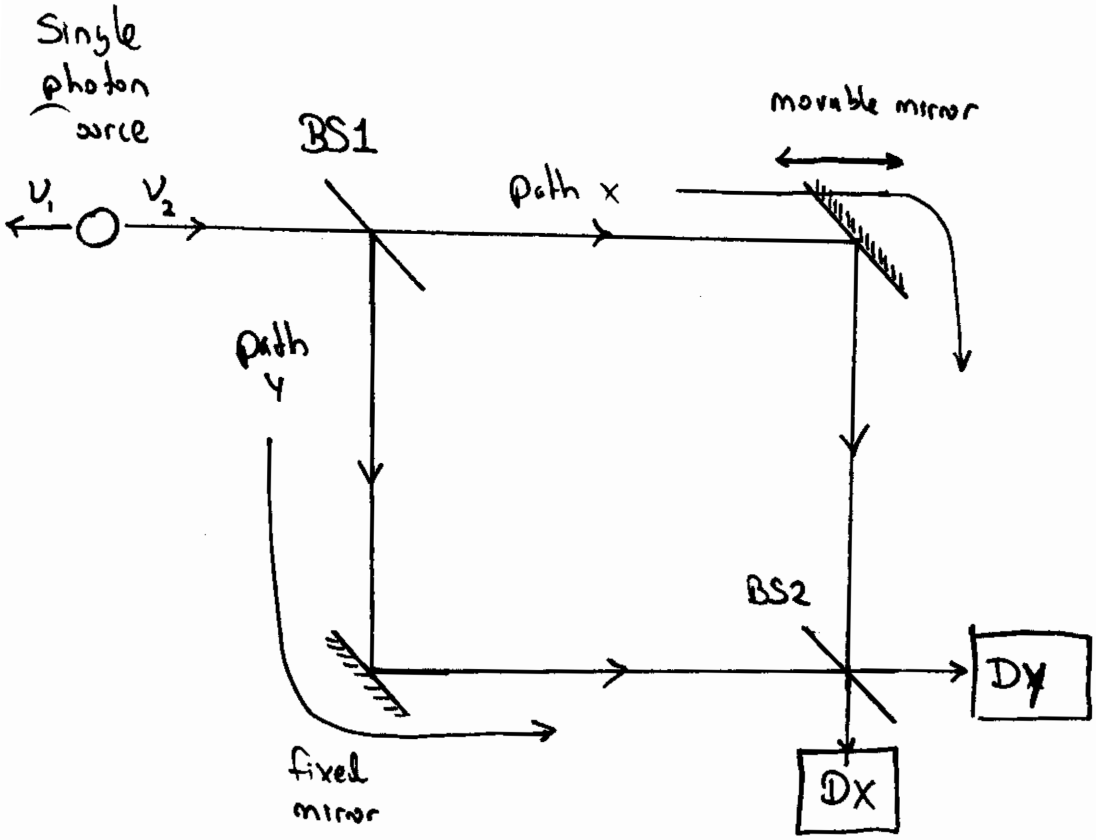
Clear evidence at last for the existance of photons.

What has been shown here?

- Individual particles from their source were either reflected or transmitted, going one way or another, but never <u>both</u>

Experiment 2 : Single photon interference (huh?!)

Allow single photons to enter a Mach - Zehnder Interferometer.

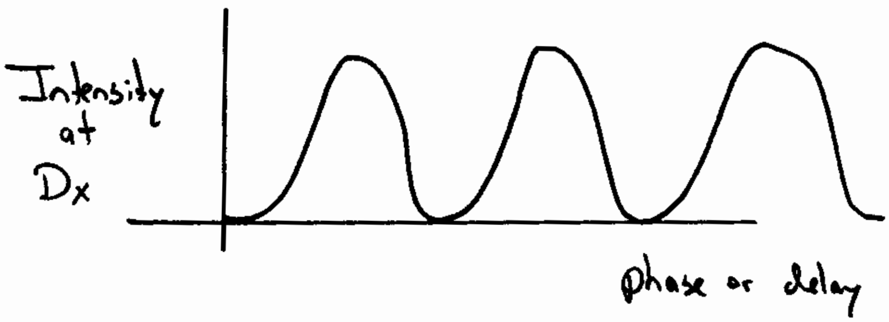


-

moving the mirror put a phase difference between the "arms" of the Mach Zehnder Interferometer.

What would happen if light is a wave?

One would see interference fringes as a function of changing the position of the movable mirror (just like for miniproject 1)

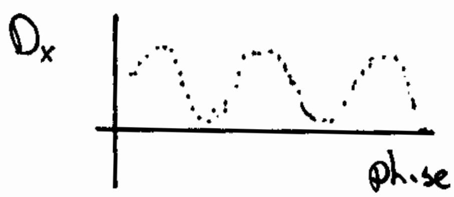


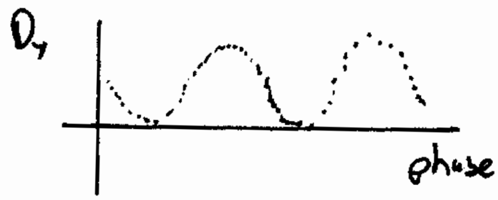
What would happen if light is particle?

From experiment #1 we know that the photon goes one way or the other at the BS. We would not expect any interference.

What happened?

They saw an interference pattern as they aquired counts!





Does the photon take both paths?!

Does the photon interfere with itself?!

Wave particle duality!! Can Wave-particle duality Explain this?

Does light "know" when to behave like a wave and when to behave like a particle?!

Review

- Aspect et al
- Ask "which path did the photon take?"
- What does the photon interfere with to get the interference pattern?
- Does it interfer with itself?!

Lecture 31 What is a photon? Part 3: Delayed Choice Experiment

Wave particle duality: Does this explain the Aspect Experiment?

Sometimes a wave / Sometimes a particle.

Photon "conspiracy theory"? to understand the wave/particle duality.

~~"senses" an experiment & should demonstrate its wave or particle properties!?~~

⟹ As the photon leaves a source, it "senses" the experiment and behaves accordingly, like a particle or wave.

Does this happen?! Test this...

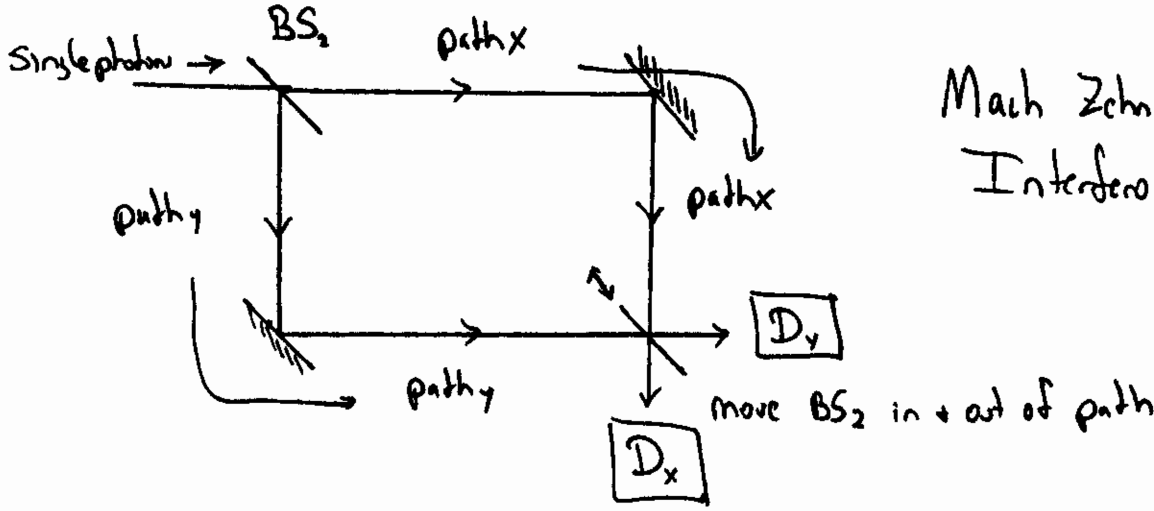
Delayed choice Experiment of John Wheeler

Ref. Wheeler "Mathematical Foundations of Quantum Mechanics" p. 9-48

- States that "conspiracy" theory is wrong. Also shows that wave-particle duality is more complicated
- Experiment of Delayed Choice
  - able to detect either particles (anti coincidences) or waves (interference)
  - Designed so that choice of which aspect to observe is delayed after the photon has "decided"

How to do this ⟹ make BS2 in the MZ movable.

Delayed Choice Experiment



Mach Zehnder (MZ) Interferometer

What to expect?

— Single photon enters MZ

— Without BS_2 Detectors D_x + D_y ascertain a specific path for the photon

$\Rightarrow$ Like Aspect Experiment #1 (anticorrelation exp.)

— With BS_2 Expect to get interference. Lose all information on the path the photon takes

The idea is to insert BS_2 after the photon has entered the MZ interferometer

According to "conspiracy theory" the last minute insertition of BS_2 should "fool" light

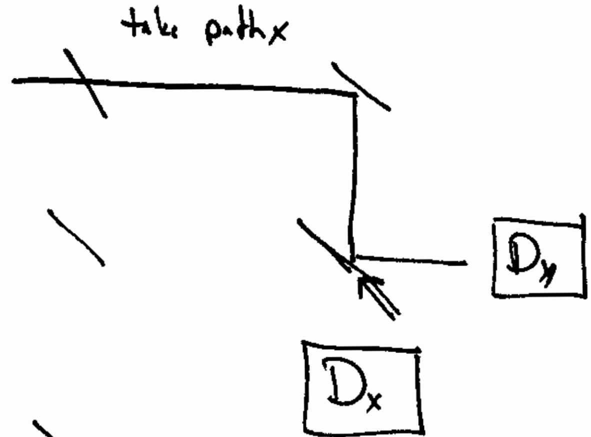
Example

1) No BS_2

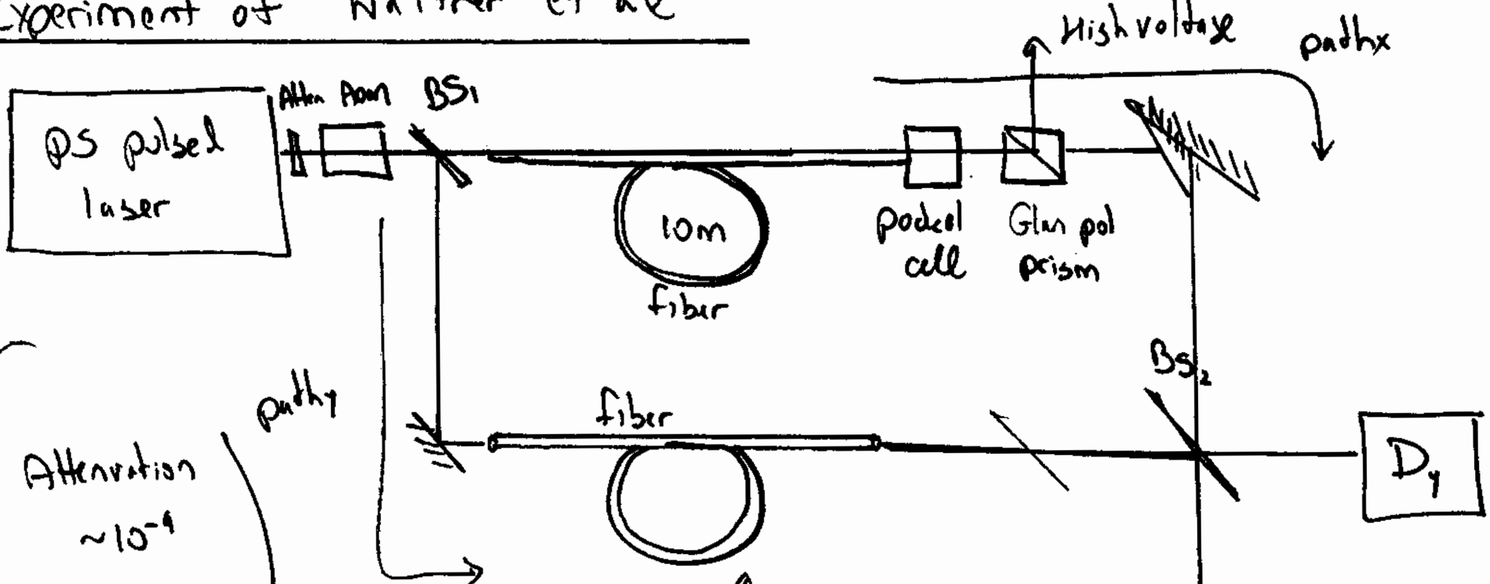
2) Light takes path x or y

3) Insert BS_2

4) Should observe interference?!!
(no according to conspiracy theory)



Experiment of Walther et al



(Attenuation $\sim 10^{-4}$

0.2 photons per pulse)

(80MHz → 1:8000)

10m (use fibers to increase propagation time

- Switching time 5ns
- Voltage High ⇒ Light reaching BS_2 came from path y
- Voltage Low ⇒ Light can take both paths ⇒ interference.

- Used pulsed light source
  Path of interferometer arms increased 30ns by 10m Fiber
  Give time for Pockels cell to operate

Speed of light

Light travels... in vacuum

in 100 fs $\Rightarrow$ 30 μm

in 500 fs $\Rightarrow$ 150 μm   Thickness of hair

in 1 ns $\Rightarrow$ 30 cm   Ruler length

In fiber

Light slower by 1.45

$$1\text{ns} \Rightarrow 30\text{cm}\,(1.45) = 43.50\text{cm}$$

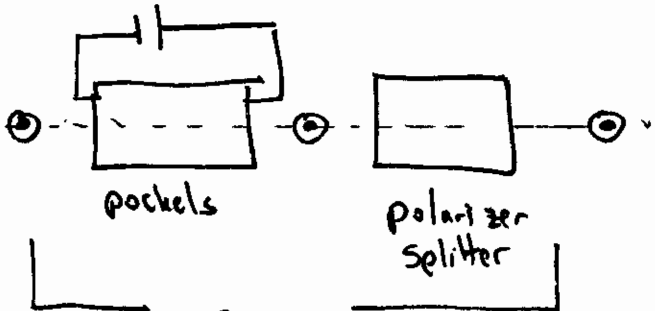
Pockels Cell

Applied electric field rotates input polarization by 90°

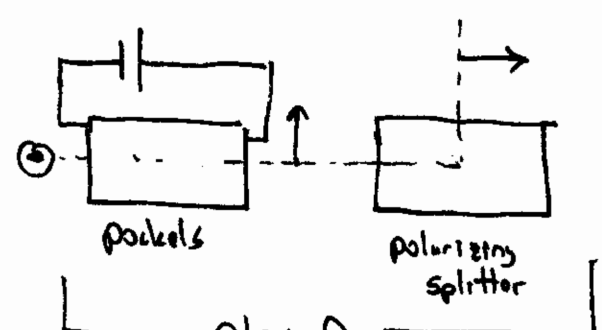
~~A polarizer/splitter~~   Birefringent crystal with electro-optic effect

Rotate polarization ellipse by applying voltage

With combination of polarizer make polarization dependent switch

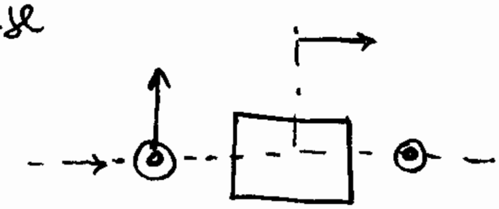


Open
no voltage



Closed
voltage

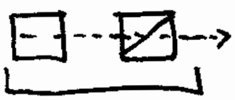
~~Polarizer~~

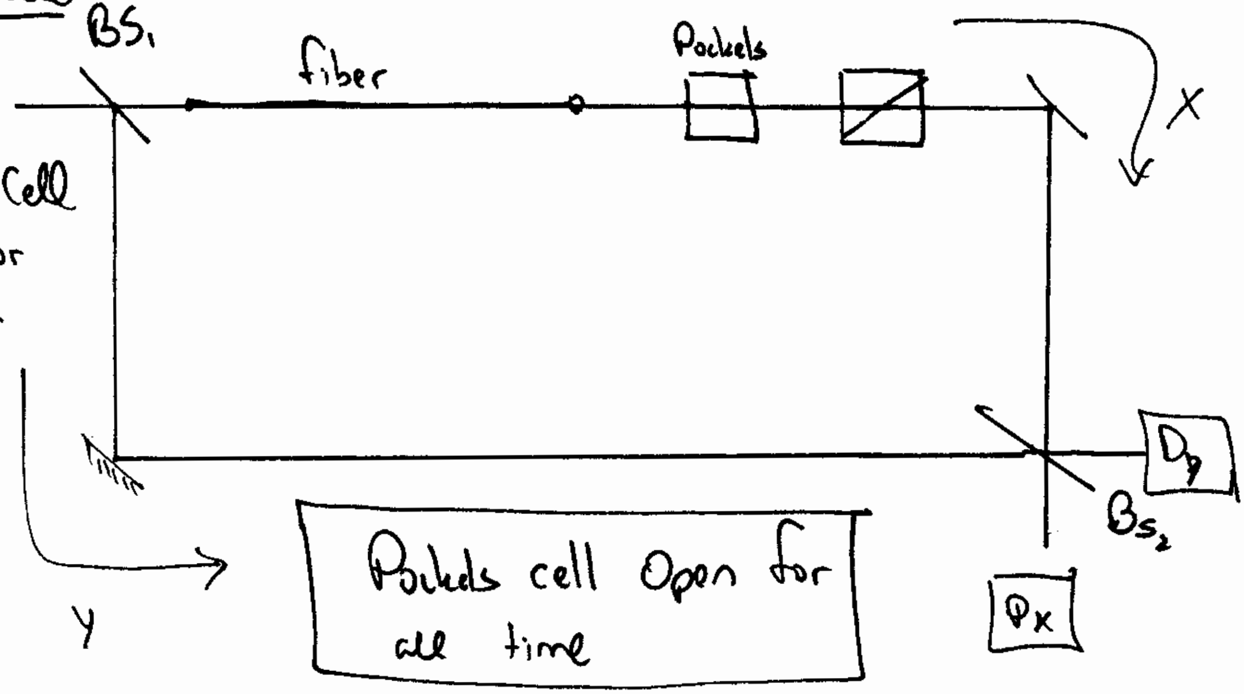


Glan Polarizing Beamsplitter

Normal Mode

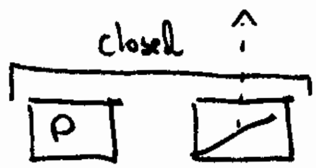
1) Turn Pockel's Cell off (open) for all time t.





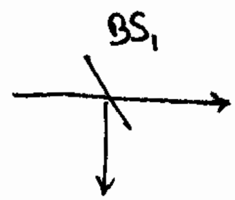
Delayed Choice Mode

1) t=0 Pockels Cell Closed ⇒



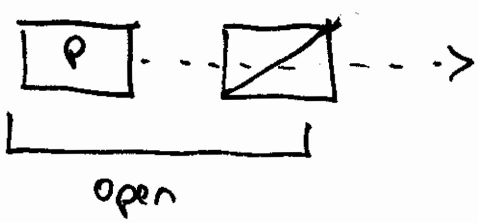
2) ~~t=5ns Open~~

t = 1ns Light passes BS_1 ⇒

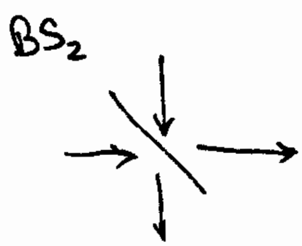


5ns

3) t = 6ns Pockel's Cell Open ⇒



4) t ≃ 30ns Light passes BS_2 ⇒



"Normal" mode Pockel's Cell open (low voltage) when pulse reaches BS_2 and for the whole experiment

"Delayed Choice" mode Pockels Cell Closed + opened 5 ns after pulse has passed BS_1 Pulse in fiber at this time

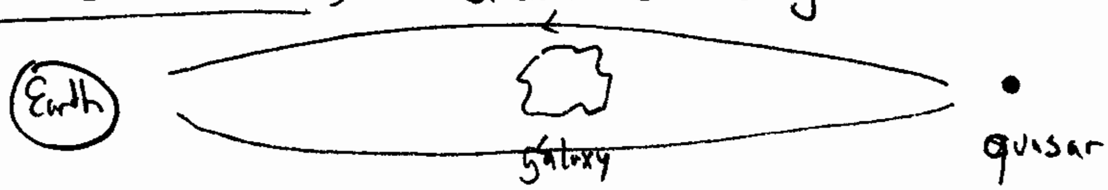
Results

- No matter when BS_2 was inserted an interference pattern was observed.
- If experiment begin with BS_2 in place then removed interference was not observed.

Light is not fooled; when the apparatus is changed after light has made its "choice" the light still makes the correct choice.

"Double" Quasars (Wheeler)

Gravitational Lensing



Two paths

Two possible paths ⇒ interfere can arise
Insert BS ⇒ see Interference
No BS see two images

BS in place
photon has already traveled billions of yrs. via both paths
What possible difference does inserting the BS make in the billion yrs

Light has already traveled for billion of years.
What difference does in serting the BS make in the history of the light?!

Do our actions at this present moment have consequences that stretch back to the ~~cosmic~~ past?

"Smoky Dragon" (J. Wheeler)

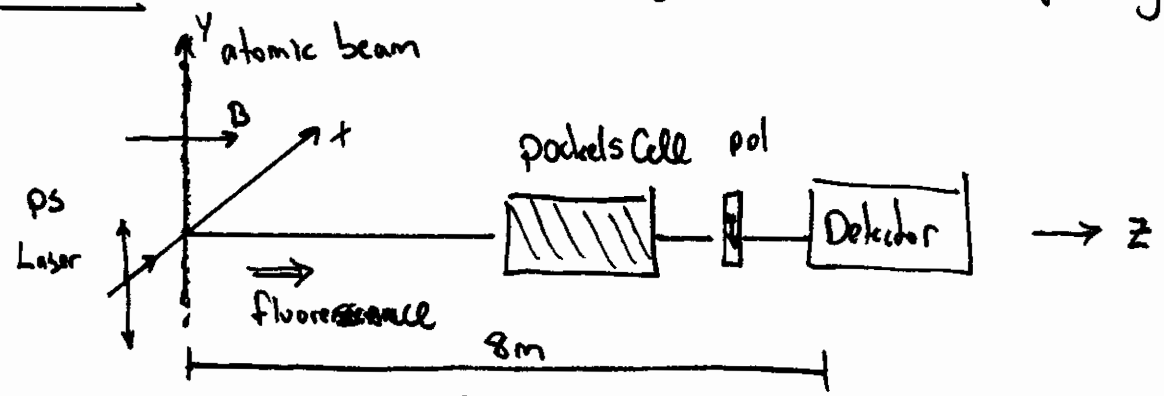
simultaneously existing in everywhere in the interferometer.
which suddenly bends to bite the detector.

Quantum Beat Measurements (Walther et al)

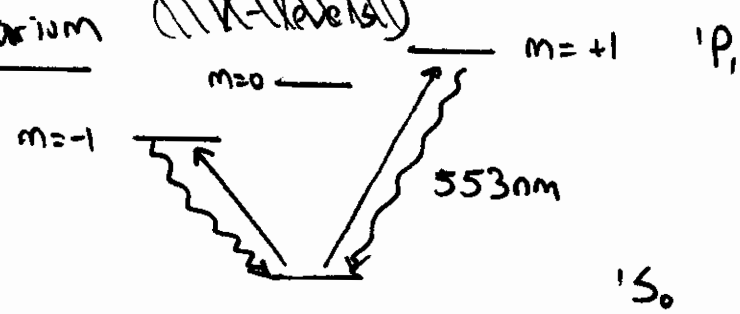
- Atomic beam of Barium atoms
- Magnetic field applied $\perp$ to direction of atomic beam
- Spectral measurement of Delayed Choice

Experimental Setup

Magnetic Field causes Zeeman splitting



States for Barium



- Less than one photon per pulse
- Interference between two paths
  1) $|0\rangle \to |+1\rangle \to |0\rangle$
  2) $|0\rangle \to |-1\rangle \to |0\rangle$
- Delayed Choice requires one path remains blocked untill the emitted photon arrives at detection system.

  Use Pockel's Cell again

What does Pockels Cell Do here?

When the pockel's cell is on:

- The light from the $|0\rangle \rightarrow |+1\rangle \rightarrow |0\rangle$ path is changed to linearly polarized light which is transmitted by the polarizer

- The light from the $|0\rangle \rightarrow |-1\rangle \rightarrow |0\rangle$ path ~~[illegible]~~ is changed to linearly polarized light and is blocked by the polarizer.

The pockels Cell effective blocks a path as in a similar manner as in the Mach Zehnder interferometer experiment.

See ~~[illegible]~~ interference when both paths are present, see no interference if one path is blocked.

