Lecture Notes for Nonlinear and Quantum Optics

PHYS 953

Fall 2010

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Kansas State University

PHYS 953 – Adv. Topics/Non-linear and Quantum Optics - Fall 2010
Lecture: T/U, 1:05-2:20 p.m. CW 145

Textbooks: *Nonlinear Optics*, Boyd; *Introductory Quantum Optics*, Gerry and Knight;

Suggested References: *Introduction to Quantum Optics, From Light Quanta to Quantum Teleportation*, Paul; *The Quantum Challenge*, Greenstein and Zajonc; *Quantum Optics*, Walls and Milburn; *Coherence and Quantum Optics*, Mandel and Wolf; *Nonlinear Optics*, Shen; *Nonlinear Fiber Optics*, Agrawal; *Handbook of Nonlinear Optics*, Sutherland; *Handbook of Nonlinear Optical Crystals*, Dmitriev, Gurzadyan, and Nikogosyan; *Electromagnetic Noise and Quantum Optical Measurements*, Haus;

Instructor: Dr. Brian R. Washburn, CW 36B, (785) 532-2263, washburn@phys.ksu.edu. Office hours: M/W/F 9:30-10:30 PM or by appt.

Prerequisites: A solid foundation in undergraduate-level quantum mechanics, electromagnetism, and optics.

Course Objective: The purpose of this course is to provide an introduction to the field of nonlinear optics, exploring the physical mechanisms, applications, and experimental techniques. Furthermore the fundamentals of quantum optics will be taught in the second half in this course. Connections between quantum and nonlinear optics will be highlighted throughout the semester. My goal is for students to end up with a working knowledge of nonlinear optics and a conceptual understanding of the foundations of quantum optics.

Grading:

Exam 1	150 pts	300 pts
Exam 2	150 pts	
Mini-Projects		500 pts
Final Project		200 pts
Total possible		**1000 pts**

Exams: There will be two exams during the semester. The format will be a take-home exam to be completed over 24 hours.

Mini-Projects: Problems in nonlinear and quantum optics are quite involved, so traditional homework assignments will not properly teach the material. So, the homework for this course will be in the form of mini-projects. The mini-projects will be a detailed solution of interconnected problems related to lecture topics. The problems will need to be solved using resources beyond the textbook and class notes. The purpose of the mini-projects is to mimic problem-solving scenarios found in a research environment.

There will be between 5-7 mini-projects, each given with two or more weeks for completion. Working on the mini-projects in groups is strongly encouraged, but you will need to write up the assignment on your own.

Final Project: There will be a final project for the class but no final exam. The final project will be an investigation of a topic or problem in the areas of nonlinear and quantum optics, that will involve a literature search and some original work. The final project will consist of three parts:

Part 1: Abstract and bibliography
Part 2: 6 page paper plus references
Part 3: 15 minute presentation

Late Projects: No project will be accepted after its due date unless prior arrangements have been made. Please inform me with possible conflicts before the due date, and other arrangements will be made.

Class Material: Extra class materials are posted on K-state Online, including papers and tutorials.

Disabilities: If you have any condition such as a physical or learning disability, which will make it difficult for you to carry out the work as I have outlined it or which will require academic accommodations, please notify me and contact the Disabled Students Office (Holton 202), in the first two weeks of the course.

Plagiarism: Plagiarism and cheating are serious offenses and may be punished by failure on the exam, paper or project; failure in the course; and/or expulsion from the University. For more information refer to the "Academic Dishonesty" policy in K-State Undergraduate Catalog and the Undergraduate Honor System Policy on the Provost's web page: http://www.ksu.edu/honor/.