Results

- Normal Mode

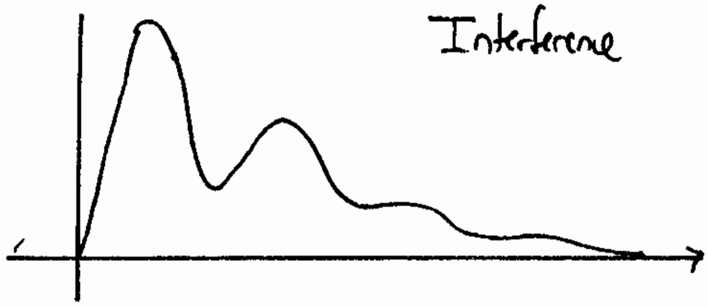
  Pockels Cell open ⇒ See interference
  (No voltage)

  Apply voltage / Pockels Cell Closed ⇒ see exponential decrease

- ~~Delayed Choice~~

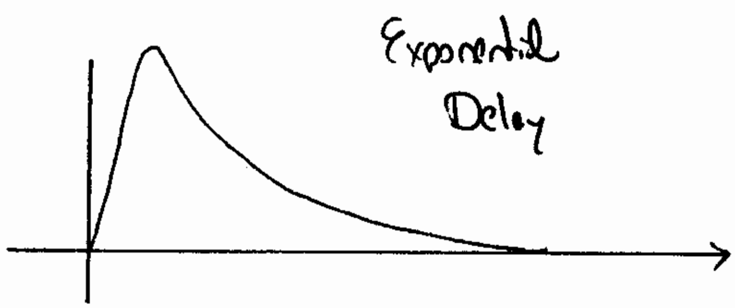
  ~~Apply voltage to pockels cell at different times.~~

Type I



pockel's cell zero (open)

Type II



pockels cell voltage high
(closed

Delayed Choice Experiment with Ba atoms

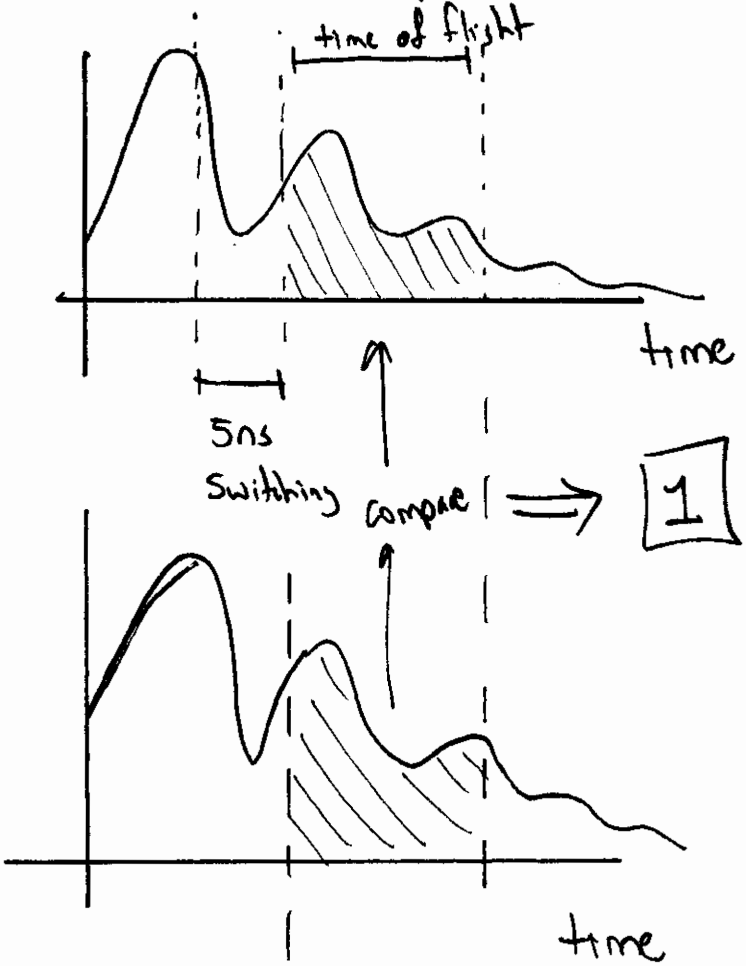
- Turn on Pockels Cell
- Turn off at different times 2ns 17ns 29ns
- See modulation of exponential decay after switching cell.

– They compare the normal operation to Delayed choice operation after swiching the pockels cell after 4ns

(Use ~~Data~~ Data from 10 - 30ns)

– They show that the data from 10 - 30ns for both the delayed choice and normal operation are basically the same.

Delayed Choice



Lecture 32 Quantization of a Single mode field

Complementarity and Quantum Beats

Principle of Complementarity

— { Classical Physics ⇒ unity
  { Quantum Mechanics ⇒ duality of two complementary pictures

particle ⟺ wave are complements

— Refers to what quantum mechanics allows us to know ⇒ x OR p, t or E

— What experiment allows us to know.

Knowledge of which path and the possibility of observing an interference pattern.

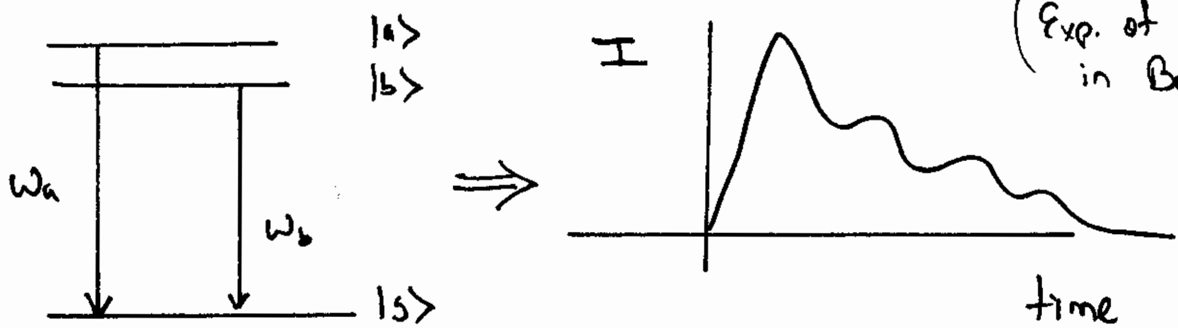
— All are mutually exclusive concepts

Complementarity + Information: Quantum Beats

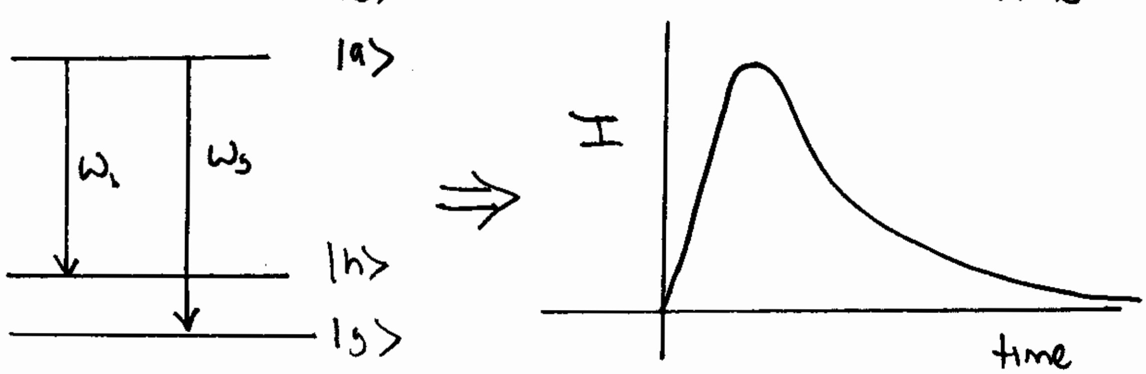
Two Experiments Fluoresence

(Exp. of Walther et al in Barium)

Type I



Type II



Why don't see beats for type II but see beats for Type I ?

Complementarity:

If it is possible to distinguish by which history the atom has gone from initial to final state, then no beats will occur.

Do not need to perform experiment, sufficient that it is only possible!

Information (or lack of) leads to detection of beats.

<u>Type I</u>

Detector Cannot tell which "path" generated a photon $\left\{ \begin{array}{l} \text{Cannot tell} \\ |g\rangle \to |a\rangle \to |g\rangle \\ \text{OR} \\ |g\rangle \to |b\rangle \to |g\rangle \end{array} \right.$

- lack of information $\Rightarrow$ interference / beats.
- paths are indistinguishable

~~Type III~~ Wavefunction

$$|\psi\rangle = C_1(t)\,|g\rangle|n_a\rangle + C_2\,|g\rangle|n_b\rangle$$

$$|\psi|^2 = |\langle\psi|\psi\rangle|^2 = (|C_1|)^2 + (|C_2|)^2 + \underbrace{C_1^* C_2\, e^{i(\omega_a-\omega_b)t}\,\langle g|g\rangle}_{\text{interference term!}}$$

<u>Type II</u>

Paths here <u>are</u> <u>not</u> indistinguishable!

Do not expect interference.

1) Excite to $|a\rangle$

2) Decay to either $|g\rangle$ or $|h\rangle$. But one <u>knows</u> the final state

Can tell difference

$$|g\rangle \to |a\rangle \to |g\rangle$$
$$|g\rangle \to |a\rangle \to |h\rangle$$

Wave function

$$|\psi\rangle = C_1\, |g\rangle\, |n_g\rangle + C_2\, |h\rangle\, |n_h\rangle$$

$$|\langle\psi|\psi\rangle|^2 = |C_1|^2 + |C_2|^2 + \underbrace{C_1^* C_2\, e^{-i(\omega_g-\omega_h)t} \langle g|h\rangle}_{=\text{ zero!}}$$

since $\langle g|h\rangle = 0$

— orthogonal states

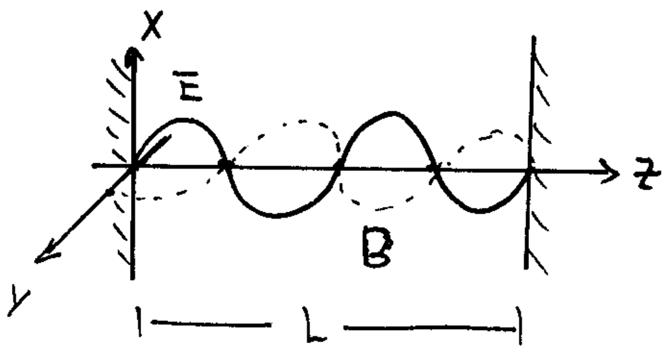
Amount of information $\Rightarrow$ Fringe Contrast / Sharpness

Go Back to the experiment of Walther et al. By turning the Pockel's Cell on (closed) one could tell the paths and thus the interference when away.

Now to the good stuff . . . .

Field Quantization

- Consider light in a cavity of perfectly conducting walls
- Get standing waves
- Electric field along $\hat{x}$, Magnetic Field along $\hat{y}$



$$E_x(z,t) = \left(\frac{2\omega^2}{V\varepsilon_0}\right)^{1/2} q(t)\,\sin(kz)$$

$$B_y(z,t) = \frac{\mu_0\varepsilon_0}{k}\left(\frac{2\omega^2}{V\varepsilon_0}\right)^{1/2} \dot{q}(t)\,\cos(kz)$$

$k = \frac{\omega n}{c}$ $n=1$

$\omega_m = c\left(\frac{m\pi}{L}\right)$

$V \equiv$ volume

$q(t) \Rightarrow$ canonical position

$\dot{q}(t) = p(t) \Rightarrow$ canonical momentum

Classical Energy or Hamilitonian

$$H = \frac{1}{2}\int dV \left(\varepsilon_0 E_x^2 + \frac{1}{\mu_0} B_y^2\right)$$

Rewrite in terms of canonical terms

$$H = \frac{1}{2}\left(p^2 + \omega^2 q^2\right)$$

The system we have described in a harmonic oscillator. Thus we can use our quantum description of the harmonic oscillator

$$[\hat{q}, \hat{p}] = i\hbar$$

Quantized fields

$$\hat{E}_x(z,t) = \left(\frac{2\omega^2}{V\epsilon_0}\right)^{1/2} \hat{q}(t)\sin(kz)$$

$$\hat{B}_y(z,t) = \left(\frac{2\omega^2}{V\epsilon_0}\right)^{1/2} \hat{p}(t)\cos(kz)$$

$$\hat{H} = \frac{1}{2}\left(\hat{p}^2 + \omega^2\hat{q}^2\right)$$

$\hat{p}$ + $\hat{q}$ are Hermitian (observables)

Introduce non Hermitian operators

$\hat{a}^+$ creation

$\hat{a}$ annihilation

$$\boxed{\begin{aligned} a^+ &= \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} - i\hat{p}) \\ a &= \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + i\hat{p}) \end{aligned}}$$

so

$$\hat{E}_x = \mathcal{E}_0(\hat{a} + \hat{a}^+)\sin kz \qquad \mathcal{E}_0 \equiv \sqrt{\hbar\omega/\epsilon_0 V}$$

$$\hat{B}_y = \mathcal{B}_0\frac{1}{i}(a - a^+)\cos kz \qquad \mathcal{B}_0 \equiv \mu_0/k\sqrt{\frac{\epsilon_0\hbar\omega^3}{V}}$$

Also $[\hat{a}, \hat{a}^+] = 1$

$$\boxed{H = \hbar\omega\left(\hat{a}^+\hat{a} + 1/2\right)}$$

Time Dependence of $\hat{a}$ + $\hat{a}^{+}$

Heisenberg picture $\Rightarrow$ Liouville Eq (as we did before)

$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar}[H, \hat{a}] = -i\omega\hat{a}$$

Solution $\boxed{\hat{a}(t) = \hat{a}(0)\, e^{-i\omega t}}$

For $\hat{a}^{+}$ $\boxed{\hat{a}^{+}(t) = \hat{a}^{+}(0)\, e^{i\omega t}}$

Number Operator $\hat{n}$

$\hat{n} \equiv a^{+}a \Rightarrow$ eigenstate $|n\rangle$ with energy E_n

Thus $\hat{H}|n\rangle = \hbar\omega(\hat{a}^{+}\hat{a} + \frac{1}{2})|n\rangle = E_n|n\rangle$

write $\hbar\omega\left(\hat{a}^{+}a + \frac{1}{2}\right)(a^{+}|n\rangle) = (E_n + \hbar\omega)(a^{+}|n\rangle)$

$$\hat{H}(a^{+}|n\rangle) = (E_n + \hbar\omega)(a^{+}|n\rangle)$$

eigenState $\hat{a}^{+}|n\rangle$ has eigenvalue $E_n + \hbar\omega$

$\hat{a}^{+} \Rightarrow$ creates quantum of energy $\hbar\omega$

Also $\hat{H}(a|n\rangle) = (E_n - \hbar\omega)(a|n\rangle)$

So $\boxed{E_n = \hbar\omega(n + \frac{1}{2})}$

Zero point energy $n=0$

$$\hat{H}|0\rangle = \hbar\omega\left(\hat{a}^{+}\hat{a} + \tfrac{1}{2}\right)|0\rangle = \frac{1}{2}\hbar\omega|0\rangle$$

$\frac{1}{2}\hbar\omega \Rightarrow$ zero energy

$$\hat{a}|0\rangle = 0 \qquad \hat{a}^{+}|0\rangle = |1\rangle$$

Normalizing Number States

$$\hat{n}|n\rangle = n|n\rangle \qquad \langle n|n\rangle = 1$$

$$\hat{a}|n\rangle = c_n|n-1\rangle$$

$$\Rightarrow c_n = \sqrt{n}$$

$$\hat{a}^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$\boxed{\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle \qquad \hat{a}^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle}$$

$$\boxed{|n\rangle = \frac{(\hat{a}^{+})^{n}}{\sqrt{n!}}|0\rangle}$$

Non vanishing elements

$$\boxed{\begin{gathered}\langle n-1|\hat{a}|n\rangle = \sqrt{n}\\ \langle n+1|a^{+}|n\rangle = \sqrt{1+n}\end{gathered}}$$

Lecture 33 Multimode Fields

$|n\rangle$ has a well defined energy but not of field since

$$\langle n|\hat{E}_x|n\rangle = 0 \qquad (\text{expectation of } \sin())$$

But the energy density which is proportional to E^2 is not zero

$$\langle n|E_x^2|n\rangle = 2\mathcal{E}_0^2 \sin^2(kz)\,(n+\tfrac{1}{2})$$

Variance of $\hat{E}$

$$\langle(\Delta\hat{E}_x)^2\rangle = \langle\hat{E}_x^2\rangle - \langle\hat{E}_x\rangle^2 =$$

so

$$\boxed{\Delta E_x = \sqrt{2}\,\mathcal{E}_0 \sin(kz)\sqrt{n+1/2}}$$

Which is valid for even $n=0$! $\Rightarrow$ Vacuum fluctuations

Commutation between $\hat{n}$ + $\hat{E}$

$$[\hat{n}, \hat{E}_x] = \mathcal{E}_0 \sin(kz)(\hat{a}^+ - \hat{a})$$

OR

$$\Delta n\,\Delta E_x \geq \frac{1}{2}\mathcal{E}_0\,|\sin(kz)|\,|\langle\hat{a}^+ - \hat{a}\rangle|$$

If field is accurately known, # of photons is uncertain

$\Rightarrow$ Amplitude + phase in QM cannot be both well defined.

In analogy with $\Delta t \, \Delta E$ we got number / phase

$$\Delta n \, \Delta \phi \geq 1$$

Quadrature Operators

Write out time dependence

$$\hat{E}_x = \mathcal{E}_0 \left(\hat{a} e^{-i\omega t} + a^{\dagger} e^{i\omega t} \right) \sin(kz)$$

Define quadrature operators

$$\hat{X}_1 = \frac{1}{2} \left(\hat{a} + \hat{a}^{\dagger} \right) \qquad \text{"In-phase"}$$

$$\hat{X}_2 = \frac{1}{2i} \left(\hat{a} - \hat{a}^{\dagger} \right) \qquad \text{"In-quadrature"} \quad (90^\circ \text{ out of phase})$$

So $[\hat{X}_1, \hat{X}_2] = i/2 \Rightarrow \langle (\Delta X_1)^2 \rangle \langle (\Delta X_2)^2 \rangle \geq 1/16$

And $\langle n | X_1^2 | n \rangle = \frac{1}{4} (2n+1)$

in-phase   in-quadrature

~~Also~~

$$\boxed{\hat{E}_x = 2 \mathcal{E}_0 \sin(kz) \left(\hat{X}_1 \cos\omega t + \hat{X}_2 \sin\omega t \right)}$$

Vacuum minimizes the uncertainty product

$$\langle \hat{X}_1^2 \rangle_{vac} = 1/4 = \langle \hat{X}_2^2 \rangle$$

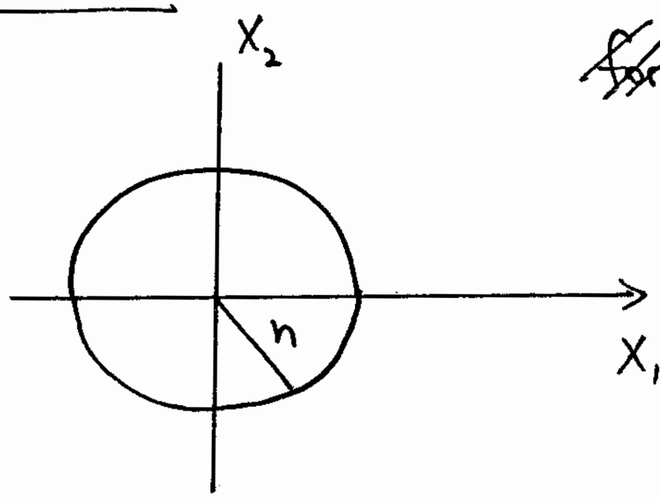
Squeezed Vacuum $\Rightarrow$ Miniproject Charmin

Phase space pictures

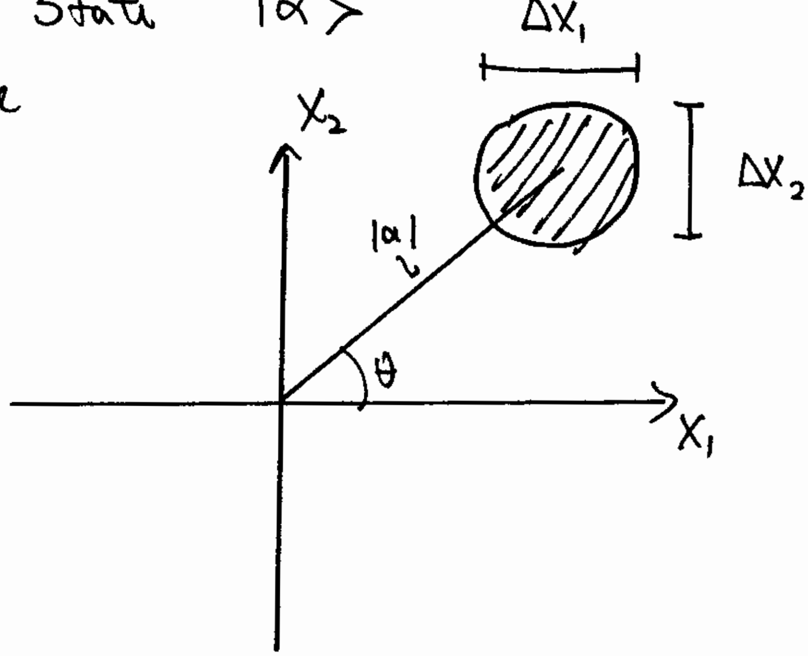
~~for Ins~~

For $|n\rangle$
(number state)

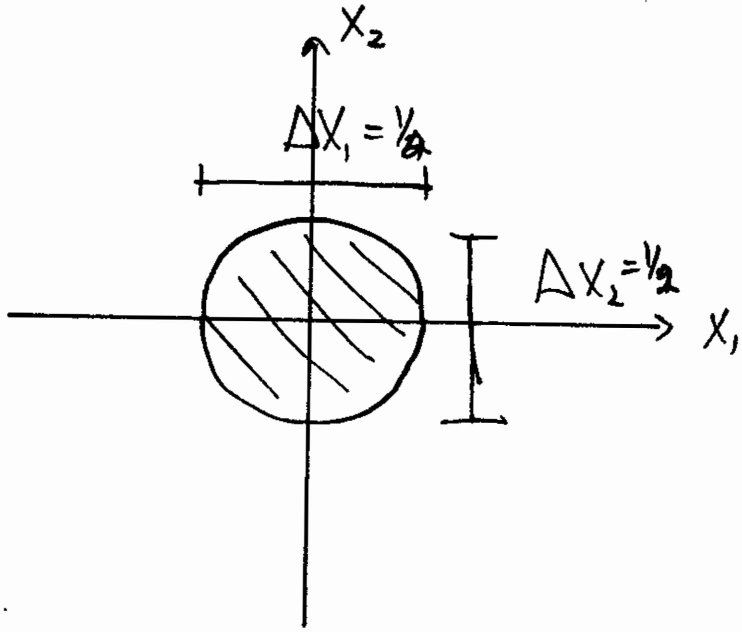
a circle



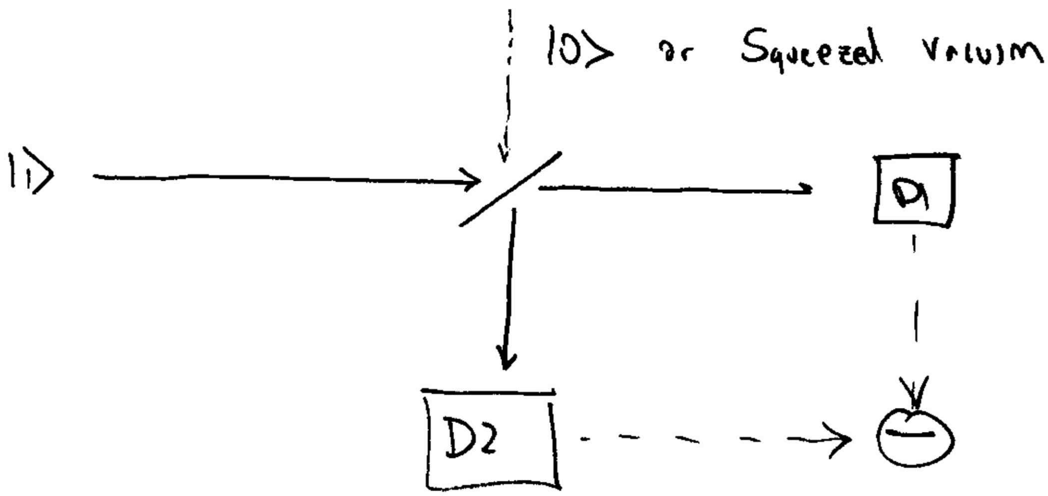
For a coherent state $|\alpha\rangle$
a filled circle



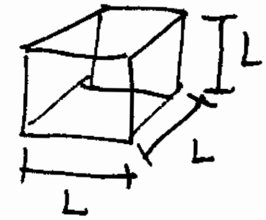
For vacuum



Balanced Detection with Squeezed Light



Multimode fields



- Generalization of single mode result

Cubical Cavity with periodic boundary conditions

Express field in terms of the vector potential $\vec{A}$

$$\vec{E} = -\partial_t \vec{A} \quad + \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

$$\nabla^2 \bar{A} - \frac{1}{c}\, \partial_t^2 \bar{A} = 0$$

The boundary conditions impose

$$k_x = \frac{2\pi}{L} m_x \qquad k_y = \frac{2\pi}{L} m_y \qquad k_z = \frac{2\pi}{L} m_z$$

- Total # of modes in k space

$$\Delta m = 2\left(\frac{L}{2\pi}\right)^3 \Delta k_x \,\Delta k_y \,\Delta k_z$$

$$dm = 2\,\frac{V}{8\pi^3}\, dk_x\, dk_y\, dk_z = 2\,\frac{V}{8\pi^3}\, k^2 dk \sin\theta \, d\theta \, d\varphi$$

The vector potential can be express a superposition of plane waves

$$\vec{A}(\bar{r},t) = \sum_{\bar{k},s} \hat{e}_{\bar{k},s} \left(A_{ks}(t)\, e^{i\bar{k}\cdot\bar{r}} \;\; A_{\bar{k}s}(t)\, e^{-i\bar{k}\cdot\bar{r}} \right)$$

Sum over k → Sum over m

Sum over s → two polarizations orthogonal

Get time dependences of A_{ks} from Wave Eq

$$A_{ks}(t) = A_{ks}\, e^{-i\omega_k t}$$

Write Electric + Magnetic fields

$$\mathbf{E}(\vec{r},t) = i \sum_{\bar{k},s} \omega_k \hat{e}_{ks} \left[A_{\bar{k}s}\, e^{i(\bar{k}\cdot\bar{r} - \omega_k t)} - A^{*}_{\bar{k}s}\, e^{-i(\bar{k}\cdot\bar{r} - \omega t)} \right]$$

$$\mathbf{B}(\bar{r},t) = \frac{i}{c} \sum_{\bar{k},s} \omega_k \left(\frac{\bar{k}}{|\bar{k}|} \times \hat{e}_{ks} \right) \left[\quad " \quad \right]$$

Write Hamiltonian by integrating over volume

$$H = 2 \varepsilon_0 V \sum_{\bar{k},s} \omega_k^2 A_{ks} A^{*}_{ks}$$

Introduce canonical variables $\bar{q}_{\bar{k}s}$ + $\bar{p}_{\bar{k}s}$

$$A_{\bar{k}s} = \frac{1}{2\omega_k \sqrt{\varepsilon_0 V}} \left(\omega_k \bar{q}_{ks} + i \bar{p}_{ks} \right)$$

Then $\boxed{H = \frac{1}{2} \sum_{\bar{k}s} \left(p_{ks}^2 + \omega_k^2 q_k^2 \right)}$ Classical field

operators quantize $\hat{p}_{\bar{k}s}$ + $\hat{q}_{\bar{k}s}$

$$\boxed{[\hat{q}_{\bar{k}s}, \hat{p}_{\bar{k}'s'}] = i\hbar\, \delta_{\bar{k}\bar{k}'} \delta_{ss'}}$$

$$\boxed{\hat{H} = \sum_{\bar{k}s} \hbar\omega_k \left(\hat{a}^{\dagger}_{\bar{k}s} \hat{a}_{\bar{k}s} + \frac{1}{2} \right) \qquad \hat{n}_{\bar{k}s} = \hat{a}^{\dagger}_{ks} \hat{a}_{\bar{k}s}}$$

For j^{th} mode and all modes

$$\hat{H} = \sum_j \hbar\omega_j \left(\hat{n}_j + 1/2\right)$$ Field Hamiltonian for all modes

where $\hat{n}_j = \hat{n}_{\vec{k}_j s_j}$

Multimode photon number state

$$|n_1\rangle|n_2\rangle|n_3\rangle \cdots \equiv |n_1\, n_2\, n_3 \cdots\rangle$$

$$= |\{n_j\}\rangle$$

So

$$\hat{H}\,|\{n_j\}\rangle = E\,|\{n_j\}\rangle$$

all modes

Multimode are orthogonal

$$\langle n_1\, n_2 \cdots n_j \cdots | n_1\, n_2 \cdots\rangle = \delta_{n_1 n_1'}\,\delta_{n_2 n_2'} \cdots$$

operation by $\hat{a}_j$

$$a_j\,|n_1\, n_2 \cdots n_j \cdots\rangle = \sqrt{n_j}\;|n_1\, n_2 \cdots n_j - 1 \cdots\rangle$$

multimode vacuum $|\{0\}\rangle = |0_1\, , 0_2 \cdots\rangle$

$$|\{n_j\}\rangle = \prod_j \frac{(\hat{a}_j^{\dagger})^{n_j}}{\sqrt{n_j}}\,|\{0\}\rangle$$

Back to the fields

Now $$\hat{A}_{\bar{k}s} = \sqrt{\frac{\hbar}{2\omega_k \varepsilon_0 V}}\; \hat{a}_{ks}$$

and $$\hat{E}(\bar{r},t) = i \sum_{\bar{k}s} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\; \hat{e}_{\bar{k}s} \left(\hat{a}_{ks}\, e^{i(\bar{k}\cdot\bar{r} - \omega_k t)} + \hat{a}^{\dagger}_{\bar{k}s}\, e^{-i(\bar{k}\cdot\bar{r} - \omega_k t)} \right)$$

Review

Show

$$\langle n|\hat{E}_x|n\rangle = 0 \qquad \langle \hat{E}(t)\rangle$$

We want some expectation value that varies sinusodally with time

$$\langle \ddot{-}|E_x|\ddot{-}\rangle \simeq \sin(\omega t)$$

Need some state that looks more like the classical harmonic osillator

Lecture 34 Coherent States

Review $\langle n|E_x|n\rangle = 0$

Brief discussion on the quantum phase

Dirac $\quad \hat{a} = e^{i\hat{\phi}}\sqrt{\hat{n}} \qquad \hat{a}^{+} = \sqrt{\hat{n}}\, e^{-i\hat{\phi}}$

$$[\hat{n}, \hat{\phi}] = i \qquad \Delta n \, \Delta\phi \geq 1/2$$

Problems with this definition:

1) $\hat{\phi}$ is not Hermitian

Eg. If $\hat{\phi}$ is Hermitian then $e^{i\hat{\phi}}$ is unitary

But $\quad (e^{i\hat{\phi}})^{+} e^{i\hat{\phi}} = 1 \quad$ <u>but</u> $\quad e^{i\hat{\phi}}(e^{i\hat{\phi}})^{+} \neq 1$

2) Is $\hat{\phi}$ an angle operator?

Its <u>not</u> periodic $\qquad -\infty < \phi < \infty \qquad \psi(\phi) \neq \psi(\phi + 2\pi)$

And $\quad \Delta\phi > 2\pi$

Another approach Susskind - Glogower operators

operators analogous to exponential phase factor $e^{i\phi}$

$$\hat{E} \rightarrow e^{i\phi} \qquad \hat{E} = \sum_{n=0}^{\infty} |n\rangle\langle n+1|$$

$$\hat{E}^{+} \rightarrow e^{-i\phi} \qquad \hat{E}^{+} = \sum_{n=0}^{\infty} |n+1\rangle\langle n|$$

not field

$$EE^{+} = 1 \qquad E^{+}E = 1 - |0\rangle\langle 0|$$

unitary for large $n \Rightarrow$ approx unitary !!

Eigenstate of $\hat{E}$

$$\hat{E}|\phi\rangle = e^{i\phi}|\phi\rangle$$

where $$|\phi\rangle = \sum_{n=0}^{\infty} e^{in\phi}|n\rangle$$

Phase distribution

$$P(\phi) \equiv \frac{1}{2\pi}|\langle\phi|\psi\rangle|^2$$

$$\int_0^{2\pi} P(\phi)\, d\phi = 1$$

How to relate this phase to experimental measurements?

$|\phi\rangle$ probability to measure phase

Measuring phase difficult classically + quantum mechanically.

- photon number states have uniform phase distribution over range 0 to 2π. No well defined phase

Distribution of Phase for $|n\rangle$

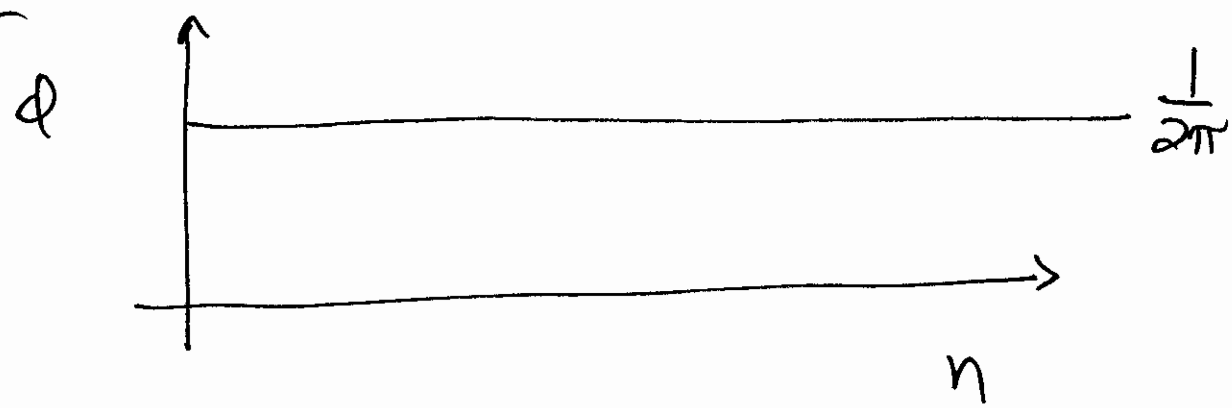
Uniform probability density

$$\Rightarrow P(\phi) = \frac{1}{2\pi} \qquad \text{all but } c_n = 1$$

with $\Delta\phi = \frac{\pi}{\sqrt{3}}$

Uniform distribution over 0 to 2π

True for all number states



uniform phase distribution : work it out

$$P(\phi) = \frac{1}{2} \left| \langle \phi | \psi \rangle \right|^2$$

$$= \frac{1}{2} \left| \sum_n \langle \phi | c_n | n \rangle \right|^2 \qquad e^{2i\phi}$$

$$= \frac{1}{2\pi} \left| \sum_n \sum_m \langle m | e^{-im\phi} c_n | n \rangle \right.$$

$m = n$

$$= \boxed{\frac{1}{2\pi}}$$

Coherent States

— How to get classical limit?

We should get classical limit as $n \to \infty$

But $\langle n | \hat{E}_x | n \rangle = 0$ even if $n \to \infty$!!

Fixed point in space in classical field oscillates sinusoidally

But $\langle n \rangle$ does not!

Coherent States

"most classical" quantum states of harmonic oscillator.

— Want non zero expectation value of $\hat{E}_x$.

Need superposition of $|n\rangle$

Seek eigenstates of $\hat{a}$

$$\boxed{\hat{a}|\alpha\rangle = \alpha|\alpha\rangle}$$

"Right" eigenstate

$$\langle\alpha|\hat{a}^\dagger = \alpha^*\langle\alpha|$$

$|n\rangle \Rightarrow$ complete set so

$$|\alpha\rangle = \sum_{n=0}^{\infty} C_n |n\rangle$$

Operate by $\hat{a}$

$$\hat{a}|\alpha\rangle = \sum_{n=1}^{\infty} c_n \sqrt{n}\, |n-1\rangle = \alpha \sum_{n=0}^{\infty} c_n |n\rangle$$

So $$c_n \sqrt{n} = \alpha\, c_{n-1} \qquad \text{Same } \underline{\underline{n}}$$

$$c_n = \frac{\alpha}{\sqrt{n}} c_{n-1} = \frac{\alpha^2}{\sqrt{n(n-1)}} c_{n-2} = \cdots \frac{\alpha^n}{\sqrt{n!}} c_0$$

So $$\boxed{|\alpha\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle}$$

Find c_0 by normalization

$$\langle\alpha|\alpha\rangle = 1 = |c_0|^2 e^{|\alpha|^2}$$

so $$c_0 = e^{-|\alpha|^2/2}$$

Thus $$\boxed{|\alpha\rangle = \exp(-\tfrac{1}{2}|\alpha|^2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle}$$

Write out electric field and expectation value

$$\langle \alpha | \hat{E}_x | \alpha \rangle = 2|\alpha| \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \sin(\omega t - \bar{k}\cdot\bar{r} - \theta)$$

using

$$E = i\sqrt{\frac{\hbar}{2\varepsilon_0 V}} \left(\hat{a}\, e^{i\bar{k}\cdot\bar{r} - \omega t} + \hat{a}^\dagger e^{-i\bar{k}\cdot\bar{r} - \omega t} \right)$$

$$\alpha = |\alpha| e^{i\theta}$$

$$\boxed{\langle \alpha | E_x | \alpha \rangle = 2|\alpha| \left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{1/2} \sin(\omega t - \bar{k}\cdot\bar{r} - \theta)}$$

And

$$\Delta E_x = \langle (\Delta \hat{E}_x) \rangle^{1/2} = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}$$

True for all ~~the~~ n.

For quadrature operators

$$\langle (\Delta \hat{X}_1)^2 \rangle_\alpha = \frac{1}{4} = \langle (\Delta \hat{X}_2)^2 \rangle_\alpha$$

Physical meaning of α?

$|\alpha| \equiv$ related to field amplitude

$$\bar{n} \equiv \langle \alpha | \hat{n} | \alpha \rangle = |\alpha|^2 \quad \text{Average photon \#}$$

$$\langle \alpha | \hat{n}^2 | \alpha \rangle = \bar{n}^2 + \bar{n}$$

so $$\Delta n = \sqrt{\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2} = \sqrt{\bar{n}}$$

For case where $\Delta n = \sqrt{\bar{n}}$

Varince = sq rt. of average

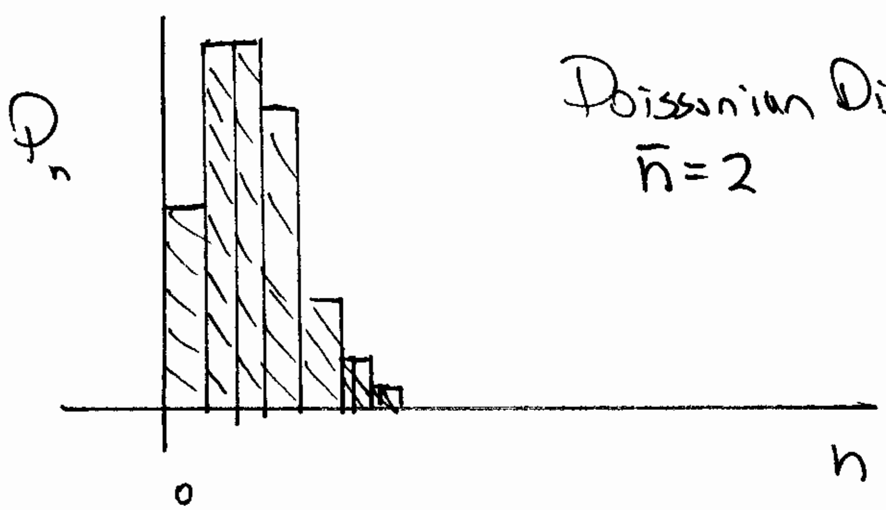
Poissonian distribution with mean $\bar{n}$

$$\frac{\Delta n}{\bar{n}} = \frac{1}{\sqrt{\bar{n}}}$$

For n photons

$$P_n = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}$$

$$= e^{-\bar{n}} \frac{\bar{n}^n}{n!}$$



Poissonian Distribution
$\bar{n} = 2$

Phase distribution

$$P(\phi) = \frac{1}{2\pi} |\langle \phi | \alpha \rangle|^2 = \frac{1}{2\pi} e^{-|\alpha|^2} \left| \sum e^{in(\phi - \theta)} \frac{|\alpha|^n}{\sqrt{n!}} \right.$$

For large n Poissonian → Gaussian

$$P(\phi) = \sqrt{\frac{2|\alpha|^2}{\pi}} \exp(-2|\alpha|^2(\phi - \theta)^2)$$

⇒ Phase distribution is Gaussian over $\underline{n}$

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

$$|\alpha\rangle = \sum_{n=0} c_n |n\rangle$$

$$\hat{a}|\alpha\rangle = \sum_{n=1} c_n\sqrt{n}\ |n-1\rangle = \alpha \sum_{n=0} c_n |n\rangle$$

$$C_{n-1}\sqrt{n} = \alpha C_n$$

$$C_n = \frac{\alpha}{\sqrt{n}} C_{n-1} = \frac{\alpha^2}{\sqrt{n(n-1)}} C_{n-2}$$

$$= \cdots \frac{\alpha^n}{\sqrt{n!}} C_0$$

$$|\alpha\rangle = \sum_n \frac{\alpha^2}{\sqrt{n!}} C_0 |n\rangle$$

Find C_0

$$\langle\alpha|\alpha\rangle = 1$$

$$\cancel{\frac{(\alpha^n)^2}{n!} |C_0|^2 \sum_n \langle n|n\rangle = 1}$$

So $\boxed{C_0 = \exp(-\tfrac{1}{2}|\alpha|^2)}$

$$1 = |C_0|^2 \sum_n \sum_m \frac{\alpha^{*n}\alpha^m}{\sqrt{n!\,m!}} \langle n|m\rangle = \sum_n |C_0|^2 \frac{|\alpha|^{2n}}{n!}$$

$$= |C_0|^2 e^{|\alpha|^2}$$

Lecture 35 More on coherent states

$$|\alpha\rangle = \exp(-\tfrac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$ Coherent States

$$|\phi\rangle = \sum_{n=0}^{\infty} e^{in\phi}|n\rangle$$ Phase states

Poissonian Distribution

$$\boxed{\frac{e^{-\lambda}\lambda^x}{x!}}$$

# Deaths x	Observed Corp-years for x deaths
0	109
1	65
2	22
3	3
4	1

200 Corp years

Bortkiewicz $\Rightarrow$ Horse Kick Deaths of Prussian cavalry

For n distribution

$$P(n) = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!} \Leftarrow \text{Poisson Distribution}$$

Phase distribution

$$P(\phi) = \frac{1}{2\pi}|\langle\phi|\alpha\rangle|^2 = \frac{1}{2\pi}e^{-|\alpha|^2}\left|\sum_{n=0}^{\infty}e^{in(\phi-\theta)}\frac{|\alpha|^n}{\sqrt{n!}}\right|^2$$

use $\alpha = |\alpha|e^{i\theta}$

For large $|\alpha|^2$ Poisson distribution becomes Gaussian

$$P(\phi) = \sqrt{\frac{2|\alpha|^2}{\pi}}\exp\left(-2|\alpha|^2(\phi-\theta)^2\right)$$

For n distribution — Poisson

$$\langle \alpha | \hat{n} | \alpha \rangle = \langle \alpha | a^\dagger a | \alpha \rangle$$

$$= \left(\left(\langle \alpha | a^\dagger \right) \left(a | \alpha \rangle \right) \right)$$

$$= \alpha^* \alpha = |\alpha|^2 \equiv \bar{n}$$

$$\langle \alpha | \hat{n}^2 | \alpha \rangle = |\alpha|^4 + |\alpha|^2 = \bar{n}^2 + \bar{n}$$

So

$$\Delta n = \sqrt{\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2} = \sqrt{\left(\bar{n}^2 + \bar{n} \right) - \bar{n}^2}$$

$$= \cancel{\bar{n}^4 + 2\bar{n}\bar{n}^2 + n^2 - n^2}$$

$$= \sqrt{\bar{n}}$$

Average value $\bar{n}$

Standard deviation $\sqrt{\bar{n}}$

Details

$$P(n) = |\langle n|\alpha\rangle|^2 = \left| \sum_{m=0}^{\infty} e^{-|\alpha|^2/2} \frac{\alpha^m}{\sqrt{m!}} \langle n|m\rangle \right|^2$$

$$= \frac{\alpha^n (\alpha^n)^*}{n!} e^{-|\alpha|^2}$$

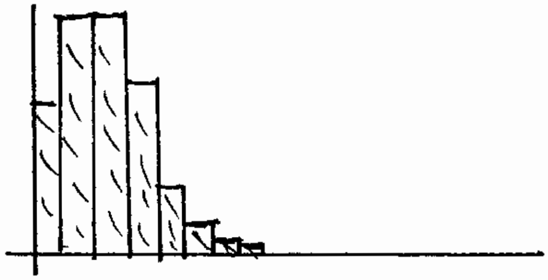
$$= \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}$$

$$\left(|\alpha| e^{+i\theta}\right)^2$$

$$P(\phi) = \frac{1}{2\pi} \left| \langle \phi|\alpha\rangle \right|^2 = \frac{1}{2\pi} \left| \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} e^{-im\phi} \langle m|n\rangle \frac{\alpha^n}{\sqrt{n!}} e^{-|\alpha|^2/2} \right|^2$$

$$= \frac{1}{2\pi} \left| \sum_{n=0} e^{+i n(\phi+\theta)} \frac{|\alpha|^n}{\sqrt{n!}} \right|^2 e^{-|\alpha|^2/2 \cdot 2}$$

Poisson Distribution



Expectation value

$$E_x = i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\left(\hat{a}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \hat{a}^{\dagger} e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\right)$$

$$\Downarrow$$

$$\langle\alpha|E_x|\alpha\rangle = \langle\alpha| \qquad\qquad |\alpha\rangle$$

$$\langle\alpha|\hat{a}|\alpha\rangle = \langle\alpha|\alpha\rangle\alpha$$

$$\langle\alpha|a^{\dagger}|\alpha\rangle = \alpha^{*}\langle\alpha|\alpha\rangle$$

Since $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, $\langle\alpha|\hat{a}^{\dagger} = \alpha^{*}\langle\alpha|$

So $\langle\alpha|E_x|\alpha\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\left(\alpha\, e^{(+)} - \alpha^{*} e^{(-)}\right)$

Write $\alpha = |\alpha| e^{i\theta}$

$$\langle\alpha|E_x|\alpha\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\,|\alpha|\left(e^{i\theta}e^{(+)} - e^{-i\theta}e^{(-)}\right)$$

$$\boxed{\langle\alpha|E_x|\alpha\rangle = 2|\alpha|\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\,\sin(\omega t - \vec{k}\cdot\vec{r} - \theta)}$$

$$\langle \alpha | E_x^2 | \alpha \rangle = \frac{\hbar\omega}{2\epsilon_0 V}\left(1 + 4|\alpha|^2 \sin^2(\;)\right)$$

$$(\Delta E_x)_\alpha = \sqrt{\langle(\Delta \hat{E}_x)^2\rangle} = \sqrt{\langle \hat{E}_x^2\rangle - \langle E_x\rangle^2} = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}$$

independent of n!!

$$\boxed{(\Delta E_x)_\alpha = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}}$$

Before

$$(\Delta E_x)_n = \sqrt{2\epsilon_0}\,\sin(kz)\sqrt{n + 1/2}$$

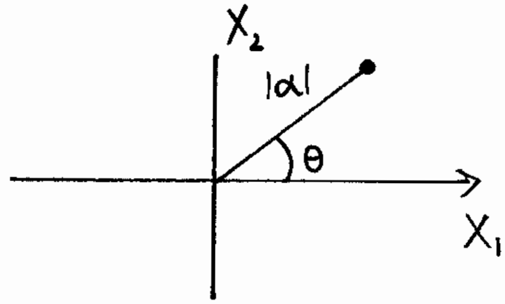
F - quadrature operators

$$\langle(\Delta \hat{X}_1)^2\rangle_\alpha = \frac{1}{4} = \langle(\Delta \hat{X}_2)^2\rangle_\alpha$$

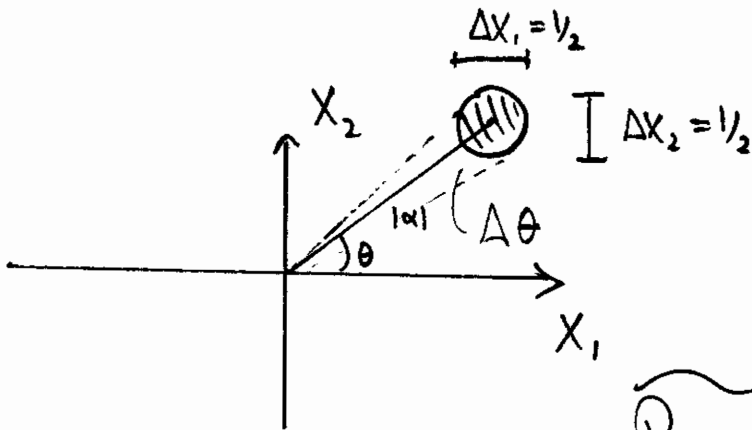
- coherent States have the fluctuations of the vacuum!!

Phase Space Pictures of Coherent States

– Classical Field



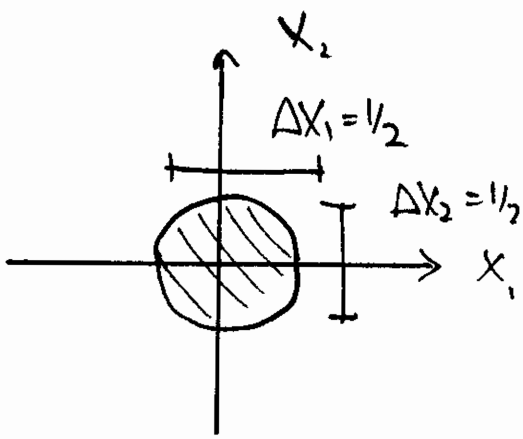
Quantum Field



– Remember

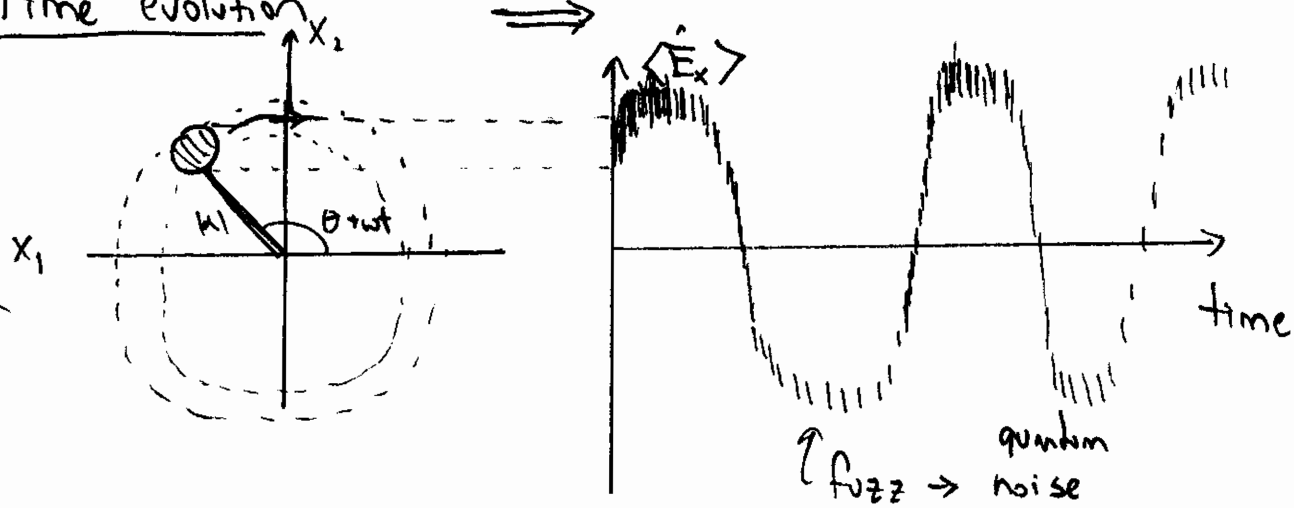
$$\langle \alpha | E_x | \alpha \rangle \cong \sin(\omega t - \bar{k} \cdot \bar{r} - \theta)$$