Tentative Course Schedule, Nonlinear and Quantum Optics, PHYS 953, Fall 2010

Date	Topic	Chapters	Projects
Aug. 24 (T)	Class overview: review of linear optics and the semi-classical treatment of light Review of material dispersion and absorption		
Aug. 26 (U)	Introduction to nonlinear optics: the nonlinear susceptibility —Formal definitions —Nonlinear optics and mechanics: analogy to anharmonic motion	B1	
Aug. 31 (T)	The Maxwell's wave equation in a nonlinear medium Symmetry and nonlinear optical properties	B1	
Sept. 1 (U)	Second order nonlinear effects —Coupled equations: Sum frequency and second harmonic generation —Phase matching in second harmonic crystals	B2	MP1 Due
Sept. 7 (T)	Second harmonic generation with ultrashort pulses —Phasematching and bandwidth issues	B2	
Sept. 9 (U)	Difference and sum frequency generation —Parametric amplification in crystals, optical parametric oscillators	B2	
Sept. 14 (T)	No Class (need to make this day up)		
Sept. 16 (U)	No Class (need to make this day up)		
Sept. 21 (T)	Applications for second harmonic generation —Ultrashort pulse measurements	B2	
Sept. 23 (U)	Applications for second harmonic generation —Carrier-envelope phase measurement: the *f*-to-2*f* interferometer		MP2 Due
Sept. 28 (T)	Catch up day!		
Sept. 30 (U)	Third order nonlinear effects: Intensity dependent refractive index; four-wave mixing Nonlinear fiber optics: fiber parametric oscillators	B4, B13	
Oct. 5 (T)	More nonlinear fiber optics —Pulse propagation in a third order nonlinear medium, soliton generation	B4, B13 Exam 1	
Oct 6 (W)	Exam 1 Due		
Oct. 7 (U)	Spontaneous and stimulated Raman scattering —Spontaneous Raman scattering —Stimulated Raman scattering in third order media	B4	
Oct. 12 (T)	More on stimulated Raman scattering: CARS spectroscopy	B9	
Oct. 14 (U)	Third order effects in gases: applications for short pulse generation	B9	MP3 Due
Oct. 19 (T)	High field processes: higher harmonic generation	B13	
Oct. 21 (U)	Introduction to quantum optics: What is a photon? —The photoelectric effect —The Hanbury-Brown and Twiss experiment	G1	
Oct. 26 (T)	No Class (need to make this day up)		
Oct. 28 (U)	What is a photon? —The photoelectric effect revisited: Lamb and Scully —The Aspect experiments	G1	
Nov. 2 (T)	What is a photon? —Wheeler's delayed choice experiment —Quantum beat experiments	G2	
Nov. 4 (U)	Field quantization and coherent states	G2	
Nov. 9 (T)	More on coherent states	G2,G3	MP4 Due
Nov. 11 (U)	Interferometry with a single photon	G2,G3	
Nov. 16 (T)	Bell's theorem and quantum entanglement —EPR Paradox and Bell's Theorem	G9 Exam 2	
Nov. 17 (W)	Exam 2 Due	Final project part 1 due	
Nov. 18 (U)	Optical tests of EPR: violations of the Bell's inequality	G9	
Nov. 23 (T)	Thanksgiving Break		
Nov. 25 (U)	Thanksgiving Break		
Nov. 30 (T)	Nonclassical light: squeezed states	G9	
Dec. 2 (U)	Optical tests of quantum mechanics	G9	MP5 Due
Dec. 7 (T)	Catch up day!	Final project part 2 due	
Dec. 9 (U)	Final Project Presentations	Final project part 3 due	
Dec. 14 (T)	Final Project Presentations: Exam Period 2:00 PM – 3:50 PM		

Books: B= Boyd, *Nonlinear Optics*, G= Gerry and Knight, *Introductory Quantum Optics*;

BRW 8/13/2010

Lecture 1

Intro to nonlinear & quantum optics

Go over Syllabus

Cover Intro to nonliner optics

Intro to quantum optics : what is a photon

Start with review

Maxwell's wave eq

Lorentz model

Define some terms

- Optics : Study of light & its interaction with matter

- nonlinear optics : Study of the interaction of intense light and matter with a nonlinear response (large # of photons)

Classical / Semiclassical description

Semiclassical : Classical electric field with quantized states of matter.

- Quantum Optics : The quantum nature of light

(Small # of photons)

Quantum Treatment

The linear susceptibility (electric)

$$\vec{P} = \epsilon_0 \sum_{i=1}^{3} \hat{x}_i \sum_{j=1}^{3} \chi_{ij}^{(1)} \mathcal{E}_j$$

$\left\{ \vec{P} \equiv \right.$ polarization
net dipoles per unit Volume.

$\vec{\mathcal{E}} \equiv$ real instantaneous field

$\hat{x}_i : \hat{x}, \hat{y}, \hat{z} \qquad \vec{\mathcal{E}} = \sum_{i=1}^{3} \hat{x}_i \mathcal{E}_i$

For linear isotropic homogeneous media we have

$$\vec{P} = \epsilon_0 \chi^{(1)} \vec{\mathcal{E}} \qquad \left(\text{really, } \vec{P}(\omega) = \epsilon_0 \chi^{(1)} \vec{\mathcal{E}}(\omega)\right)$$

(could talk about M where $\vec{M} = \chi_m \vec{\mathcal{H}}$)