– Vacuum

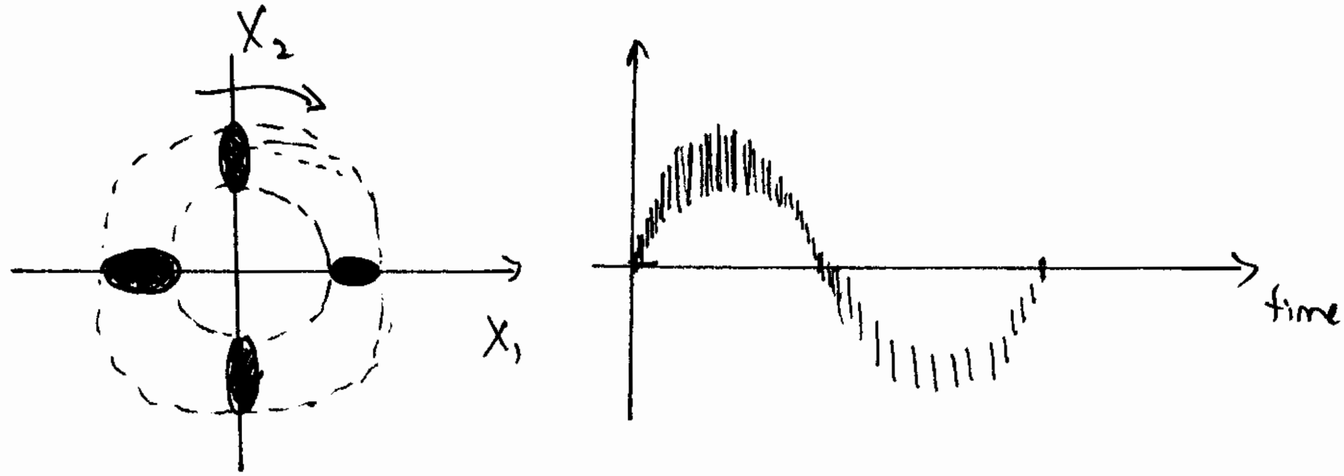


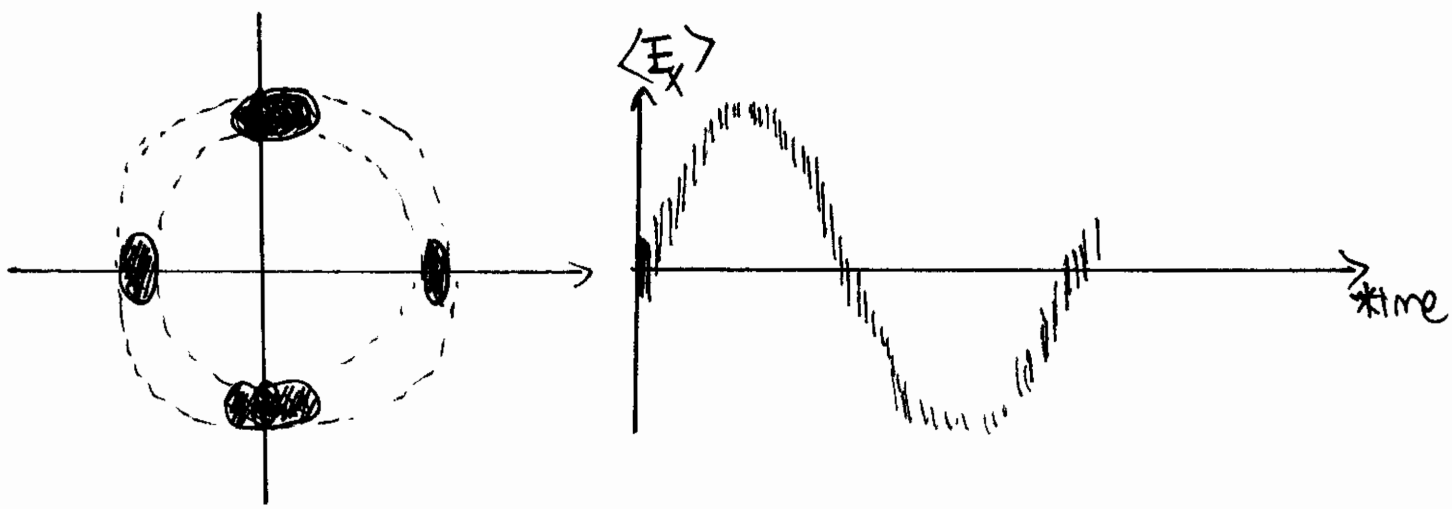
<u>Time evolution</u>

Projection $\Longrightarrow$



Squeezed States





$$\langle \alpha e^{-i\omega t} | \hat{E}_x | \alpha e^{i\omega t} \rangle = 2\mathcal{E}_0 \sin(kz)\, \alpha \cos\omega t$$

Time evolution + flucuations $\Rightarrow$ projection on $\langle X_1 \rangle$ axis

Coherent States as Quantum "Classical" States

1) Expectation value has form of classical
2) Flucuations of E are same as vacum
3) Flucuations of fractional uncertainty for n decrease with increasing $\bar{n}$ $\quad \frac{\Delta n}{\bar{n}} = \frac{1}{\sqrt{\bar{n}}} \quad \Delta n = \sqrt{\overline{n^2} - \bar{n}^2}$
4) States become well localized in phase with $n \to \infty$.

Lecture 36 Even More on Coherent States

Displacement operator

Define coherent states

- "right" eigenstates of the annihilation operator
- states that minimizes the uncertainty relation for two orthogonal field quadratures

  $\rightarrow$ uncertainties equal to vacuum
- Displacement of vacuum

Displacement operator

$$\hat{D}(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a})$$

where $$|\alpha\rangle = \hat{D}(\alpha)|0\rangle$$

Write $$\hat{D}(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}) = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}}$$

But $$e^{-\alpha \hat{a}} = \sum_{l=0}^{\infty} \frac{(-\alpha \hat{a})^l}{l!}$$

zero except for $l=0$

So $$e^{-\alpha^* \hat{a}} = \sum \frac{(-\alpha^* \hat{a})^l}{l!}|0\rangle = |0\rangle \Bigg\} = |0\rangle$$

So $D(\alpha)|0\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{-\alpha a^\dagger} e^{-\alpha^*}|0\rangle$

$$= e^{-\frac{1}{2}|\alpha|^2} e^{-\alpha \hat{a}^\dagger}|0\rangle$$

But

$$e^{-\alpha \hat{a}^\dagger}|0\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}(a^\dagger)^n|0\rangle$$

$$= \sum \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

So
$D(\alpha)|0\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum \frac{\alpha^n}{\sqrt{n!}}|n\rangle = |\alpha\rangle$

coherent state

Unitary matrix

Density operators + Phase space probability Functions

For number states

$$\hat{\rho} = \sum_n \sum_m |m\rangle \rho_{mn} \langle n|$$

For coherent states

$$\hat{\rho} = \iint \langle \alpha' | \hat{\rho} | \alpha'' \rangle |\alpha'\rangle\langle\alpha''| \frac{d^2\alpha' \, d^2\alpha''}{\pi^2}$$

OR

$$\hat{\rho} = \int P(\alpha) \, |\alpha\rangle\langle\alpha| \, d\alpha^2$$

Glauber - Sudarshan function $\Rightarrow$ Quasi Probability function
Phase space distribution

$P(\alpha)$ can be negative unlike a "normal probability"
measure of "non classical" light
All states a quantum mechanical
Some states are more quantum mechanical than others

All states are quantum mechanical, but some states are more quantum mechanical than others.

Wigner function

$$W(q,p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \langle q + \tfrac{1}{2}x \,|\, \hat{\rho} \,|\, q - \tfrac{1}{2}x \rangle e^{ipx/\hbar} \, dx$$

$|q \pm \tfrac{1}{2}x\rangle \equiv$ eigenstates of position operator

For pure state

$$W(q,p) = \frac{1}{2\pi\hbar} \int \psi^*(q - \tfrac{1}{2}x)\, \psi(q + \tfrac{1}{2})\, e^{ipx/\hbar} \, dx$$

$$= |\psi(q)|^2$$

over q

$$\int W(q,p)\, dq = |\phi(p)|^2$$

Q Function

$Q(\alpha) = \langle \alpha | \hat{\rho} | \alpha \rangle / \pi$ phase space probability always positive

Why different distributions?

Normal order $\Rightarrow$ creation left / annihilation right

$$\hat{n} = \hat{a}^{+} a$$

Not normal order $\qquad \hat{n}^2 = \hat{a}^{+} \hat{a} \hat{a}^{+} \hat{a}$

Normal ordered

$$:\hat{n}: = (a^{+})^2 \hat{a}^2$$

Anti normal $\qquad \hat{a}^2 (\hat{a}^{+})^2$

$P(\alpha) \equiv$ Expectation value for operators expressed normally

$Q(\alpha) \equiv$ " " antinormally

$W(\alpha) \equiv$ Symmetrized Products

$P(\alpha) \equiv$ Square law detection

$W(\alpha) \equiv$ homodyne detection

More properties on coherent states

- Time evolution $\Rightarrow$ Coherent state remains a coherent state

$$|\alpha, t\rangle = \exp(-i\hat{H}t/\hbar)|\alpha\rangle \qquad \left\{ \begin{array}{l} \hat{H} = (\hat{a}^\dagger\hat{a} + 1/2)\hbar\omega \\ \hat{H} = (\hat{n} + 1/2)\hbar\omega \end{array} \right.$$

$$= e^{-i\omega t/2} e^{-i\frac{t\hbar\omega}{\hbar}\hat{n}} |\alpha\rangle$$

$$= e^{-i\omega t/2} e^{-i\omega t\hat{n}} |\alpha\rangle$$

$$= e^{-i\omega t/2} e^{-i\omega t|\alpha|^2} |\alpha\rangle \Rightarrow \text{another coherent state}$$

- Coherent States are <u>not</u> orthogonal

number states are orthogonal and complete

$$|\langle\beta|\alpha\rangle|^2 = \exp(-|\beta-\alpha|^2) \neq 0 \qquad \boxed{\langle\alpha|\beta\rangle = \exp(-1/2|\alpha|^2 - 1/2|\beta|^2 + \alpha^*\beta)}$$

nearly orthogonal for large $|\beta-\alpha|^2$

- Completeness $\boxed{\int |\alpha\rangle\langle\alpha| \frac{d^2\alpha}{\pi} = 1}$

- Coherent States not linearly independent

Overcomplete more than enough states

Lecture 37 Quantum Mechanical Description of Beamsplitters

Phase Space Probability Distributions

- P function $$\hat{\rho} = \iint P(\alpha)\, |\alpha\rangle\langle\alpha|\, d^2\alpha$$

$\boxed{\text{$P(\alpha) \equiv$ probability of finding coherent state α}}$

Glauber Sudarshan function P function

$P(\alpha)$ must be real and $\int P(\alpha)\, d^2\alpha = 1$

Phase space probability function Can be negative

- Q Function $$Q(\alpha) = \int d^2\beta\, P(\beta)\, e^{-|\alpha-\beta|^2}$$ (Convolution)

$$Q(\alpha) = \langle\alpha|\hat{\rho}|\alpha\rangle / \pi$$

$$\int Q(\alpha)\, d^2\alpha = 1$$

Can be positive

$Q(\alpha)$ and $P(\alpha)$ coincide as $\bar{n} \to \infty$

- Wigner distribution

$$W(q,p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \langle q + \tfrac{1}{2}x | \hat{\rho} | q - \tfrac{1}{2}x \rangle\, e^{ipx/\hbar}\, dx$$

eigenket of position operator

Marginals $$W(q,p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \psi^*(q - \tfrac{1}{2}x)\, \psi(q + \tfrac{1}{2}x)\, e^{ipx/\hbar}\, dx$$

P function used for expectation of normal ordered operators

$$\boxed{\langle G^{(N)}(a, a^\dagger) \rangle = \int P(\alpha)\, G^{(N)}(\alpha, \alpha^*)\, d^2\alpha}$$

Properties of Wigner function

a) Real but positive & negative

b) Marginals for pure states

$$|\phi(p)|^2 = w(q) = \int W(q,p)\, dp$$

$$|\psi(q)|^2 = w(p) = \int W(q,p)\, dq$$

Exact quantum mechanical distributions of $\hat{p}$ + $\hat{q}$

c) Expectation value of $F(\hat{q},\hat{p})$ if symmetrically ordered, can be computed. Used to calculate averages.

$$\langle F(\hat{q},\hat{p})\rangle = \iint F(q,p)\, W(q,p)\, dq\, dp$$

d) $$\iint W(q,p)\, dq\, dp = \underline{1}$$

e) Compute purestate $$W(q,p) = \frac{1}{2\pi\hbar} \int \exp\left(\frac{i p y}{\hbar}\right) \psi\left(x-\frac{y}{2}\right) \psi^*\left(x+\frac{y}{2}\right) dy$$

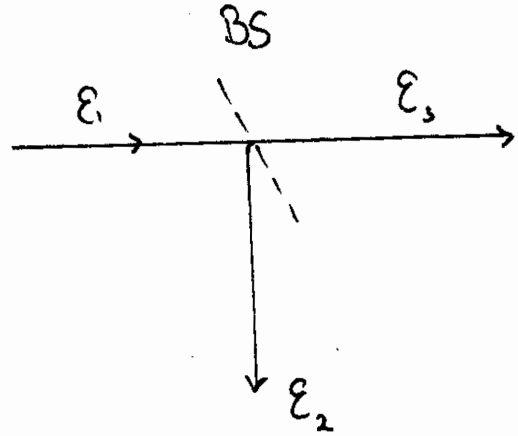
Ordering

- Normal ordered — $\hat{a}$ on right of $\hat{a}^+$ — $(\hat{a}^+)^m (\hat{a})^n$
- Antinormal ordered — $\hat{a}^+$ on right of $\hat{a}$ — $(\hat{a})^n (\hat{a}^+)^m$
- Symmetric ordered (Weyl)

$$\hat{a}\hat{b} = (\hat{a}\hat{b} + \hat{b}\hat{a})/2$$

Quantum Mechanics of Beam Splitters

- Classically



$$\mathcal{E}_2 = r\,\mathcal{E}_1$$
$$\mathcal{E}_3 = t\,\mathcal{E}_1$$

50/50 $\quad |r| = |t| = \frac{1}{\sqrt{2}}$

$$|\mathcal{E}_1|^2 = |\mathcal{E}_2|^2 + |\mathcal{E}_3|^2 \quad \text{or} \quad |r|^2 + |t|^2 = 1$$

- Quantum Mechanics

Can we write

$$\hat{a}_2 = r\,\hat{a}_1 \qquad \hat{a}_3 = t\,\hat{a}_1$$

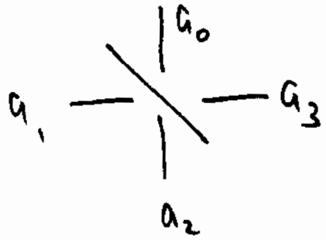
but we have $\quad [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$

so

$$[\hat{a}_2, \hat{a}_2^\dagger] = |r|^2 [\hat{a}_1, \hat{a}_1^\dagger] = |r|^2$$
$$[\hat{a}_3, \hat{a}_3^\dagger] = |t|^2$$
$$[\hat{a}_2, \hat{a}_3^\dagger] = r t^* \neq 0$$

Problem: Beam Splitter treated as 3 port device while it <u>is</u> a four port device

Unused port $\Rightarrow$ vacuum



Write

$$\hat{a}_2 = r\hat{a}_1 + t' a_0 \qquad \hat{a}_3 = t\hat{a}_1 + r' a_0$$

$$\begin{pmatrix} \hat{a}_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} t' & r \\ r' & t \end{pmatrix} \begin{pmatrix} \hat{a}_0 \\ \hat{a}_1 \end{pmatrix}$$

Get

$$|r'| = |r| \qquad |t| = |t'|$$

$$|r|^2 + |t|^2 = 1 \qquad r^* t' + r' t^* = 0$$

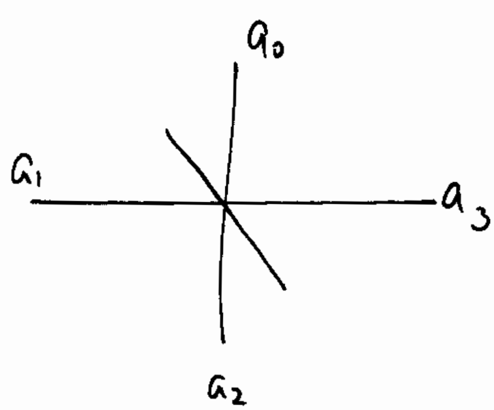
$$r^* t + r' t'^* = 0$$

<u>Example</u> 50/50

transmitted & reflected $\exp(\pm i\pi/2) = \pm i$ $\pi/2$ phase shift

So
$$\boxed{\hat{a}_2 = \frac{1}{\sqrt{2}}(\hat{a}_0 + i\hat{a}_1) \qquad \hat{a}_3 = \frac{1}{\sqrt{2}}\left(i(\hat{a}_0) + \hat{a}_1\right)}$$

from 50/50



Solve for $\hat{a}_0$ & $\hat{a}_1$

$$\boxed{\hat{a}_1 = (i\hat{a}_2^{\dagger} + \hat{a}_3^{\dagger})1/\sqrt{2}}$$

$$\boxed{\hat{a}_0 = (\hat{a}_2 + i\hat{a}_3)\frac{1}{\sqrt{2}}}$$

For a given input what is the output?!

- Vacuum

$$|0\rangle_0 |0\rangle_1 \xrightarrow{BS} |0\rangle_2 |0\rangle_3$$

- one port light

$$|0\rangle_0 |1\rangle_1 \Rightarrow \hat{a}_1^\dagger |0\rangle_0 |0\rangle_1$$

So

But $$\boxed{\hat{a}_1 = \left(i\,\hat{a}_2^\dagger + \hat{a}_3^\dagger\right)/\sqrt{2}}$$

$$|0\rangle_0 |1\rangle_1 \xrightarrow{BS} \frac{1}{\sqrt{2}}\left(i\,\hat{a}_2^\dagger + \hat{a}_3^\dagger\right) |0\rangle_2 |0\rangle_3$$

$$= \boxed{\frac{1}{\sqrt{2}}\left(i\,|1\rangle_2 |0\rangle_3 + |0\rangle_2 |1\rangle_3\right)}$$

What does this say? A single photon with vacuum on the other port will be transmitted or reflected with equal probability.

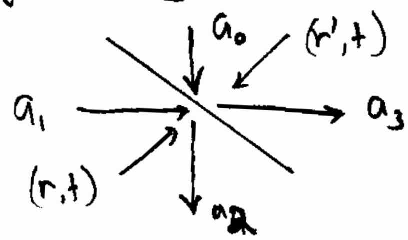
Aspect Experiment!!

This is an entangled state: Cannot be written as a simple product of states of individual modes 2 + 3.

Schrödinger Cat State $\Rightarrow$ $|\psi_{cat}\rangle = \frac{1}{\sqrt{2}}\left(|\text{Dead}\rangle_1 |\text{Alive}\rangle_2 + |\text{Alive}\rangle_2 |\text{Dead}\rangle\right.$

Lecture 38 Interferometry with a single photon

Review Beam splitter gives entangled state



Vacuum

$$\boxed{|0\rangle_0 |0\rangle \xrightarrow{BS} |0\rangle_2 |0\rangle_3}$$ use this

Beam splitter

$\hat{a}_2 = r\hat{a}_1 + t'\hat{a}_0 = \frac{1}{\sqrt{2}}(ia_1 + a_0)$

$\hat{a}_3 = t\hat{a}_1 + r'\hat{a}_0 = \frac{1}{\sqrt{2}}(a_1 + ia_0)$

OR for 50/50

$\hat{a}_1^+ = (i\,\hat{a}_2^+ + \hat{a}_3)\frac{1}{\sqrt{2}}$

$\hat{a}_0^+ = (\hat{a}_2^+ + i\hat{a}_3^+)\,1/\sqrt{2}$

single photon state on port 1

$$|0\rangle_0 |1\rangle_1 = \hat{a}_1^+ |0\rangle_0 |0\rangle_1$$

$$|0\rangle_0 |1\rangle_1 \xrightarrow{BS} \frac{1}{\sqrt{2}}(i\hat{a}_2^+ + \hat{a}_3^+)|0\rangle_2 |0\rangle_3$$

$$= \frac{1}{\sqrt{2}}\left(i\,|1\rangle_2 |0\rangle_3 + |0\rangle_2 |1\rangle_3 \right)$$

could be here

Probability to find photon in one arm or the other

$$|{}_2\langle 0|\,{}_3\langle 1|\psi\rangle|^2 = \frac{1}{2}$$ where $|\psi\rangle = \frac{1}{\sqrt{2}}(i\,|1\rangle_2 |0\rangle_3 + |0\rangle_2 |1\rangle_3)$

$$|{}_2\langle 1|\,{}_3\langle 0|\psi\rangle|^2 = 1/2$$

At Beamsplitter one has equal probability of being transmitted or reflected.

Result of Aspect experiment with anticorrelation factor of $A = 0$:

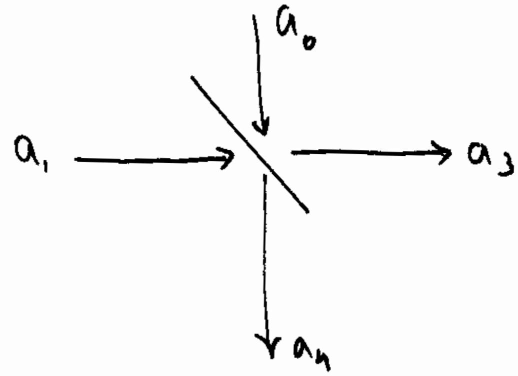
Another way of saying this

$$|0\rangle_0 |1\rangle_1 = \hat{a}_1^\dagger |0\rangle_0 |0\rangle_1 \xrightarrow{BS} \hat{a}_1^\dagger |0\rangle_2 |0\rangle_3$$

$$= (i\hat{a}_2^\dagger + \hat{a}_3)\frac{1}{\sqrt{2}} |0\rangle_2 |0\rangle_3$$

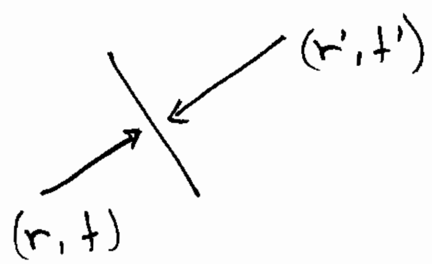
$$= \frac{1}{\sqrt{2}}\left(i |1\rangle_2 |0\rangle_3 + |0\rangle_2 |1\rangle_3 \right)$$

Note on beam splitter



- Dielectric Beam splitter

Coatings on two sides : two different transmission + reflectivities



Phase shift of $\pi/2$ on reflection!

Different than metal beam splitter

Phase shift on reflection

- Half silvered (metal) beam splitter π
- Fiber 3dB coupler $\pi/2$
- Dielectric Beam splitter $\pi/2$

Results from Conservation of Energy

Entangled State

- Cannot be written as simple product of states of the individual modes 2 + 3.

- No classical analog

Density matrix

Input $\hat{\rho}_{in} = |1\rangle_1 |0\rangle_0 \langle 0|_0 \langle 1|_1 = |1\rangle_1 \langle 1|_1 \; |0\rangle_0 \langle 0|_0$

Probability to find photon at port 1

$\cancel{Tr(\rho_{in})}$ $\quad \cancel{\rho_{nm} = \langle n|\hat{\rho}_{in}|m\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$

$\cancel{\rho_{11} = {}_0\langle 0| {}_1\langle 1| \; |1\rangle_1 |0\rangle_0 \; {}_0\langle 0| {}_1\langle 1| \; |1\rangle_1 |0\rangle_0 = 1}$

$\cancel{\rho_{00} =}$ ~~Probability to find photon in port 1 = 1~~

$\Rightarrow$ Show Probability to find vacuum in port 0 = 1

$\cancel{Tr_{2}(\rho_{01}) = \sum_{n=0}^{\infty} {}_0\langle n|\rho_{01}|n\rangle_0 = 1}$

~~Probability to find~~

$$\hat{\rho}_0 = \mathrm{Tr}_1(\hat{\rho}_{01}) = \sum_{n=0}^{\infty} {}_1\langle n | \hat{\rho}_{01} | n \rangle_1$$

$$= {}_1\langle 1 | 1 \rangle_1 \, {}_1\langle 1 | 1 \rangle_1 \quad |0\rangle_0 {}_0\langle 0|$$

$$\boxed{\hat{\rho}_0 = |0\rangle_0 {}_0\langle 0|}$$

Then

Prob to find vacuum in ~~state zero~~ port zero

$$\cancel{\mathrm{Tr}_0(\hat{\rho}_0) = 1} \qquad \boxed{{}_0\langle 0 | \hat{\rho}_0 | 0 \rangle_0 = 1} \checkmark$$

Show probability to find photon in port 1 = 1

$$\hat{\rho}_1 = \mathrm{Tr}_0(\hat{\rho}_{01}) = |1\rangle_1 {}_1\langle 1|$$

So $\cancel{\mathrm{Tr}_1(\hat{\rho}_1) = 1}$ $\qquad \boxed{{}_1\langle 1 | \hat{\rho}_1 | 1 \rangle_1 = 1} \checkmark$

Density matrix after beam splitter

$$\hat{\rho}_{23} = \frac{1}{2}\Big[|1\rangle_2 |0\rangle_3 \; {}_2\langle 1| \; {}_3\langle 0| + |0\rangle_2 |1\rangle_3 \; {}_2\langle 0| \; {}_3\langle 1|$$

$$+ \, i \, |1\rangle_2 |0\rangle_3 \; {}_2\langle 0| \; {}_3\langle 1| - i \, |0\rangle_2 |1\rangle_3 \; {}_2\langle 1| \; {}_3\langle 0| \Big]$$

Make measurement at port 2

$$\hat{\rho}_2 = \mathrm{Tr}_3(\hat{\rho}_{23}) = \sum_{n=0}^{\infty} {}_3\langle n| \hat{\rho}_{23} |n\rangle_3$$

$$= \frac{1}{2}\Big[|0\rangle_2 \; {}_2\langle 0| + |1\rangle_2 \; {}_2\langle 1| \Big]$$

No terms like $|0\rangle\langle 1|$ or $|1\rangle\langle 0| \Rightarrow$ off diagonal

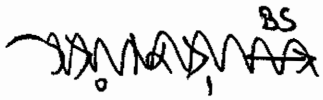
$$\rho_2 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Probability to find photon at port 2

$${}_2\langle 1| \hat{\rho}_2 |1\rangle_2 = 1/2$$

Beamsplitting a coherent state : Use displacement operator

Classical Domain



What to expect? $\Rightarrow$ Beam splitter fields by $\frac{1}{\sqrt{2}}$ with $\pi/2$ phase shift

– Initial state

$$|0\rangle_0 \, |\alpha\rangle_1 = \hat{D}_1(\alpha) \, |0\rangle_0 \, |0\rangle_1$$

displacement operator

$$D_1(\alpha) = \exp(\alpha \hat{a}_1^\dagger - \alpha^* \hat{a}_1)$$

Beam splitter

$$|0\rangle_0 |\alpha\rangle_1 \xrightarrow{BS}$$

$$\exp\left[\frac{\alpha}{\sqrt{2}} (i \hat{a}_2^\dagger + a_3^\dagger) - \frac{\alpha^*}{\sqrt{2}} (-i \hat{a}_2 + \hat{a}_3)\right] |0\rangle_2 \, |0\rangle_3$$

$$= \sum_{n=0}^{\infty} \left(\frac{i\alpha}{\sqrt{2}}\right)^n \frac{1}{\sqrt{n!}} \exp\left(-\frac{1}{2} |i\alpha/\sqrt{2}|^2\right) |n\rangle_2$$

$$* \sum_{m=0}^{\infty} \left(\frac{\alpha}{\sqrt{2}}\right)^m \frac{1}{\sqrt{m!}} \exp\left(-1/2 \, |\alpha/\sqrt{2}|^2\right) |m\rangle_3$$

$$= \left|\frac{i\alpha}{\sqrt{2}}\right\rangle_2 \left|\frac{\alpha}{\sqrt{2}}\right\rangle_3 \Longrightarrow \text{output is } \underline{\underline{\text{not}}} \text{ entangled}$$

Both are coherent states that ~~diffe~~ have same amplitude but are $\pi/2$ ~~ou~~ out of phase

Two photons into Beam splitter 

$$|1\rangle_0 |1\rangle_1 \rightarrow \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left(\hat{a}_2^{\dagger} + i\hat{a}_3\right)\left(i\,\hat{a}_2^{\dagger} + \hat{a}_3\right)|0\rangle_2 |0\rangle_3$$

~~$\hat{a}_0^{\dagger}\hat{a}_1^{\dagger}|0\rangle_2|0\rangle_3$~~

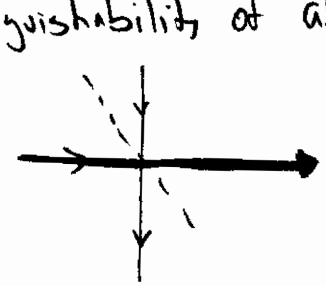
$$= \frac{i}{2}\left(\hat{a}_2^{\dagger}\hat{a}_2^{\dagger} + \hat{a}_3^{\dagger}\hat{a}_3^{\dagger}\right)|0\rangle_2|0\rangle_3$$

$$= \boxed{\frac{i}{2}\left(|2\rangle_2|0\rangle_3 + |0\rangle_2|2\rangle_3\right)}$$

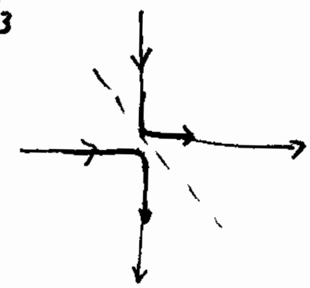
Two photons out one port only

Two indistinguishable processes causes interference

Indistinguishability of "absent" state $|1\rangle_2|1\rangle_3$



OR



Do not observe simutaneous counts due to interference.

This interference comes about due to the indistinguishable states.

Two photons out the same port!!

Density operator for above state

$$\hat{\rho}_{23} = \frac{1}{2}\Big[\left(|1\rangle_2|0\rangle_3\,{}_2\langle 1|\,\langle 0| + |0\rangle_2\,|1\rangle_3\,{}_2\langle 0|\,{}_3\langle 1|\right)$$

$$+\, i\left(|1\rangle_2\,|0\rangle_3\ \ {}_2\langle 0|\,{}_3\langle 1| - |0\rangle_2\,|1\rangle_3\,{}_2\langle 1|\ \ {}_3\langle 0|\right)\Big]$$

Make measurement of mode 2

$$\hat{\rho}_2 = \mathrm{Tr}_3(\hat{\rho}_{23}) = \sum_{n=0}^{\infty} \langle n|\hat{\rho}_{23}|n\rangle$$

$$= \frac{1}{2}\left(|0\rangle_{2\,2}\langle 0| + |1\rangle_{2\,2}\langle 1|\right)$$

No off diagonal terms $|0\rangle\langle 1|$ or $|1\rangle\langle 0|$

No interference

Example: Coherent State

$$|0\rangle_0\,|\alpha\rangle_1 = \hat{D}_1(\alpha)\,|0\rangle_0\,|0\rangle_1$$

$$\hat{D}_1(\alpha) = \exp(\alpha\hat{a}_1^{\dagger} - \alpha^*\hat{a}_1)$$

Displacement operator

$$|0\rangle_0\,|\alpha\rangle_1 \xrightarrow{BS} \exp\left(\alpha\frac{1}{\sqrt{2}}(i\hat{a}_2^{\dagger} + \hat{a}_3^{\dagger}) - \frac{\alpha^*}{\sqrt{2}}(-i\hat{a}_2 + \hat{a}_3)\right]|0\rangle_2\,|0\rangle_3$$

$$= \left|\frac{i\alpha}{\sqrt{2}}\right\rangle_2 \left|\frac{\alpha}{\sqrt{2}}\right\rangle_1$$

From $|i\alpha/\sqrt{2}\rangle|\alpha/\sqrt{2}\rangle$ we can say

- Average photon number $\frac{|\alpha|^2}{(\sqrt{2})^2}$

- phase difference $e^{i\frac{\pi}{2}} = i$

- No interference + no entanglement

Case: Two single photons injected in BS

$$|1\rangle_1 |1\rangle_0 = \hat{a}_0^\dagger \hat{a}_1^\dagger |0\rangle_1 |0\rangle_0$$

$$|1\rangle_1 |1\rangle_0 \xrightarrow{BS} \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} (\hat{a}_2^\dagger + i\hat{a}_3^\dagger)(i\,\hat{a}_2^\dagger + \hat{a}_3^\dagger)\, |0\rangle_2 |0\rangle_3$$

$$= \frac{i}{2}\left(\hat{a}_2^\dagger a_2^\dagger + \hat{a}_3^\dagger \hat{a}_3^\dagger\right) |0\rangle_2 |0\rangle_3$$

$$= \frac{i}{\sqrt{2}}\left(|2\rangle_2 |0\rangle_3 + |0\rangle_2 |2\rangle_3\right)$$

Interference Entanglement

The Plagiarism Resource Site

Charlottesville, Virginia

"The Importance of Writing"

by Louis Bloomfield, Professor of Physics, University of Virginia, Charlottesville, VA 22904

Originally published on the Commentary Page of the ***Philadelphia Inquirer*** on Sunday, April 4, 2004, edited by John Timpane.

Writing is hard work and all the marvels of modern technology haven't made it any easier. Vast resources now lie just keystrokes away, but the basic art of assembling one's thoughts into engaging prose is little changed since the days of paper and pencil. While mindless information doubles every three years, thoughtful writing still proceeds at an old fashioned pace.

Unfortunately, the timeless nature of writing isn't shared by its fraudulent imitation: plagiarism. Though nearly as ancient as writing itself, plagiarism adapts quickly to new technology. With a web full of seemingly ownerless prose, plagiarism is as easy as cut-and-paste. And if you don't see exactly what you want for free, you can buy it online at any number of "paper mills."

But a more insidious way in which technology has fostered plagiarism is by shifting our attention from content to appearance. A well-written student paper is no longer "A" work unless it's printed in color on glossy paper, with fonts and images and an accompanying multimedia presentation. Students feel expected to turn in the best papers ever written, not the best papers they can write themselves. So they assemble those papers. With hours invested in the decorations, students feel justified in stealing some or all of the text. After all, they "couldn't have said it any better" themselves.

In addition to its easy rationalization by people seeking the rewards of writing without the associated effort, plagiarism is also widely misunderstood. It isn't limited to the theft of another person's words; it also includes the theft of their ideas. More generally, plagiarism is any form of dishonesty about authorship. A reader or listener should always know whose thoughts they're hearing.

Plagiarism isn't a victimless crime. It deprives its readers of their time and trust, and its true authors of their good names. In academia, plagiarism inflates grades relative to education and devalues honest scholarship. Among authors and journalists, plagiarism cheapens the very art of writing, much as performance enhancing drugs cheapen so many sports. Plagiarism is as much a problem of morale as it is of ethics.

Prosecuting plagiarists is a miserable undertaking. It brings joy to no one, as I know from sad experience at the University of Virginia. After uncovered extensive plagiarism in my large introductory physics class in 2001, I spent two years dealing with endless honor cases. But I view that episode as an anti-scandal—as an enlightened community taking action against a misbehaving few in order to maintain its own intellectual integrity. Eliminating plagiarism isn't about the plagiarists; it's about supporting the honest people by giving them a fair environment.

Plagiarism isn't an obscure tweed-collar crime. It's a sorry fact of life everywhere and any school or organization that feels untainted is probably in denial. With plagiarism so commonplace, an organization that deals openly with it deserves our support, not our condemnation. There is no scandal in cleaning house. The scandal is in tolerating or covering up plagiarism.

Unfortunately, plagiarism is openly tolerated in the most public sectors of modern life. It wasn't always that way. Lincoln didn't just perform his Gettysburg address; he actually wrote it. What happened to that tradition of intellectual honesty in public speech? With ghostwriting so ubiquitous among the rich and powerful, it's no wonder that young people see little value in learning to write well. They view writing the way they view cleaning their rooms—an unpleasant chore they'll do only until they can afford to hire someone else.

When students believe that writing assignments are merely hazing rituals, hurdles on the path to success in life, some will inevitably plagiarize. And when instructors assign writing that has no clear educational goals, how can the students value it? Having explicitly stated goals is both good discipline and a way to avoid misunderstandings. If students believe an assignment is "busy work," some will be busy cheating.

Finally, students need to be taught that the act of writing is intrinsically valuable *to them*. It crystallizes one's thoughts in a way that nothing else can. As a physicist, I find that I often learn more from writing papers and proposals than I do from working in the laboratory. I rarely find writing easy, but I always find it rewarding.

Optics Letters vol 30, 8 p 813-813 (2005)

From the Board of Editors: on Plagiarism

Dear Colleagues:

There has been a significant increase in the number of duplicate submissions and plagiarism cases reported in all major journals, including the journals of the Optical Society of America. Duplicate submissions and plagiarism can take many forms, and all of them are violations of professional ethics, the copyright agreement that an author signs along with the submission of a paper, and OSA's published Author Guidelines. There must be a significant component of new science for a paper to be publishable. The copying of large segments of text from previously published or in-press papers with only minor cosmetic changes is not acceptable and can lead to the rejection of papers.

Duplicate submission: Duplicate submission is the most common ethics violation encountered. Duplicate submission is the submission of substantially similar papers to more than one journal. There is a misperception in a small fraction of the scientific community that duplicate submission is acceptable because it sometimes takes a long time to get a paper reviewed and because one of the papers can be withdrawn at any time. This is a clear violation of professional ethics and of the copyright agreement that is signed on submission. Duplicate submission harms the whole community because editors and reviewers waste their time and in the process compound the time it takes to get a paper reviewed for all authors. In cases of duplicate submission, the Editor of the affected OSA journal will consult with the Editor of the other journal involved to determine the proper course of action. Often that action will be the rejection of both papers.

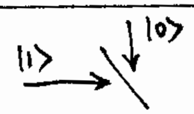
Plagiarism: Plagiarism is a serious breach of ethics and is defined as the substantial replication, without attribution, of significant elements of another document already published by the same or other authors. Two types of plagiarism can occur – self-plagiarism and plagiarism from others' works:

> **Self-plagiarism** is the publication of substantially similar scientific content of one's own in the same or different journals. Self-plagiarism causes duplicate papers in the scientific literature, violates copyright agreements, and unduly burdens reviewers, editors, and the scientific publishing enterprise.
>
> **Plagiarism from others' works** constitutes the most offensive form of plagiarism. Effectively, it is using someone else's work as if it is your own. Any *text, equations, ideas,* or *figures* taken from another paper or work must be specifically acknowledged as they occur in that paper or work. Figures, tables, or other images reproduced from another source normally require permission from the publisher. Text or concepts can, for example, be quoted as follows: "As stated by xxx (name of lead author), "text" [reference]."

Action on Notification of Allegations of Plagiarism:

OSA identifies an act of plagiarism in a published document to be the substantial replication, without appropriate attribution, of significant elements of another document already published by the same or other authors. OSA has implemented a process for dealing with cases of plagiarism. When the Editor-in-Chief of a journal is notified of an instance of either of the two possible forms of plagiarism discussed above, he or she will make a preliminary investigation of the allegations, including a request for the accused authors to explain the situation. If further action is justified, then the Editor-in-Chief will convene a panel consisting of the Editor-in-Chief of the OSA journal involved, the Chair of the Board of Editors, and the Senior Director of Publications. Their unanimous decision confirming that an act of plagiarism has occurred requires the insertion of the following statement in the official OSA electronic record of the plagiarizing article:

Lecture 39 More on single pulse interferometry

– Number state on beam splitter 

$$\boxed{\begin{aligned} |1\rangle_1 |0\rangle_0 &\xrightarrow{BS} \frac{1}{\sqrt{2}}\left(i\hat{a}_2^{\dagger} + \hat{a}_3\right)|0\rangle_2 |0\rangle_3 \\ &= \frac{1}{\sqrt{2}}\left(i|1\rangle_2 |0\rangle_3 + |0\rangle_2 |1\rangle_3\right) \end{aligned}}$$

Write density matrix ~~probability to find number state at port 3~~

$$\hat{\rho}_2 = Tr_3(\hat{\rho}_{23}) = \sum_{n=0}^{\infty} \langle n | \hat{\rho}_{23} | n \rangle$$

$$= \boxed{\frac{1}{2}\left(|0\rangle_2\, {}_2\langle 0| + |1\rangle_2\, {}_2\langle 1|\right)}$$

$\cancel{\text{Then } \left|{}_2\langle 1|\hat{\rho}_2|1\rangle\right| = \frac{1}{2}}$

$\boxed{\text{No coherence terms so no interference}}$

So: Terms of matrix $\hat{\rho}_2$

${}_2\langle 1|\hat{\rho}_2|1\rangle_2 = 1/2$ ${}_2\langle 0|\hat{\rho}_2|0\rangle_2 = 1/2$

All other terms are zero

In matrix form

$$\hat{\rho}_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

No off diagonal terms

Probability to find photon at port 2

$$\left| {}_2\langle 1|\,{}_3\langle 0| \left[\frac{1}{\sqrt{2}} \left(i\,|1\rangle_2 |0\rangle_3 + |0\rangle_2 |1\rangle_3 \right) \right] \right|^2$$

$$= \frac{1}{2}$$

Probability to find photon at port 2 <u>and</u> 3.

$$\left| {}_2\langle 1|\,{}_3\langle 1| \left[\frac{1}{2} \left(i\,|1\rangle_2 |0\rangle_3 + |0\rangle_2 |1\rangle_3 \right) \right] \right|^2$$

$$= 0$$

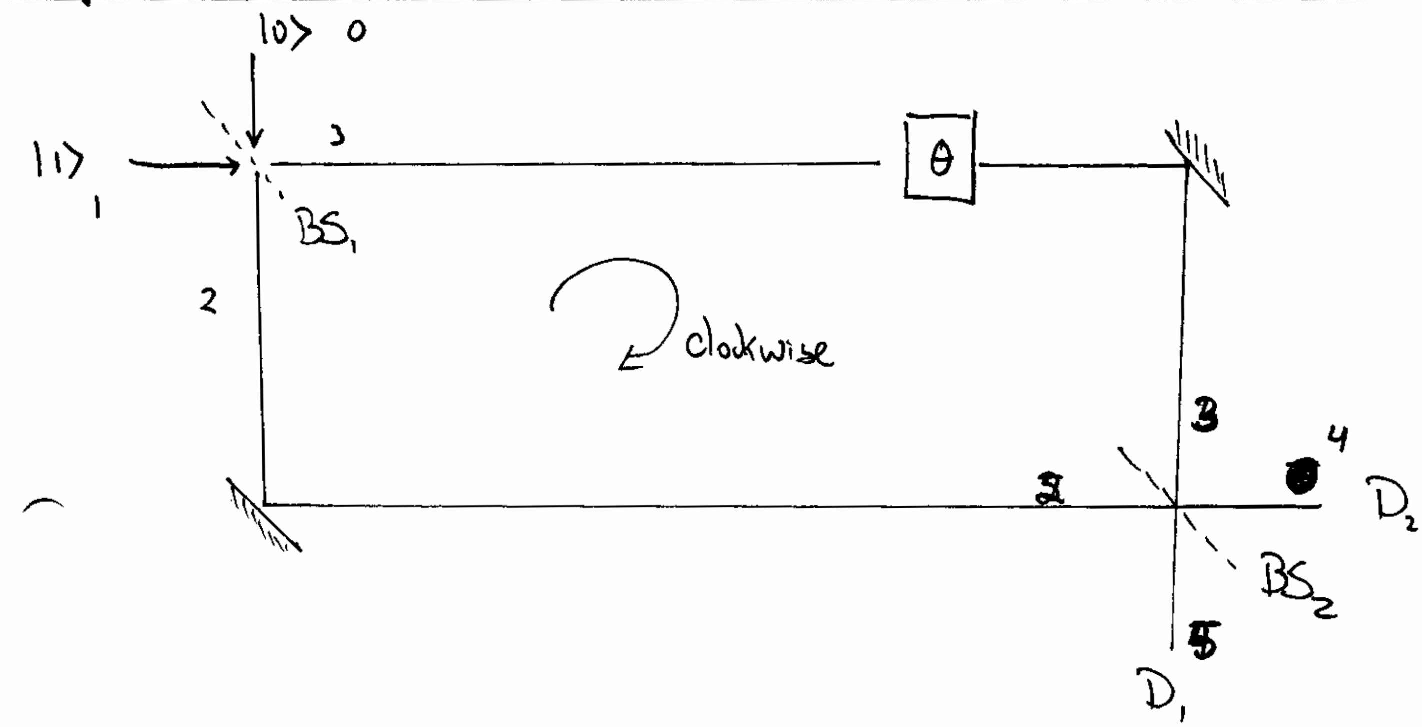
Probability to find photon at port 3

$$\left| {}_2\langle 0|\,{}_3\langle 1| \left[\frac{1}{\sqrt{2}} \left(i\,|1\rangle_2 |0\rangle_3 + |0\rangle_2 |1\rangle_3 \right) \right] \right|^2 = \frac{1}{2}$$

The density matrix represents the statistical nature of the system.

- The detector at port 2 has a 50% chance of detecting a "click".

Aspect Experiment #2 : Mach-Zehnder Interferometer.



BS_1 gives

$$|0\rangle_0 |0\rangle_1 \xrightarrow{BS_1} \frac{1}{\sqrt{2}}\left(i|1\rangle_2|0\rangle_3 + |0\rangle_2|1\rangle_3 \right)$$

Clockwise path causes phase change on port 3 component only

$$\frac{1}{\sqrt{2}}\left(|0\rangle_2|1\rangle_3 + i|1\rangle_2|0\rangle_3\right) \xrightarrow{\theta} \frac{1}{\sqrt{2}}\left(\underbrace{e^{i\theta}|0\rangle_2|1\rangle_3}_{\text{only on this term}} + i|1\rangle_2|0\rangle_3 \right)$$

phase shift on clockwise arm only

BS_2 gives individually

$$|0\rangle_2 |1\rangle_3 \xrightarrow{BS_2} \frac{1}{\sqrt{2}}\left(|0\rangle_4 |1\rangle_5 + i\,|1\rangle_4 |0\rangle_5\right)$$

$$|1\rangle_2 |0\rangle_3 \xrightarrow{BS_2} \frac{1}{\sqrt{2}}\left(|1\rangle_4 |0\rangle_5 + i\,|0\rangle_4 |1\rangle_5\right)$$

$5 \to D1$

$4 \to D2$

So put it all together

$$\frac{1}{\sqrt{2}}\left(e^{i\theta}|0\rangle_2 |1\rangle_3 + i\,|1\rangle_2 |0\rangle_3\right) \xrightarrow{BS_2}$$

$$\frac{1}{2}\left[\left(e^{i\theta}-1\right)|0\rangle_4 |1\rangle_5 + i\left(e^{i\theta}+1\right)|1\rangle_4 |0\rangle_5\right]$$

~~Probability that only D_1 "clicks"~~

~~So the probability that D_1 clicks and not D_2~~

$$\cancel{{}_5\langle 1|\,{}_4\langle 0|\left[\;"\quad"\;\right]}$$

Set $|\psi\rangle = \frac{1}{2}\left[\left(e^{i\theta}-1\right)|0\rangle_4 |1\rangle_5 + i\left(e^{i\theta}+1\right)|1\rangle_4 |0\rangle_5\right]$

Probability to see click at D_1

$$\left| {}_5\langle 1| {}_4\langle 0|\psi\rangle \right|^2 = \left|\tfrac{1}{2}(e^{i\theta}-1)\right|^2$$

$$= \frac{1}{4}\left(e^{i\theta}-1\right)\left(e^{-i\theta}-1\right)$$

$$= \frac{1}{4}\left(1-e^{-i\theta}-e^{+i\theta}+1\right) \quad \Big\} \quad e^{i\theta}+e^{i\theta} = 2\cos\theta$$

$$= \frac{1}{4}\left(2-(e^{i\theta}+e^{-i\theta})\right)$$

$$= \frac{1}{4}\left(2-2\cos\theta\right)$$

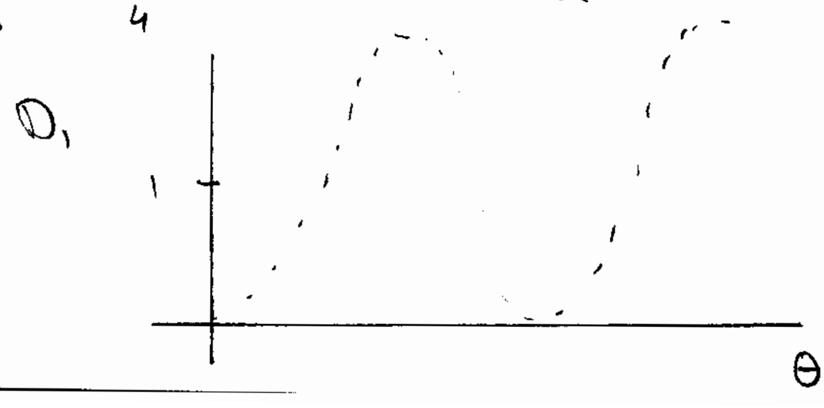
$$= \frac{1}{2}\left(1-\cos\theta\right)$$

Probability to see click at D_1 <u>and</u> D_2

$$\left| {}_5\langle 1| {}_4\langle 1|\psi\rangle \right|^2 = 0$$

Probability to see a click at D_2

$$\left| {}_5\langle 0| {}_4\langle 1|\psi\rangle \right|^2 = \frac{1}{2}\left(1+\cos\theta\right)$$



} Result from Aspect experiment

Interaction-free measurement

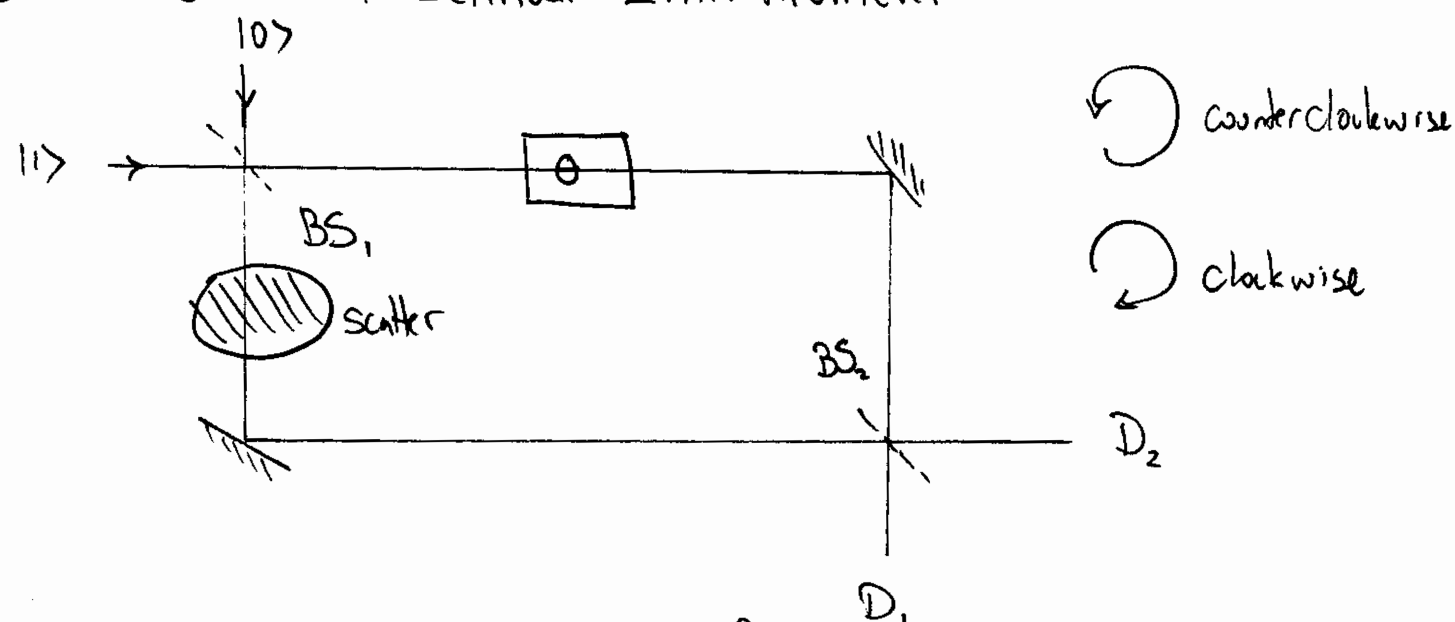
- The capability of detecting the presence of a object without scattering any quanta off it.

Exposes feature of quantum mechanics

$\Rightarrow$ Nonlocality

- the apparent instanteous effects of certain kinds of influences.

Consider the Mach Zehnder Interferometer



1) Set $\theta = 0$ Expect probability for D_1 to fire $= 1$, prob for D_2 to fire $= 0$

2) Put in the scattering object

3) If we put a detector around the object we can detect "which-path" the photo took thus destroying the interference.

4) However, we do not need a detector. The experiment is set up to determine which path irregardless whether we measure the scattered photons or not.

5) After beamsplitter 1, a photo may be

50% in clockwise arm

50% in counterclockwise arm

After beamsplitter 2 the same

But if the clockwise path is "choosen", there is a 50% chance it will go to either D_1 or D_2

6) In the end

Probability 50% that neither detector fires

Probability 25% that D_1 or D_2 fires

7) Initially the interferometer was set up so D_1 fires 100%

~~If D_1 does not fire 100%~~ then

So when D_2 fires at all then we know that something is in the arm of the interferometer.

But the photon we measure has not been scattered!! It never comes in contact with the scatter!! $\Rightarrow$ Nonlocal behavior

Lecture Schedule

(F) November 30 Entanglement, EPR, Bell's thm

(M) No Dec 3 Bells thm, Optical tests of local theories of QM

(W) Dec. 5 Catch up!

Dec 7. Final project

Dec. 10 Final projects

$12 \times 10 \Rightarrow 120$ minutes

$+ \; 2 \times 12 = 24$

144 minutes 2 hrs 24 min

Lecture 40 Entanglement

Generation of Entangled States (Polarization Entangled states)

1. Spontaneous Parametric Down-conversion in a $\chi^{(2)}$ crystal

(Degenerate form of difference frequency generation
$\omega_s = \omega_i$)

$$\hat{H}_1 \sim \chi^{(2)} \hat{a}_p \hat{a}_s^\dagger \hat{a}_i^\dagger + \chi^{(2)*} \hat{a}_p^\dagger \hat{a}_s \hat{a}_i \qquad \text{Nondegenerate case}$$

Generate signal and idler photon from pump

$$|1\rangle_p |0\rangle_s |0\rangle_i \xrightarrow{\chi^{(2)}} \hat{a}_p \hat{a}_s^\dagger \hat{a}_i^\dagger |1\rangle_p |0\rangle_s |0\rangle_i = |0\rangle_p |1\rangle_s |1\rangle_i$$

- process is spontaneous since modes are originally from vacuum.
- Signal and idler photons are generated simultaneously
- Must satisfy both energy conservation and momentum conservation (i.e. phase matching)

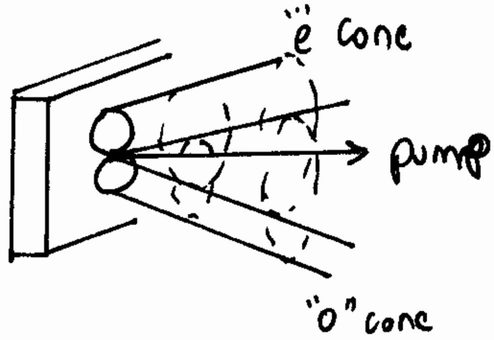
Type II Down conversion (in BBO or KDP)

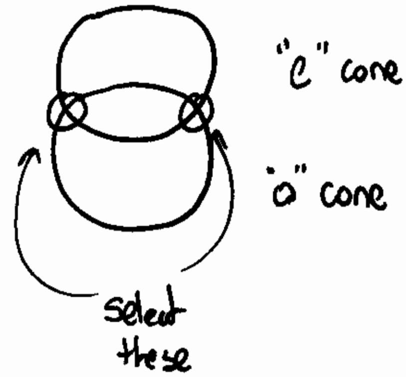
This crystal exhibits birefringence

Photons are emitted in two different cones

one for "o" axis

one for "e" axis





- Intersection of cones produce polarization entangled states

Use notation to represent polarization of single photon states

$$|V\rangle \;\; + \;\; |H\rangle$$

$$\hat{H} \cong \chi^{(2)}\left(\hat{a}^{\dagger}_{V_s} \hat{a}^{\dagger}_{H_i} + \hat{a}^{\dagger}_{H_s} \hat{a}^{\dagger}_{V_i} \right)$$

$$+ \;\; \chi^{(2)*}\left(\hat{a}_{V_s} \hat{a}_{H_i} + \hat{a}_{H_s} \hat{a}_{V_i} \right)$$

Initial state

$$|\psi_0\rangle = |0\rangle_{V_s} |0\rangle_{H_s} |0\rangle_{V_i} |0\rangle_{H_i}$$

Final states (see text for renormilization procedure)

$$|\psi(t)\rangle = \exp(-i H_I t/\hbar)\, |\psi_0\rangle = \boxed{\frac{1}{\sqrt{2}}\left(|H\rangle_1 |V\rangle_2 + e^{i\theta} |V\rangle_1 |H\rangle_2 \right)}$$

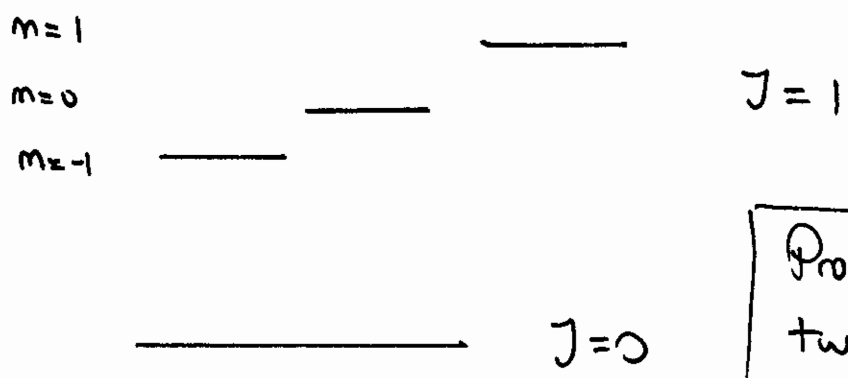
~~$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|H\rangle_s|V\rangle_i \pm |V\rangle_s|H\rangle_i)$ are 1 of 4 Bell States~~

$$\boxed{\begin{aligned} |\psi^{\pm}\rangle &= \frac{1}{\sqrt{2}}\left(|H\rangle_1 |V\rangle_2 \pm |V\rangle_1 |H\rangle_2 \right) \\ |\Phi^{\pm}\rangle &= \frac{1}{\sqrt{2}}\left(|H\rangle_1 |H\rangle_2 \pm |V\rangle_1 |V\rangle_2 \right) \end{aligned}}$$

By a choice of phase θ one ~~complete set~~ can complete the set.

2. Cascade Emission for generation of polarization entangled states

Transition $\quad J \rightarrow 0 \rightarrow 1 \rightarrow 0$



$\boxed{\text{Process produces two photons}}$

Output State

$$\boxed{|\psi\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle_1 |H\rangle_2 + |V\rangle_1 |V\rangle_2 \right)}$$

rewrite in terms of circularly polarized light $|+\rangle$ $|-\rangle$

$$\boxed{|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle_1 |-\rangle_2 + |-\rangle_1 |+\rangle_2 \right)}$$

$|\Psi^{\pm}\rangle$ + $|\Phi^{\pm}\rangle$ form a complete set (basis) in the Hilbert space

They are known as Bell states (for reasons we will come to later)

Type II down conversion is a very important process since it can produce all 4 Bell states. This process provides an experimental optical tool to test quantum mechanics.

. . . . But what shall we test?

locality + the EPR argument

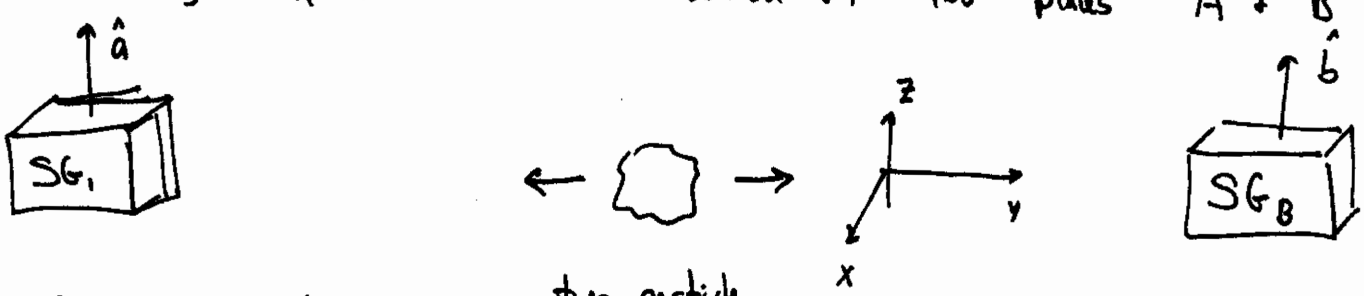
The Einstein Podolsky + Rosen Argument (EPR) (1932)

Einstein never liked quantum mechanics because he believed it was an incomplete theory. He posed a gedanken experiment to illustrate a possible fault with QM. Here, we will discuss David Bohm's version of the EPR argument. Bohm's argument is structured around entangled electrons but a similar argument can be constructed for photon states.

We will express Bohm's version of the EPR argument in terms of ~~polarization entangled states of light~~ electron spins. ~~These~~ This is isomorphic to polarization states of light (see page 228).

EPR Argument for electrons

Consider a process that produces two particles with opposite spin. A Stern-Gerlach analyzer, which measures the component of spin along a specific axis is located at two places A + B



A ("Alice")

B ("Bob")

$$\boxed{|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B\right)}$$

For spin, it is important to note that $[S_x, S_z] \neq 0$

1. Alice orients here SG along $\hat{z}$ $\qquad \hat{a} \parallel \hat{z}$

   She reads "spin up" or "spin down" say spin up

Since the two particles are pairs then Bob's particle must be spin down along $\hat{z}$.

Alice has measured the component of Bob's particle.

2. Point A can be very far from point B so what goes on at A cannot have no effect on B (this is the locality assumption). Even though Alice experiment may have had an effect on her particle it should not affect Bob's!

Thus EPR conclude that Bob's particle must had spin down before Alice made her measurement!

3. Now Alice aligns her SG along $\hat{x}$. The same argument can be made about Bob's particle.

Thus the two complementary variables S_z and S_x exist and have definite values

<u>Conclusions from EPR Argument</u>

The EPR argument is based on <u>locality</u>.

(~~justified if "something" can move faster than light~~)

The EPR ~~experiment~~ argument ~~demonstrates~~ trys to show that quantum mechanics is an incomplete theory since there are "hidden variables" to quantum mechanics that are well defined (according to the EPR Argument). Thus a better theory, a <u>local</u> <u>hidden</u> <u>variable</u> <u>theory</u> is needed.

However, the EPR argument is in error due to its locality assumption. Can we set up a condition to test the EPR argument? Thus we are testing locality.

<u>Bell's theorem (Bell's inequality) 1964</u>

John Bell devised a logical argument in the form of an inequality as a method to test the conclusions from the EPR argument.

Lecture 41 Bell's Inequality and the EPR Argument

Bell's theorem (Bell's Inequality)

John Bell devised a logical argument in the form of an inequality as a method to test the conclusions of the EPR Argument.

Note there are many versions of the inequality. We will discuss two important versions

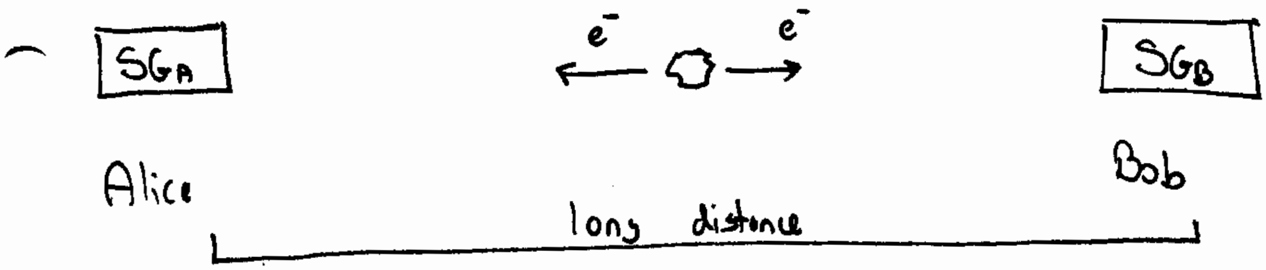
1) Bell's "original" inequality

$$\boxed{\left| C_{HV}(\hat{a}, \hat{b}) - C_{HV}(\hat{a}, \hat{c}) \right| \leq 1 + C_{HV}(\hat{b}, \hat{c})}$$

2) The Clauser Horn Shimony + Holt version (CHSH)

$$\boxed{-2 \leq C_{HV}(\theta, \phi) + C(\theta', \phi) + C(\theta, \phi') - C(\theta', \phi') \leq 2}$$

Revisit EPR



$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right)$$

1) Alice orientates her SG along $\hat{z}$

   Measures up $S_z = +1$

2) So Bob's particle must have $S_z = -1$ even if he does not measure it.

3) But Bob has a choice, so instead he aligns his SG along $\hat{x}$ and measures $+1$

4) Then Alice's measurement and Bob's we have measured both S_x and S_z precisely.

~~By Quantum mechanics S_x and S_z do not commute but we have measured both of them~~

But QM tells us that S_x and S_z do not commute so we cannot measure them precisely!

In the language of EPR, since we have determined S_x and S_z precisely, they exist <u>before</u> the measurement. But QM cannot tell us values of both S_x & S_z so QM must be an incomplete theory.

Bohr's Response to EPR

- Complementarity applied after long distances
- The context needed to think about the $\hat{z}$ component of B is not compatible with what is needed to think about $\hat{x}$ component
- Even though we can predict B without disturbing B there is no experimental situation where both S_x & S_z have meaning.

Nonlocality "hidden" in Bohr's Response

The measurement ~~of~~ at A "collapses the wavefunction" which predetermines the result a B without the experiment of A interacting with B.

Is there a way to test this nonlocality $\Rightarrow$ Bell's inequality

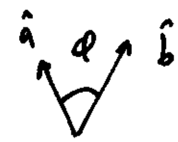
Back to Alice and Bob

Alice orients her SG along $\hat{a}$
Bob orients his SG along $\hat{b}$

Alice measures $A = +1$ "up"
$A = -1$ "down"

Bob measures $B = +1$ "up"
$B = -1$ "down"

Let's create the product AB which depend on the orientation of $\hat{a}$ to $\hat{b}$. One can show



$$\langle AB \rangle_{am} = -\hat{a} \cdot \hat{b} = -\cos\varphi$$

Write as $\boxed{C(\hat{a}, \hat{b}) = -\cos\varphi}$

Expectation value

However if we have a local hidden variable theory we can calculate $C_{HV}(\hat{a}, \hat{b})$

Bells inequality shows that <u>one</u> <u>can</u> <u>never</u> devise a theory where

$$\boxed{\begin{gathered} E_{HV}(\hat{a}, \hat{b}) = E_{am}(\hat{a}, \hat{b}) \\ \text{for all } \hat{a} \text{ \& } \hat{b} \end{gathered}}$$

"No physical theory of local hidden variables can ever produce the predictions of quantum mechanics"

Bell's Inequality

$$\boxed{\left| E_{HV}(\hat{a},\hat{b}) - E_{HV}(\hat{a},\hat{c}) \right| \leq 1 + E_{HV}(\hat{b},\hat{c})}$$

~~(Bell wrote this as $1 + C(b,c) \geq |C(a,b) - C(a,c)|$ where $C(x,y) = E(xy)$)~~

> Quantum optical measurements show a violation of the Bell's Inequality (up to 242σ!)

This demonstrates that no hidden variable theory can predict the measured results, which <u>are</u> predicted by Quantum mechanics.

~~More about the inequality~~

~~Proof of the inequality~~

~~(Consider three parameters A B C~~
~~Bells inequality states~~
~~Number (A, not B) + Number (B not C) $\geq$ Number (A not C))~~

Back to Alice + Bob : <u>Proof of the inequality</u>

A does not depend on $\hat{b}$ (locality)

B does not depend on $\hat{a}$

$A = A(\hat{a}, \lambda)$ not $A(\hat{a}, \hat{b}, \lambda)$

$B = B(\hat{b}, \lambda)$ not $B(\hat{a}, \hat{b}, \lambda)$

$\lambda \equiv$ "hidden" variable

We can compute the expectation value from the hidden variable theory

$$\boxed{E_{HV}(\hat{a}, \hat{b}) = \int AB\, d\lambda}$$

(Technically should be $C_{HV}(\hat{a}, \hat{b}) = \int d\lambda\, p_A(\hat{a}, \lambda)\, p_B(\hat{b}, \lambda)\, \rho(\lambda)$)

EPR case states $\Rightarrow A(\hat{a}, \lambda) = -B(\hat{a}, \lambda)$ {Analyzers are ∥ opposite answers}

Derivation

$$E_{HV}(\hat{a}, b) - E_{HV}(\hat{a}, \hat{c}) = \int \left(A(\hat{a}, \lambda) B(\hat{b}, \lambda) - A(\hat{a}, \lambda) B(\hat{c}, \lambda)\right) d\lambda$$

⇓ use EPR Case

$$= -\int \left(A(\hat{a}\lambda) A(\hat{b}, \lambda) - A(a\lambda) A(c, \lambda)\right) d\lambda$$

⇓ factor using $|A(b,\lambda)|^2 = 1$

$$= -\int A(\hat{a}, \lambda) A(\hat{b}, \lambda) \left(1 - A(\hat{b}, \lambda) A(\hat{c}, \lambda)\right) d\lambda$$

Since $A(\hat{a}, \lambda) A(\hat{b}, \lambda) = +1$ or -1

We can write the inequality

Case II say $\hat{b}$ makes angle $60°$ to $\hat{a}$ + $\hat{c}$

$$\hat{a}\cdot\hat{b} = \hat{b}\cdot\hat{c} = \frac{1}{2} \qquad \hat{a}\cdot\hat{c} = -1/2$$

Then

$$\left|\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\right| \overset{?}{\leq} 1 + \left(-\frac{1}{2}\right)$$

$$|1| \overset{?}{\leq} \frac{1}{2}$$

which violates the Bell inequality

So
$$\boxed{C_{QM} \neq C_{HV} \quad \text{for all } \hat{a}\ \hat{b} \text{ and } \hat{c}}$$

$$|E_{HV}(\hat{a},\hat{b}) - E_{HV}(\hat{a},\hat{c})| \le \left| \int (1 - A(\hat{b},\lambda)A(\hat{c},\lambda))\, d\lambda \right|$$

Do integration over λ

$$\boxed{|E_{HV}(\hat{a},\hat{b}) - E_{HV}(\hat{a},\hat{c})| \le 1 + E_{HV}(\hat{b},\hat{c})}$$

<u>Show that QM violate the inequality</u>

<u>Case I</u>

Choose $\hat{a} = \hat{b} = \hat{c}$

$$E_{QM}(\hat{a},\hat{a}) = -1$$

then

$$|(-1) - (-1)| \le +1 - 1$$

$$0 \le 0 \qquad \text{good!}$$

This choice <u>does</u> <u>not</u> <u>violate</u> the Bell's inequality

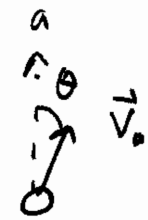
Bell's Inequality : Review

Bell's inequality states that any ~~[illegible]~~ local hidden variable theory is <u>not</u> consistant with quantum mechanics for an EPR-like experiment

$$C_{HV}(\hat{a}, \hat{b}) \neq C_{QM}(\hat{a}, \hat{b})$$

for all $\hat{a}$ + $\hat{b}$

Example of a local hidden variable theory



Define θ angle between $\hat{a}$ and $\vec{V}_a$

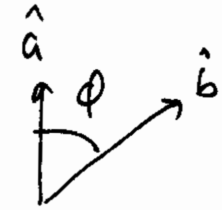
$\vec{V}_a$ spin vector of particle A

Treat the electrons as actually rotating particles with spin $\vec{V}$.

For each electron θ will change

Hidden variable $\Rightarrow$ orientation of $\vec{V}_a$ $\quad \vec{V}_b = -\vec{V}_a$

Thus $\quad A = \text{sign}(\cos\theta)$



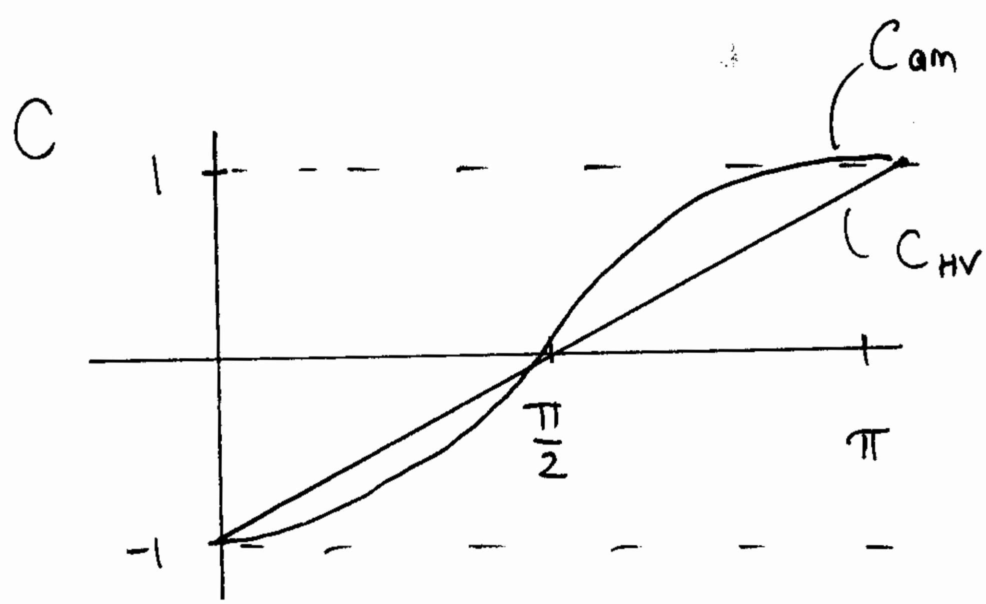
$$B = \text{sign}(-\cos(\theta - \phi))$$

$$AB = \text{sign}(-\cos\theta \cos(\theta - \phi))$$

$$C_{HV}(\hat{a}.\hat{b}) = \frac{1}{2\pi}\int_{-\pi/2}^{3/2\pi} AB\, d\theta = \boxed{\frac{2}{\pi}\phi - 1}$$

$$\boxed{C_{qm}(\hat{a},\hat{b}) = -\cos\phi}$$

Compare Two results



As Aspect put it :

" The clear violation of the Bell's Inequalities leads to the conclusive rejection of theories that are simultaneously realistic and local "

Objections on the concept of nonlocality

Newton	"philosophical absurdity"
Einstein	"Spooky" action at a distance
Bohm	" cannot see any well-founded reason for such objections ... "
Aspect	See Nature Paper

Lecture 42 — Optical Tests of EPR: Violations of the Bell's inequality

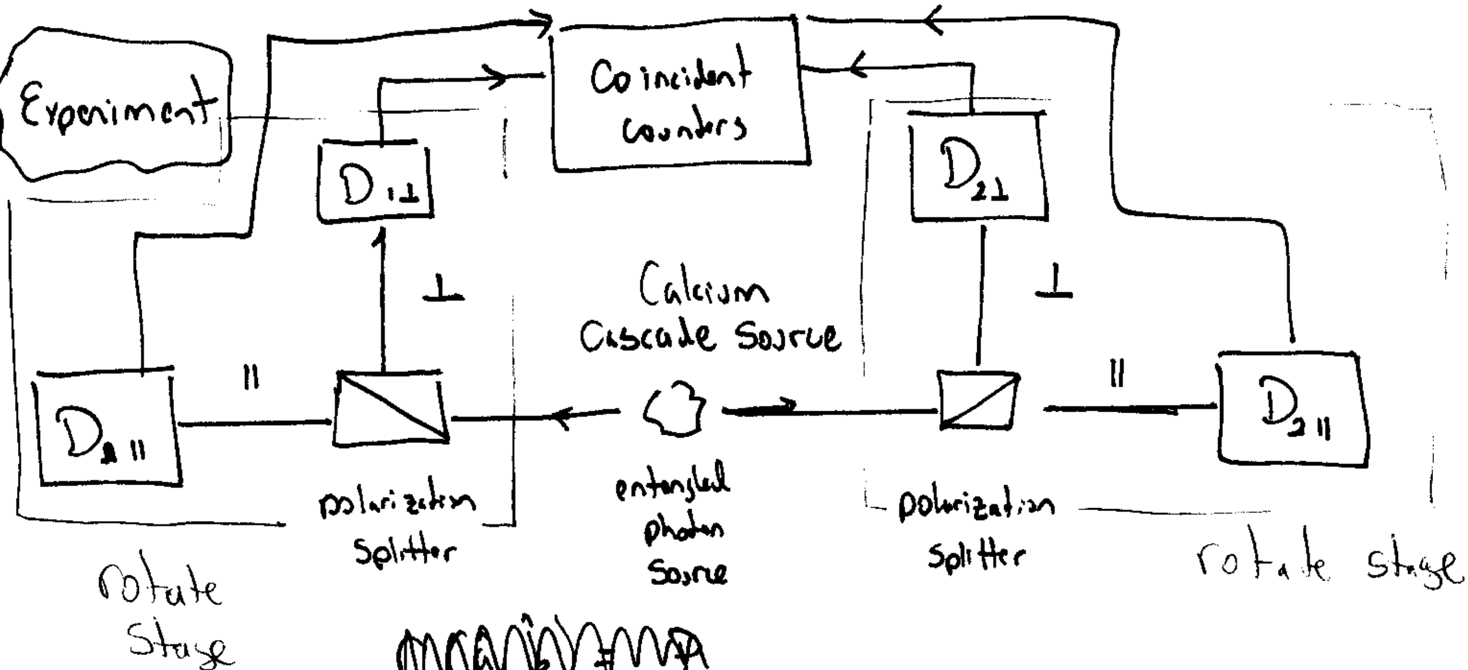
Wish to discuss two experiments:

1) Aspect et al PRL 49 (2) 1982
(Grangier, Roger)
Note there are actually three papers in 1982 by this group that show a violation of Bell's inequality
Used atomic cascade source for polarization entangled photons

2 Ou + Mandel PRL 61 (1) 1988
Use spontaneous down conversion for polarization entangled photons

Aspect et al "Experimental Realization of EPR ... "

- PRL 49 2 1982



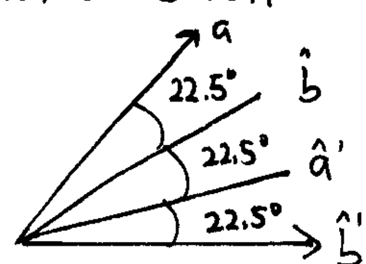
- Tested CHSH Inequality

Measured $S = C(\hat{a},\hat{b}) - C(\hat{a},\hat{b}') + C(\hat{a}'\hat{b})$
$+ C(\hat{a}',\hat{b}')$

Where $-2 \leq S_{HV} \leq 2$

Quantum mechanics predicts for 22.5° angle between all $\hat{a}$ + $\hat{b}$

$\boxed{S_{QM} = \pm 2\sqrt{2}}$ in violation to CHSH



-

$S_{QM} = 2.8284$

They ~~expected (?)~~ ~~S(0.834)~~ ≅ ~~2.70 ± 0.05~~

Note on detector efficiency: Possible ~~[illegible]~~ Black Systematic Errors,

$$C(\theta, \phi) = -\eta_{det}^2 \cos(2(\theta - \phi)) \quad \left(\text{Model and } 0.\right)$$

Why? $\Rightarrow$ Joint probability of two measurements

So

$$S_{qm} = \eta^2 \, 2\sqrt{2}$$

Bells (CHSH) inequality is violated when $S > 2$

So $\eta_{det} > 1/\sqrt[4]{2} \simeq 0.84$ 84%!!

Now Aspect et al modify the expect result from QM based on actual values of their polarizers. Given the transmission of polarizer 1 + 2, QM would predict

$$C(\hat{a}, \hat{b}) = F \frac{(T_1^{\parallel} - T_1^{\perp})(T_2^{\parallel} - T_2^{\perp})}{(T_1^{\parallel} + T_1^{\perp})(T_2^{\parallel} - T_2^{\perp})} \cos(2(\hat{a}, \hat{b}))$$

where $F \simeq 0.984$

Thus they expected $\boxed{S_{QM} = 2.70 \pm 0.05}$

Experimental Results

In the experiment, the choose $\hat{a}$ $\hat{b}$ $\hat{a}'$ + $\hat{b}$ such that

$$(\hat{a}, \hat{b}) = (\hat{b}, \hat{a}') = (\hat{a}', \hat{b}) = 22.5°$$

$$(a, \hat{b}') = 67.5°$$

They measured $\boxed{S_{exp} = 2.697 \pm 0.015}$

Using

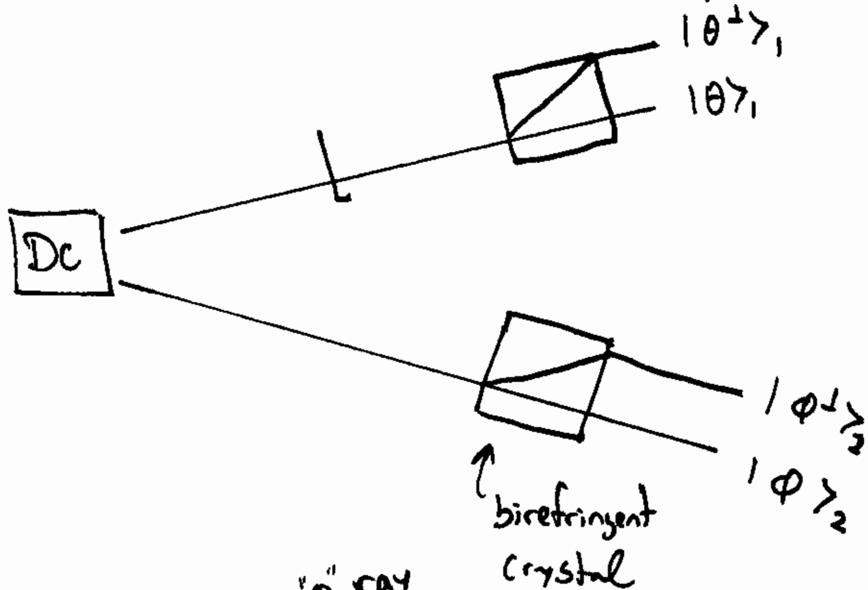
$$C(\hat{a}, \hat{b}) = \frac{R_{++}(\hat{a}, \hat{b}) + R_{--}(\hat{a}, \hat{b}) - R_{+-}(\hat{a}, \hat{b}) - R_{-+}(\hat{a}, \hat{b})}{R_{++}(\hat{a}, \hat{b}) + R_{--}(\hat{a}, \hat{b}) + R_{+-}(\hat{a}, \hat{b}) + R_{-+}(\hat{a}, \hat{b})}$$

Fourfold coincidence counting $R_{\pm\pm}(\hat{a}, \hat{b}) \equiv$ coincidence

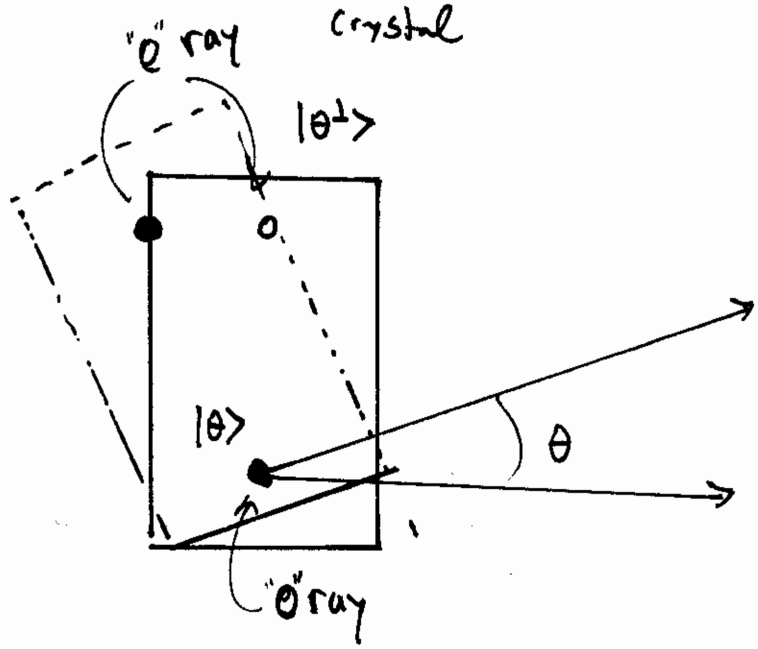
Mandel and Ou

Use spontaneous down conversion

Use similar way to measure the different polarizations



Rotate crystal



"e" ray has polarization $\perp$ to "o" ray

Initial state

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle_1 |V\rangle_2 - |V\rangle_1 |H\rangle_2 \right) \qquad \text{Bell State}$$

Write
$$|\theta\rangle_1 = \cos\theta\, |H\rangle_1 + \sin\theta\, |V\rangle_1$$
$$|\theta^\perp\rangle_1 = -\sin\theta\, |H\rangle_1 + \cos\theta\, |V\rangle_1$$

Reduce to

$$\boxed{|\psi^-\rangle = \frac{1}{\sqrt{2}}\left(|\theta\rangle_1 |\theta^\perp\rangle_2 - |\theta^\perp\rangle_1 |\theta\rangle_2 \right)}$$

Now, Back to EPR

If photon in mode 1 is $|\theta\rangle_1$ then other photon will be in $|\theta^\perp\rangle_2$

$\Rightarrow$ Strong correlation in $|\psi\rangle$

Is locality violated here

Further more $\Rightarrow$ the polarization states are isomorphic to spin ½ states

Define
$$\hat{\Sigma}_1 \equiv |\theta\rangle\langle\theta^\perp| + |\theta^\perp\rangle\langle\theta|$$
$$\hat{\Sigma}_2 \equiv -i\left(|\theta\rangle\langle\theta^\perp| - |\theta^\perp\rangle\langle\theta| \right)$$
$$\hat{\Sigma}_3 \equiv |\theta\rangle\langle\theta| - |\theta^\perp\rangle\langle\theta^\perp|$$

These satisfy

$$[\hat{\Sigma}_i , \hat{\Sigma}_j] = 2i \, \varepsilon_{ijk} \hat{\Sigma}_k \quad \Rightarrow \text{ Spin states}$$

Heisenberg uncertainty come into play with components of polarization.

Test of Bell's Inequality

Define correlation function

$$C(\theta, \phi) = \text{Average} [A(\theta) B(\phi)]$$

photon 1 $\quad A(\theta) = +1 \quad$ photon out "o" beam $\quad |\theta\rangle_1$

$\quad = -1 \quad$ photon out "e" beam $\quad |\theta^\perp\rangle$

photon 2 $\quad B(\phi) = +1 \quad$ " "

Product $A(\theta) B(\phi)$

$+1 \quad |\theta\rangle_1 |\phi\rangle_1 \quad$ or $\quad |\theta^\perp\rangle_1 |\phi^\perp\rangle_2$

$-1 \quad |\theta\rangle_1 |\phi^\perp\rangle_2 \quad$ or $\quad |\theta^\perp\rangle_1 |\phi\rangle_2$

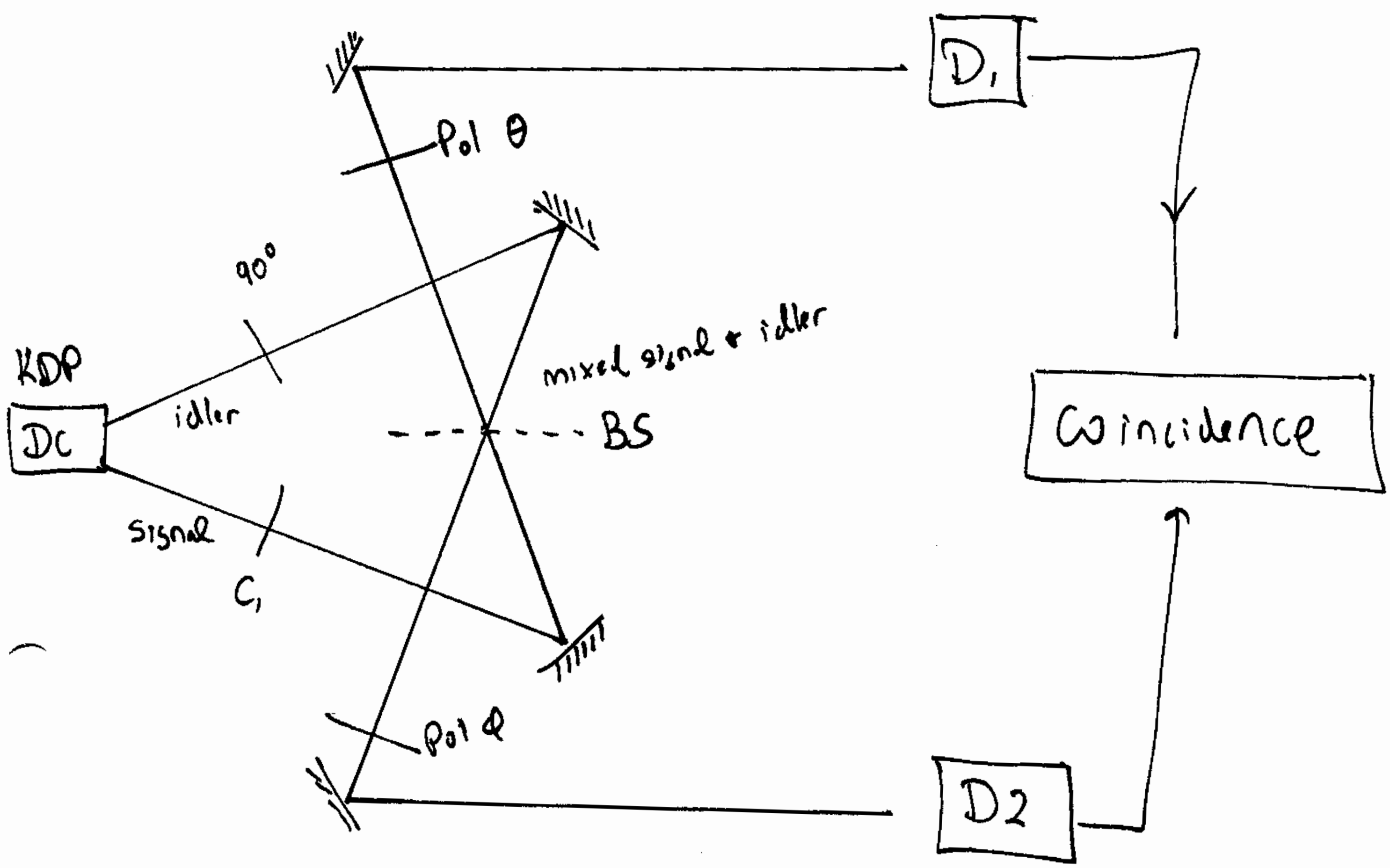
Average $\quad C(\theta, \phi) = -\cos(2(\theta - \phi))$

; same for

$$C(\theta, \phi) = \langle \psi^- | \Sigma_3^{(1)}(\theta) \hat{\Sigma}_3^{(2)}(\phi) | \psi^- \rangle = -\cos(2(\theta - \phi))$$

Experimental Setup Mandel + Ou

- A bit more complicated than described



Results

Bell's Inequality

$$S \equiv C(\theta, \phi) - C(\theta, \phi') + C(\theta', \phi')$$
$$+ C(\theta', \phi) - C(\theta', \underset{\text{no polarizer}}{-}) - C(-, \phi) \leq 0$$

~~The Aspect experiment Should have no counts without the polarizers to get R_{++} and R_{-+}~~

Set $\theta = 22.5°$ $\quad \theta' = 67.5°$

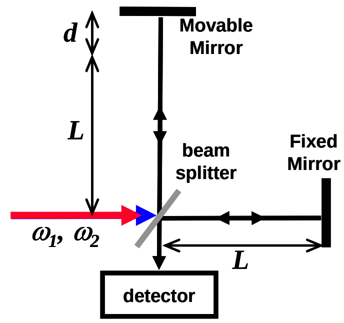
$\phi = 45°$ $\quad \phi' = 0$

Measured $\boxed{\tilde{S}_{exp} = (11.5 \pm 2.0) \quad \text{min}^{-1}}$

The purpose of the mini-projects is to offer problems in nonlinear and quantum optics in a format that mimics problem-solving scenarios found in a research environment. Buried in the mini-projects are questions that I do not expect you to know or are the solution easily found in the book. This mini-project consists of problems that should be a review of topics that will be important for our initial introduction to nonlinear optics.

1. Interference of two continuous wave lasers in a Michelson interferometer

We wish to build a Michelson interferometer to measure the frequency difference between two continuous wave lasers. The first laser has a center frequency of ω_1 (632.8 nm) and the second has a close but unknown center wavelength ($\omega_2=\omega_1+\delta\omega$). Both lasers have a power of 1 mW, and the polarizations are both vertical. The first step in building our interferometer is to choose a proper beam splitter that has 50% power reflectivity at an orientation of 45 degrees with respect to the input beam.



In the lab we have a beam splitter is made of fused silica of thickness 0.5 cm with an unknown transmission. We need to determine if this beam splitter will work for our interferometer.

1. Is the laser's polarization TE or TM?
2. Compute the power transmitted and reflected from the first surface of the beam splitter. Ignore any absorption.
3. Compute the power transmitted and reflected from the second surface of the beam splitter.
4. Give reasons why this beam splitter a bad choice for our Michelson interferometer.

To make our interferometer a second splitter is used that is very thin and has a power reflectivity of 50% at 632.8 nm for an angle of 45 degrees. If only one laser for frequency ω is used and the beam splitter is 50%, the interference is given by

$$I(\tau) = I_0\left[1+\cos(\omega\tau)\right] \tag{1}$$

where τ is the optical delay between the two arms in the interferometer, I_0 is the intensity of the input laser,.

5. Show that τ relates to d in the figure by $\tau=2d/c$, where c is the speed of light.
6. Derive the interference equation for the Michelson interferometer defining I_1 as the intensity in the fixed arm of the interferometer and I_2 as the intensity in the variable arm. Then derive Equation (1) by setting $I_1= I_2=1/2\ I_0$.

Now the two lasers of frequencies ω_2 and ω_1 are input into the Michelson interferometer. The resulting interference relation will be

$$I(d) = I_0\left[1+\cos\left(\frac{\omega_1+\omega_2}{2}\frac{2d}{c}\right)\cos\left(\frac{\omega_1-\omega_2}{2}\frac{2d}{c}\right)\right] \tag{2}$$

7. Using (2) determine a method by varying d to measure the $\delta\omega$ between the two lasers input into the interferometer.

2. Quartz as a birefringent material

Crystalline quartz is a birefringent material used in many polarization optics.

1. Describe the crystal type and its birefringent properties.
2. Plot the ordinary and extraordinary indices of refraction at from 500 nm to 1000 nm.
3. To make a quartz quarter-wave zero-order retardation plate at 800 nm, how thick does the plate need to be?

3. Ultrashort pulse dispersion in fused silica

A train of ultrashort optical pulses is produced by a mode-locked Ti:sapphire laser. Each pulse has an electric field profile of hyperbolic secant, is transform limited, and each have a duration of 10 fs full-width half maximum (FWHM). The laser's repetition rate is 100 MHz and the average power from the laser is 100 mW.

1. What is the pulse energy? The peak power?
2. Plot the temporal intensity and phase of the pulse.
3. Plot the spectral intensity and phase of the pulse.

The pulse propagates through a fused-silica window of thickness 1 cm. The dispersion of the fused-silica causes the pulse duration to increase. Consider only quadratic phase distortion (β_2) due to the fused-silica window.

4. Compute and plot final temporal intensity $I_{out}(t)$ and phase $\varphi_{out}(t)$ after propagation through the window.
5. Compute and plot final spectral intensity $I_{out}(\omega)$ and phase $\varphi_{out}(\omega)$ after propagation through the window.
6. Is the "chirp" of the pulse positive or negative?
7. Does the pulse have the same spectral bandwidth before and after the window?
8. What is the final pulse duration (FWHM) after the fused silica window?

Now, ignore the quadratic phase distortion but let the fused silica window have only cubic phase distortion β_3.

9. Compute and plot final temporal intensity $I_{out}(t)$ and phase $\varphi_{out}(t)$ after propagation through the window. Consider only quadratic phase distortion due to the fused-silica window.
10. Compute and plot final spectral intensity $I_{out}(\omega)$ and phase $\varphi_{out}(\omega)$ after propagation through the window.
11. Set β_3=-$\beta_{3,\text{fused silica}}$ and find $I_{out}(t)$. How is the temporal intensity different than in Question 9?

4. One dimensional anharmonic oscillator

The Lorentz model of the atom, which treats a solid as a collection of harmonic oscillators, is a good classical model that describes the linear optical properties of a dielectric material. This model can be extended to nonlinear optical media by adding anharmonic terms to the atomic restoring force. In the lecture we will look closely at this model but let's first solve the differential equations for a one-dimensional anharmonic oscillator.

Consider a one-dimension anharmonic oscillator of mass *m* under the influence of the nonlinear restoring force:

$$F(x) = -kx - \alpha x^2 - \beta x^3$$

where $\omega_0^2 = k/m$ is the natural frequency sans any anharmonic terms. Let $m = 1$ kg and k=0.1 N/m .

1. Plot the potential energy for the above force using $\alpha = 0.01\ \text{N/m}^2$ and $\beta = 0\ \text{N/m}^3$. Compare it to the potential energy of a simple harmonic oscillator.
2. Plot the potential energy for the above force using $\alpha = 0\ \text{N/m}^2$ and $\beta = 0.01\ \text{N/m}^3$.
3. Now, let $\alpha = 0.01\ \text{N/m}^2$ and $\beta = 0.01\ \text{N/m}^3$. Numerically solve the 2nd order differential equation of motion, solving for $x(t)$ for t=0 to 20 seconds assuming that $x_0 \equiv x(0) = 0.1$ m and $\dot{x}(0) = 0$ m/s. By plotting $x(t)$ determine the frequency of oscillation ω. How does it compare to ω_0?
4. Find $x(t)$ for $x_0 = 10$ m and $\dot{x}(0) = 0$ m/s, letting $\alpha = 0.01\ \text{N/m}^2$ and $\beta = 0.01\ \text{N/m}^3$. What is the new frequency of oscillation and how does it compare to ω_0.
5. An analytic approximation for $\omega(x_0)$, derived using the method of successive approximations (see Landau's *Mechanics*), is given by

$$\omega(x_0) = \omega_0 + \left(\frac{3\beta}{8\omega_0} - \frac{5\alpha^2}{12\omega_0^3}\right)x_0^2$$

   Compare your numerical $\omega(x_0)$ to the analytic approximation expression for $x_0 = 0.1$ m to 10 m .
6. The Fourier series for $x(t)$ is given by the expression

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi}{T}t\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi}{T}t\right)$$

   where the period $T = 2\pi/\omega$ and the Fourier series coefficients are given by

$$a_n \equiv \frac{2}{T}\int_0^T x(t)\cos\left(\frac{2n\pi}{T}t\right)dt \text{ and } b_n \equiv \frac{2}{T}\int_0^T x(t)\sin\left(\frac{2n\pi}{T}t\right)dt$$

   Numerically solve for $x(t)$ with $x_0 = 10$ m using $\alpha = 0\ \text{N/m}^2$ and $\beta = 0.01\ \text{N/m}^3$. Find the first five Fourier series coefficients a_n (where n=0, … , 4) of the solution $x(t)$. Explain why $b_n = 0$ for all n.
7. Numerically solve for $x(t)$ with $x_0 = 10$ m using $\alpha = 0.01\ \text{N/m}^2$ and $\beta = 0\ \text{N/m}^3$. Find the first five Fourier series coefficients a_n of the solution $x(t)$.
8. Compare the odd terms of a_n for the case where $\alpha = 0, \beta \neq 0$. Compare the even terms of a_n for the case where $\alpha \neq 0, \beta = 0$. How does the symmetry of the restoring force predetermine which order harmonics are produced by the nonlinear oscillator?

Informal Survey for PHYS953: Name:___________________________________

Please answer the following questions completely.
What classes in optics and quantum mechanics have you taken? Where have you taken these classes?

Briefly describe your research interests.

Why do want to take this class?

How many hours per week can you spend on homework for this class?

Which of the topics listed in the syllabus seem most interesting to you?

Are there other topics that we should cover in this class?

1. Second Harmonic Generation in Potassium Dihydrogen Phosphate (KDP)

You wish to produce second harmonic generation (SHG) of a continuous wave Nd:YAG laser centered at 1064 nm. To do this you will use a KDP crystal that is cut to produce the second harmonic using Type $I^{(-)}$ (ooe) phase matching. A single laser provides the fundamental fields for the E_1 and E_2 fields at frequency $\omega = \omega_1 = \omega_2$ (corresponding to 1064 nm), the second harmonic field will be the E_3 field at $\omega_3 = 2\omega$. Thus, $E_1 = E_2$ and half of the total power is shared among these fields. The laser power is P=0.2 W and beam diameter (assuming a "top-hat" spatial profile) of the laser in the crystal is 10 μm. The length of the crystal is L=1.0 cm

Type $I^{(-)}$ phase matching implies that the fundamental fields ($E_1 = E_2$) are both orientated along the ordinary (o) axis and the second harmonic (E_3) is orientated along the extraordinary (e) axis of the negative uniaxial KDP crystal. The ordinary and extraordinary indices of refraction as a function of wavelength for KDP are given by the following Laurent series expressions (where λ is expressed in μm):

$$n_o^2(\lambda) = 2.2576 + \frac{1.7623\lambda^2}{\lambda^2 - 57.898} + \frac{0.0101}{\lambda^2 - 0.0142} \qquad n_e^2(\lambda) = 2.1295 + \frac{0.7580\lambda^2}{\lambda^2 - 127.0535} + \frac{0.0097}{\lambda^2 - 0.0014}$$
$$n_e(\theta,\lambda) = \left[\frac{\sin^2\theta}{n_e^2(\lambda)} + \frac{\cos^2\theta}{n_o^2(\lambda)}\right]^{-1/2} \tag{1}$$

For this process d_{eff} will have the form $d_{eff} = d_{ooe} = d_{36}\sin\theta\sin 2\phi$ where d_{36} =0.39 pm/V for KDP.

1. What is the wavelength of the second harmonic generated field (E_3)?
2. Assuming the phase matching process is Type $I^{(-)}$ (ooe), find the phase matching angle θ_{pm} where $\Delta k=0$.
3. Assuming the phase matching process is Type $I^{(-)}$ (ooe), what are the values of n_1, n_2, n_3 where $n_j = n(\lambda_j)$. Make sure to use the proper index n_j (either $n_e(\theta,\lambda)$ or $n_o(\lambda)$) when computing n_1, n_2, n_3

You try to orientate the crystal for perfect phase matching, however you make an error and set the crystal at angles $\theta = 0.995\theta_{pm}$ and $\varphi = 45°$.

4. Find d_{eff} under these conditions in units of pm/V
5. Compute the phase mismatch Δk under these conditions. Use the proper n_1, n_2, n_3.
6. Determine the initial electric field amplitudes $A_1(z=0)$ and $A_2(0)$ in V/m from the given total input power of P=0.2 W. Remember that irradiance (intensity) has units of W/m^2 and is given by

$$I_j = 2\varepsilon_0 n_j c A_j A_j^* \text{ in units of W/m}^2 \tag{2}$$

7. What is the initial amplitude of $A_3(0)$?
8. Numerically solve the three coupled differential equations derived in class for the amplitudes $A_1(z)$, $A_2(z)$ and $A_3(z)$. Assume the possibility of pump depletion, $\theta = 0.995\theta_{pm}$ and $\varphi = 45°$. Plot $I_3(z)$ and $I_1(z)$ for z=0 to L.
9. Is the fundamental power depleted at $z=L$?
10. Using your numerical solution, determine the output SHG power in Watts at $z=L$=1.0 cm. Is the power at $z=L$ the maximum SHG power produced at any position z in the crystal?
11. Determine the SHG conversion efficiency $\eta_{SHG}(z) \equiv I_3(z)/[I_1(0)+I_2(0)]$ at $z=L$.

Now you set the angle θ for perfect phasematching $\theta = \theta_{pm}$ thus setting the phase mismatch Δk to zero.

12. Solve the coupled differential equations again with $\theta = \theta_{pm}$ and $\Delta k=0$, using the correct values of n_1, n_2, n_3.
13. What SHG power and SHG conversion efficient at $z=L$? Is it larger than before?

We can define a nonlinear length L_{NL} which is a length scale that determines the strength of the nonlinearity. Note that $\eta_{SHG}(z = L_{NL}) \simeq 0.58$ for perfect phase matching. A form for the nonlinear length is given by

$$L_{NL} = \frac{1}{4\pi d_{eff}}\sqrt{\frac{2\varepsilon_0 n_1 n_2 n_3 c\lambda_1^2}{I_1(0)}} \tag{3}$$

14. Compute L_{NL} using Eq. 3. How does it compare to L=1 cm?
15. Solve the coupled differential equations setting $L=4L_{NL}$ for ΔkL=10, ΔkL=1 and ΔkL=0. Plot the conversion efficiencies $\eta_{SHG}(z)$ and $\eta(z) \equiv [I_1(z)+I_2(z)]/[I_1(0)+I_2(0)]$ as a function of z for the three cases. Which case produced the most SHG power and the largest $\eta_{SHG}(L)$?

2. Second Harmonic Generation (SHG) of an Ultrashort Pulse

You wish to build a experiment to accurately measure the pulse duration of ultrashort pulses produced by a Chromium: Forsterite (Cr:F) laser. You do not need to know the details of the experiment, only that it needs second harmonic generated light to work. Thus a nonlinear crystal is needed to produce this SHG: the fundamental pulse (the pulse from the Cr:F laser) will be used to produce a SHG pulse using a nonlinear crystal. Phasematching in this nonlinear crystal will be obtained using angle tuning.

The Cr:F laser center wavelength is at 1275 nm, and it produces an average power of 0.5 W. The second harmonic light will be at 637.5 nm. The beam diameter is 50 μm in the crystal (assuming a "top-hat" spatial profile). A single pulse exits the laser every 10 ns thus the laser has a repetition rate of 100 MHz. An estimate of the pulse duration is roughly 20 fs full width at half maximum (FWHM).

Your job is choose a nonlinear crystal to generate second harmonic light at 637.5 nm from fundamental Cr:F laser pulses at 1275 nm.

1. What is the name of the crystal you would use? Find a common and easily purchased crystal that has the smallest absorption α (in units of 1/m) at the fundamental wavelength of 1275 nm.
2. Where could you buy this crystal? If you cannot find a vendor choose a different crystal. Use the internet.
3. Is the crystal uniaxial or biaxial? If your answer is biaxial, choose a different crystal.
4. Is the crystal negative or positive uniaxial?
5. What type of phase matching would you use? Type I or Type II? ooe or oeo or something else?
6. Given your choice of crystal and phase matching type, what would be the phase matching angle θ_{PM}?
7. What would be d_{eff} for your crystal in pm/V?

As discussed in class, each crystal has a finite phase matching bandwidth for pulsed SHG depending on the thickness of the crystal. This means that a given crystal cannot simultaneously phase match all spectral components of the pulse. For pulsed SHG you wish to have the longest crystal possible in order to get the most SHG power *but not at the cost of severely filtering the SHG spectrum*!

8. Given that the pulse duration approximately 20 fs FWHM, estimate the transform-limited spectral FWHM bandwidth of the fundamental pulse spectrum $I(\lambda)$ in nanometers?
9. Using the above pulse as the fundamental, what is the SHG spectral bandwidth (FWHM) in nanometers. The SHG spectrum $I_{SHG}(\omega)$ is proportional to the autoconvolution of the fundamental spectrum:

$$I_{SHG}(\omega) \propto \int I(\eta - \omega) I(\eta) d\eta \tag{4}$$

10. Make an educated guess for the optimal crystal thickness L needed for proper phase matching. Make your choice based on the longest crystal that does not severely filter the SHG spectrum. (Hint: the thickness should be between 0.001 and 1 mm). Remember, the spectral filter function $H(\omega)$ due to the phase mismatch is given by

$$H(\lambda) = \left(\frac{\sin\left(\Delta k(\theta,\lambda) L\right)}{\Delta k(\theta,\lambda) L} \right)^2 \text{ where } L \text{ is the crystal thickness.} \tag{5}$$

11. Determine the spectral width of the filtered SHG spectrum $H(\lambda) I_{SHG}(\lambda)$ in nanometers.

1. Soliton Propagation in a Single-Mode Optical Fiber

An optical soliton forms due to the interplay of anomalous group velocity dispersion (GVD) and self-phase modulation (SPM) in an optical fiber. For an ultrashort pulse injected into the fiber, GVD causes the pulse temporal envelope to broaden while SPM causes the spectral width to increase. A soliton forms when the two effects are balanced, which happens when the total amount of dispersion and nonlinearity is just right. We can define the nonlinear length (L_{NL}) and dispersion length (L_D) in the fiber in terms of the peak power P_0 , the pulse duration FWHM Δt , group velocity dispersion β_2 , and the effective nonlinearity γ by

$$L_{NL} = \frac{1}{\gamma P_0} \text{ and } L_D = \frac{T_0^2}{|\beta_2|} \quad \text{where } \mathrm{T}_0 = \frac{\Delta t}{2\ln(1+\sqrt{2})} \text{ and } \gamma = \frac{n_2 \omega}{c\pi r^2}.$$

A first order soliton occurs when $L_{NL}/L_D = 1$.

A hyperbolic secant pulse with center wavelength λ_0=1550 nm and pulse duration Δt=100 fs FWHM propagates through a length L_D of a single-mode optical fiber. The optical fiber has a core radius of r=4.1 μm and an index difference Δn=0.008 between the core and cladding index or refraction. The value for the nonlinear index of refraction is n_2=3 10^{-20} $\mathrm{m^2/W}$. The fiber core consists of germanium-doped fused silica whose index of refraction is given by the three term Sellmeier equation (valid for wavelength in μm):

$$n^2(\lambda) = 1 + \sum_{i=1}^{3} \frac{B_i \lambda^2}{\lambda^2 - C_i^2} \text{ where } \begin{array}{l} \mathrm{B}_1 = 0.711040,\ \mathrm{B}_2 = 0.451885,\ \mathrm{B}_3 = 0.704048 \\ \mathrm{C}_1 = 0.064270,\ \mathrm{C}_2 = 0.129408,\ \mathrm{C}_3 = 9.45478 \end{array} \tag{1}$$

(The fiber cladding consists of fused silica, which has a smaller index of refraction than germanium-doped fused silica. We will not need to use its Sellmeier equation for the problem.) The wave guiding due to the fiber geometry changes the total dispersion that the pulse experiences. The propagation constant $\beta(\omega)$ for the fiber, which represents the z component of the wavevector $\mathbf{k}(\omega)$, is given by

$$\beta(\omega) = n(\omega)\sqrt{1 + 2\Delta n b(\omega)} \ .$$

The propagation constant is expressed where Δn is the index difference between core and cladding, r is the core radius, and $b(\omega)$ is the normalized mode propagation constant due to the fiber geometry given in terms of the normalized frequency $V(\omega)$. An approximate form for $b(\omega)$ is given by

$$b(\omega) = 1 - \left(\frac{1+\sqrt{2}}{1+\sqrt[4]{4+V(\omega)}} \right)^2 \text{ where } V(\omega) \equiv \frac{r\omega}{c} n(\omega) \sqrt{2\Delta n}.$$

1. Show that the value of the second order propagation constant β_2 (i.e. group velocity dispersion) at λ_0=1550 nm is -0.0000180 $\mathrm{fs^2/nm}$. β_2 can be determined from

$$\beta_2(\omega_0) = \left. \frac{d^2\beta(\omega)}{d\omega^2} \right|_{\omega=\omega_0}$$

2. What is L_D ? Determine the peak power P_0 for where $L_{NL}/L_D = 1$.
3. Consider the pulse propagating through L_D of fiber experiencing only group velocity dispersion (no nonlinear effects). Plot the temporal chirp $\omega_{GVD}(t) = \omega_0 - \partial_t \varphi_{GVD}(t)$ of the pulse due only to GVD after L_D .
4. Consider the pulse propagating through L_D of fiber experiencing only self-phase modulation (no dispersion). Plot the temporal chirp $\omega_{SPM}(t) = \omega_0 - \partial_t \varphi_{SPM}(t)$ of the pulse due only SPM after L_D .
5. By comparing $\omega_{GVD}(t)$ and $\omega_{SPM}(t)$, explain how the interaction of SPM and GVD leads to soliton formation.

2. Partially Degenerate Four Wave Mixing in a Single-Mode Optical Fiber

We wish to determine the pump, signal, and idler frequencies for partially degenerate four-wave mixing (FWM) in an optical fiber. Partially degenerate FWM is described by

$$2\omega_p - \omega_i - \omega_s = 0$$

where we use the terms pump (p), signal (s), and idler (i) as for difference frequency generation. Here we define $\omega_i > \omega_s$.

A strong continuous wave laser serves as the pump at ω_p of power P_0=0.5 MW. The pump is injected into an optical fiber with a germanium-doped fused silica core. The fiber has a core radius 4.1 μm and index difference Δn=0.008 between the core and cladding indices (as in Problem 1).

1. Determine the signal and idler wavelengths produced through partial degenerate four wave mixing for pump wavelengths from λ_p=900 to 2000 nm. To determine this for a given pump frequency ω_p you will need to find the signal ω_s and idler ω_i frequencies that satisfies both energy conservation and phase matching:

$$2\omega_p - \omega_i - \omega_s = 0$$
$$\Delta k = \Delta k_m + \Delta k_w + \Delta k_{NL} = 0$$

where

$$\Delta k_m = c^{-1}\left(n(\omega_s)\omega_s + n(\omega_i)\omega_i - 2n(\omega_p)\omega_p\right)$$
$$\Delta k_w = \Delta n c^{-1}\left(b(\omega_s)\omega_s + b(\omega_i)\omega_i - 2b(\omega_p)\omega_p\right)$$
$$\Delta k_{NL} = 2\gamma P_0$$

   The phase mismatch Δk has contributions due to material dispersion (Δk_m), waveguide dispersion (Δk_w), and the fiber nonlinearity (Δk_{NL}). To determine the phase mismatch, you will need to use the Sellmeier equation and $b(\omega)$ from the previous problem.
2. Plot λ_s and λ_i versus λ_p.

The zero group velocity dispersion wavelength λ_{zGVD} is ~ 1345 nm for this fiber, which is determined using $\beta(\omega)$. Notice that the behavior of λ_s versus λ_p and λ_i versus λ_p is different on the long and short wavelength sides of λ_{zGVD}

The purpose of this Mini-project is to expose you to a seminal or groundbreaking paper in nonlinear optics, and to see how this significant paper lead to new research and discoveries. There will be two parts to this Mini-project: Writing the Summary and Reviewing the Summary

1. Writing the Summary

You will need to write a short summary of two journal papers. This first paper you will have chosen (by random ballot) from the list below. You will need to pick the second paper, however the second paper must be a relatively recent paper that cites the first paper in its reference section. Example:

Paper 1: Franken, P.A. *et al*, "Generation of Optical Harmonics", Phys Rev Lett, Vol. 7, 4, 1961
Time Cited: 564

Paper 2 which references Paper 1: Deng L, Hagley EW, Wen J, *et al*., "Four-wave mixing with matter waves", Nature, Vol. 398, 6724 Pages: 218-220 Published: MAR 18 1999
Times Cited: 260

When writing this summary, your target audience will be your fellow classmates and not your instructor. The Summary will consist of a one or two page summary of Paper 1 and a one or two page summary of Paper 2. In the first summary, you must discuss the major results of Paper 1 and the importance of the paper. In the second summary you must discuss the major results Paper 2 and how the results of Paper 1 contributed to these results. The format of the paper should be as follows:

<u>Summary Format</u>
Page 1: Title page with your name
Pages 2-3: Summary of Paper 1 (summary may be one page only)
Pages 4-5: Summary of Paper 2 (summary may be one page only)
The Summary needs to be typed and turned in electronically as a PDF file to me at washburn@phys.ksu.edu. Use 10 or 12 pt font, Times New Roman Font, 1 inch margins. Only put your name on page 1.

Please pay attention to the Review Criteria before writing your Summary. See below.

2. Reviewing the Summary

For Part Two you will evaluate your classmate's summary in a similar fashion as for the review of a journal. The manuscript will be given to you in an anonymous fashion and you must complete your review in an anonymous fashion. You will judge the Summary using the criteria below.

<u>Review Criteria</u>
How well does the Summary cover the important results of Paper 1?
How well does the Summary cover the important results of Paper 2?
How well does the Summary show a connection (or show a lack of a connection) between the results of Paper 1 to the result of Paper 2?
Are there any significant formatting, spelling or grammatical errors?

Then make a final decision on the Summary:
____ Summary is excellent, accept as is with no revisions
____ Summary needs minor revision
____ Summary needs major revision
____ Summary is poor, reject

Complete your review by writing a brief statement answering the following questions and then make a final decision on the Summary. Email the review to me. To be a responsible referee, you will need to read (or at least skim) the papers that the Summary is reviewing. Do not put your name on the review since it will go back to the author. Grades will be given based on the result of the Summary Review and on the quality of your review.

3. Due dates

Summary Due: 11/09/07
Review Due: 11/16/07

Paper List

1. THEORY OF STIMULATED BRILLOUIN AND RAMAN SCATTERING
Author(s): SHEN YR, BLOEMBER.N
Source: PHYSICAL REVIEW Volume: 137 Issue: 6A Pages: 1787-& Published: 1965

2. Experimental evidence for supercontinuum generation by fission of higher-order solitons in photonic fibers
Author(s): Herrmann J, Griebner U, Zhavoronkov N, et al.
Source: PHYSICAL REVIEW LETTERS Volume: 88 Issue: 17 Article Number: 173901 Published: APR 29 2002

3. SUPERCONTINUUM GENERATION IN GASES
Author(s): CORKUM PB, ROLLAND C, SRINIVASANRAO T
Source: PHYSICAL REVIEW LETTERS Volume: 57 Issue: 18 Pages: 2268-2271 Published: NOV 3 1986

4. SURFACE-PROPERTIES PROBED BY 2ND-HARMONIC AND SUM-FREQUENCY GENERATION
Author(s): SHEN YR
Source: NATURE Volume: 337 Issue: 6207 Pages: 519-525 Published: FEB 9 1989

5. OBSERVATION OF SELF-PHASE MODULATION AND SMALL-SCALE FILAMENTS IN CRYSTALS AND GLASSES
Author(s): ALFANO RR, SHAPIRO SL
Source: PHYSICAL REVIEW LETTERS Volume: 24 Issue: 11 Pages: 592-& Published: 1970

6. OPTICAL INVESTIGATION OF BLOCH OSCILLATIONS IN A SEMICONDUCTOR SUPERLATTICE
Author(s): FELDMANN J, LEO K, SHAH J, et al.
Source: PHYSICAL REVIEW B Volume: 46 Issue: 11 Pages: 7252-7255 Published: SEP 15 1992

7. QUASI-PHASE-MATCHED OPTICAL PARAMETRIC OSCILLATORS IN BULK PERIODICALLY POLED LINBO3
Author(s): MYERS LE, ECKARDT RC, FEJER MM, et al.
Source: JOURNAL OF THE OPTICAL SOCIETY OF AMERICA B-OPTICAL PHYSICS Volume: 12 Issue: 11 Pages: 2102-2116 Published: NOV 1995

8. Phase-matched generation of coherent soft X-rays
Author(s): Rundquist A, Durfee CG, Chang ZH, et al.
Source: SCIENCE Volume: 280 Issue: 5368 Pages: 1412-1415 Published: MAY 29 1998

9. DISCRETE SELF-FOCUSING IN NONLINEAR ARRAYS OF COUPLED WAVE-GUIDES
Author(s): CHRISTODOULIDES DN, JOSEPH RI
Source: OPTICS LETTERS Volume: 13 Issue: 9 Pages: 794-796 Published: SEP 1988

10. MODE-LOCKING OF TI-AL2O3 LASERS AND SELF-FOCUSING - A GAUSSIAN APPROXIMATION
Author(s): SALIN F, SQUIER J, PICHE M
Source: OPTICS LETTERS Volume: 16 Issue: 21 Pages: 1674-1676 Published: NOV 1 1991

11. EXPERIMENTAL-OBSERVATION OF PICOSECOND PULSE NARROWING AND SOLITONS IN OPTICAL FIBERS
Author(s): MOLLENAUER LF, STOLEN RH, GORDON JP
Source: PHYSICAL REVIEW LETTERS Volume: 45 Issue: 13 Pages: 1095-1098 Published: 1980

12. 2-PHOTON EXCITATION IN CAF2 - EU2+
Author(s): KAISER W, GARRETT CGB
Source: PHYSICAL REVIEW LETTERS Volume: 7 Issue: 6 Pages: 229-& Published: 1961

13. Bragg grating solitons
Author(s): Eggleton BJ, Slusher RE, deSterke CM, et al.
Source: PHYSICAL REVIEW LETTERS Volume: 76 Issue: 10 Pages: 1627-1630 Published: MAR 4 1996

14. Compression of high-energy laser pulses below 5 fs
Author(s): Nisoli M, DeSilvestri S, Svelto O, et al.
Source: OPTICS LETTERS Volume: 22 Issue: 8 Pages: 522-524 Published: APR 15 1997

Mini-Project 5: Nonlinear Optical Processes in Quantum Optics
KSU PHYS953, NQO Due 12/7/07

Nonlinear Processes for the Generation of Quadrature Squeezed Light

This project investigates the use of a nonlinear optical process for the generation of nonclassical light. Do the first three questions for full credit. The other questions will be extra credit.

Consider the superposition state $|\psi\rangle = a|0\rangle + b|1\rangle$ where *a* and *b* are complex and satisfy the relationship $|a|^2 + |b|^2 = 1$.

1. Calculate the variances of the quadrature operators $\hat{X}_1$ and $\hat{X}_2$ (see Eq. 2.52 and Eq. 2.53). The variance of an operator is given by

$$\left\langle (\Delta \hat{X}_i)^2 \right\rangle = \left\langle \hat{X}_i^{\,2} \right\rangle - \left\langle \hat{X}_i \right\rangle^2$$

   Remember that $\hat{X}_1$ is called the in-phase component and $\hat{X}_2$ is the in-quadrature component.
2. Show that there exits values of the parameters *a* and *b* for which either of the quadrature variances become *less* than for a vacuum state. Hint: let $b = \sqrt{1-|a|}e^{i\varphi}$ and $a^2 = |a|^2$ (this is done without the loss of generality). Plot the variance as a function of $|a|^2$ for different φ.
3. For the cases where the quadrature variances become less than for a vacuum state, check to see if the uncertainty principle is violated.
4. Verify that the quantum fluctuations of the field quadrature operators are the same for the vacuum when the field is in coherent state (*i.e.* verify Eq. 3.16).

The above result illustrate a case where the expectation value of the quadrature operator becomes less than a vacuum state, even though the quadrature operators must satisfy the minimum uncertainty relationship. Squeezing is the process when one canonical (conjugate) variable has a variance less than the vacuum state but the other canonical variable will have a larger variation in order to satisfy the uncertainty principle. The quadrature operators $\hat{X}_1$ and $\hat{X}_2$ are canonical variables and do not commute, thus they have an uncertainty relationship given by Eq. 2.56. Quadrature squeezing occurs when

$$\left\langle (\Delta \hat{X}_1)^2 \right\rangle < \frac{1}{4} \text{ or } \left\langle (\Delta \hat{X}_2)^2 \right\rangle < \frac{1}{4}.$$

We can plot a phase space diagram of a normal and squeezed state (see below and on page 154). The area in phase space remains must constant to maintain the minimum uncertainty relationship. However, we can "squeeze" the circle into an ellipse while keeping the area constant (like squeezing the Charmin done in class).

Not Squeezed Squeezed

Quadrature squeezed light can be produced by the second order nonlinear effect known as **degenerate parametric down-conversion**. This process involves two signal (*s*) waves produced by one pump wave (*p*), *i.e.* $\omega_s + \omega_s - \omega_p = 0$. This process is a degenerate form of difference frequency generation with the signal wave equal to the idler wave. The Hamiltonian for this degenerate parametric down-conversion is given by

$$\hat{H} = \frac{\hbar}{2}\left(\chi^{(2)}\hat{a}_s^\dagger \hat{a}_p \hat{a}_s^\dagger + \chi^{*(2)}\hat{a}_s \hat{a}_p^\dagger \hat{a}_s\right)$$

5. Use the Heisenberg equations of motion (Eq. 2.19) to derived two coupled first order differential equation for $\frac{d\hat{a}_s}{dt}$ and $\frac{d\hat{a}_s^\dagger}{dt}$.
6. What are the solutions to these differential equations if we assume a non-depleted pump? Integrate from time 0 to *T*.
7. Show that the quadrature operators have the solution

$$\begin{bmatrix} \hat{X}_1(T) \\ \hat{X}_2(T) \end{bmatrix} = \begin{bmatrix} e^{-\delta T} & 0 \\ 0 & e^{\delta T} \end{bmatrix} \begin{bmatrix} \hat{X}_1(0) \\ \hat{X}_2(0) \end{bmatrix} \text{ where } \delta \equiv i\chi^{(2)}\hat{a}_p$$

8. Consider a coherent state $|\alpha\rangle$ Show the mean square fluctuations (variance) result in

$$\begin{bmatrix} \langle\alpha|(\hat{X}_1(T))^2|\alpha\rangle - \langle\alpha|\hat{X}_1(T)|\alpha\rangle^2 \\ \langle\alpha|(\hat{X}_2(T))^2|\alpha\rangle - \langle\alpha|\hat{X}_2(T)|\alpha\rangle^2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} e^{-2\delta T} \\ e^{+2\delta T} \end{bmatrix}$$

This result states that the mean square fluctuations of the in-phase component $\hat{X}_1$ is exponentially smaller by $e^{-2\delta T}$ and mean square fluctuations of the in-quadrature component $\hat{X}_2$ are exponentially larger by $e^{2\delta T}$. So the above picture depicts the squeezing performed by the nonlinear process. The bizarre thing about this analysis is that it is also true for a **vacuum state**. One can have vacuum and squeezed vacuum.

9. A third order nonlinear can also be used to produce squeezed light instead. Name a third order nonlinear process that will give rise to squeezed light (Hint: we discussed a third order process that “looks” like difference frequency generation. What was that process?).

Squeezed light and squeezed vacuum has many important applications, specifically for light detection at levels below the quantum noise (*i.e.* shot noise) level. See Henry *et al*, Amer. J. Phys. Vol 56 (4) p. 318 (1988) for more information.

Final Project: Research Paper **KSU PHYS953, NQO**

The final project will be an investigation of a topic or problem in the areas of nonlinear and quantum optics, that will involve a literature search and some original work. The purpose of the paper is to pose a question about your chosen topic and try to answer that question. Please keep in mind you do not to answer the question you have posed. Your paper will be evaluated on a complete literature search, a good discussion on the question, and a well-executed attempt in answering the question.

The final project will consist of three parts:

Part 1: Abstract and bibliography	Due November 19, 2007
Part 2: Six to eight page paper	Due December 3, 2007
Part 3: 10 minute presentation	Starting December 7, 2007

1. Part 1

For Part 1, you will need to provide a draft title, abstract, and bibliography. In your abstract, you will need to state a draft question that the paper will try to answer. I will look over your topic and approve it so you can do the rest of the project. On the back is a short list of research topics. Feel free to pick any topic in quantum and nonlinear optics you wish.

2. Part 2

Part 2 is a six to eight page paper on your topic. The paper should include:

The title and abstract
An introduction to the topic
A discussion of prior work
A section stating the question your paper wishes to answer
A section of your own work investigating the question
A summary comparing your conclusions with respect to prior work
A list of references

Paper Format: 10 or 12 pt font, Times New Roman Font, 1 inch margins

3. Part 3

For Part 3 you will need to give a 10 minute talk about your topic, with 3 minutes for questions. The talk will be given in class at the times listed below. For your talk, **you will only have the white/black board at your disposal**; do not prepare a computer-based talk. You are encouraged to provide handouts to the class for your talk. Also, you are encouraged to practice your talk using a white/black board before you give your talk. Your talk will be evaluated on the clarity of presentation as well as the use of time (in other words, do not go over time!).

List of Sample Topics and Questions

- Self focusing in a rare gas with estimations of focusing versus pulse intensity
- Explaining how to describe the Compton effect semi-classically
- Quantum mechanically description of stimulated Raman scattering
- Explaining how to describe the photoelectric effect semi-classically
- Discuss how quantum entanglement can be used for secure communications
- Discuss the theory and operation of an optical parametric chirped-pulse amplification, OPCPA
- Investigate the thermodynamics of laser mode-locking and how nonlinear effects are involved
- Self similar behavior in optical fiber and the third order nonlinearity
- Quantum optics in cold atoms: how does one generate entangled states?
- How to generate entangled light using second order nonlinear processes in crystals.
- What is the quantum eraser and how can one demonstrate this?
- How is two-photon absorption used for biological imaging?
- Discuss the role of electrons and holes in the nonlinear optics of III-V semiconductors
- The quantum mechanics of electromagnetic noise: Shot and thermal noise
- The quantum description of heterodyne and homodyne optical detection
- The quantum theory of a laser: the master equation.
- The role of higher order nonlinear effects in laser mode-locking
- Quantum optical description of electromagnetically induced transparency in atomic systems
- Quadrature squeezing in optical fibers
- Applications of squeezed noise in gravity wave detection
- Nonlinear spectroscopy of gases: theory of saturated absorption
- Entanglement and quantum teleportation of states

Ask me if you want more topics.

Schedule to the end of the Semester

Nov. 7 (W)	Field Quantization: single mode fields	G2	
Nov. 9 (F)	Field Quantization: multimode fields	G2	MP4 Summary
Nov. 12 (M)	Quadrature Operators and the Quantum Phase: Zero Point Energy	G2	
Nov. 14 (W)	Coherent States	G3	
Nov. 16 (F)	More on coherent states	G3 Exam 2	MP4 Review
Nov. 19 (M)	Phase-space pictures of coherent states	G3 Exam 2 due Final Project Part 1Due	
Nov. 21 (W)	No Class		
Nov. 23 (F)	No Class		
Nov. 26 (M)	Quantum mechanics of beam splitters: revisit the Aspect experiment	G6	
Nov. 28 (W)	Entanglement	G6	
Nov. 30 (F)	Optical Tests of Quantum Mechanics: EPR Paradox and Bell's Theorem	G9	
Dec. 3 (M)	Optical Tests of Quantum Mechanics: Bell's Theorem and the Aspect experiment	G9 Final Project Part 2 Due	
Dec. 5 (W)	Catch-up…TBA		
Dec 7 (F)	Final Project Presentation	Final Project Part 3 Due	
Dec 10 (M)	Final Project Presentation, Technically Exam Period 4:10 p.m. - 6:00 p.m. Actual time TBA: Possibly 4:00 p.m. - 7:00 p.m: Cardwell 119 or 220 (I will bring pizza)	Final Project Part 3 Due	