The nonlinear susceptibility

$$\vec{P} = \epsilon_0 \sum_i \hat{x}_i \left[\sum_j \chi_{ij}^{(1)} \mathcal{E}_j + \sum_{jk} \chi_{ijk}^{(2)} \mathcal{E}_j \mathcal{E}_k + \sum_{jkl} \chi_{ijkl}^{(3)} \mathcal{E}_j \mathcal{E}_k \mathcal{E}_l + \cdots \right]$$

$\chi^{(n)} \equiv n+1$ rank tensor with 3^{n+1} terms

$\chi^{(1)} \equiv 9$ terms $\quad \chi^{(2)} \equiv 27$ terms $\quad \chi^{(3)} \equiv 81$ terms

The number of terms get quite large, For $\chi^{(2)}$ we have the mixing of three fields $\omega_1\ \omega_2\ \omega_3 \qquad \omega_3 = \omega_2 + \omega_1$

There are $3! * 2$ ways to make this product. So the # of elements needed to describe a $\chi^{(2)}$ effect is

$$3! \times 2 \times 27 = 324$$

For $\chi^{(3)}$ its 1944!

Induced dipole moment

$$\vec{P} \sim -\underbrace{N}_{\text{dipoles per unit volume}} e\vec{r}$$

Linear material in freq domain

$$\vec{\mathcal{P}}(\omega) = \epsilon_0 \chi^{(1)} \vec{\mathcal{E}}(\omega) \qquad \text{isotropic}$$

$$\mathcal{P}_i(\omega) = \sum_j \epsilon_0 \chi_{ij} \mathcal{E}_j(\omega) \qquad \text{anisotropic}$$

Time domain

$$\vec{\mathcal{P}}(t) = \epsilon_0 \int \chi(t-t') \cdot \mathcal{E}(r, t-t')\, dt'$$

$$\chi \rightarrow \delta(t-t') \quad \text{for fast materials}$$

What do nonlinear effects do?

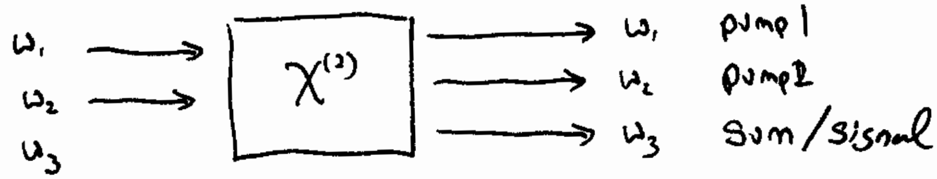
$$\boxed{\text{Create "new" spectral components}}$$

Keeping energy conservation in mind

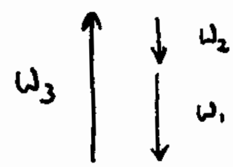
$$\boxed{\text{Nonlinear effects create new spectral components by shifting spectral energy to new frequencies}}$$

Examples of nonlinear effects $\chi^{(2)}$

<u>Sum frequency generation.</u> $\omega_3 = \omega_1 + \omega_2$



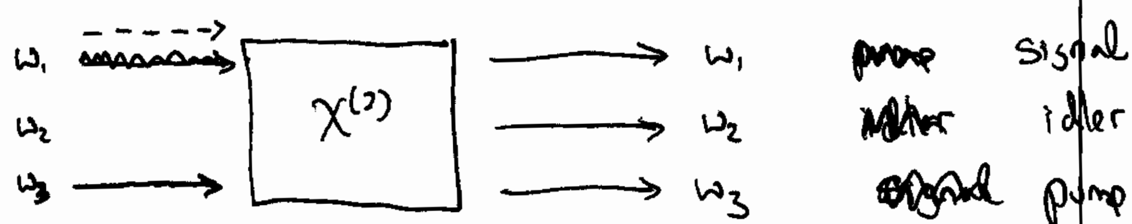
Photon process



one photons $\omega_2 + \omega_1$ are destroyed to create ω_3

Degenerate sum frequency generation $\omega_1 = \omega_2 \Rightarrow$ second harmonic generation

<u>Difference frequency generation</u> $\omega_2 = \omega_3 - \omega_1$



a photon at ω_2 is created thru the destruction of ω_3 and creation of ω_1

(amplification of ω_1)

Degenerate DFG $\Rightarrow \omega_3 - \omega_1 \Rightarrow$ optical rectification

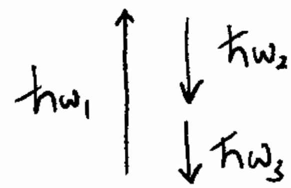
$\pm\omega_m = \pm\omega_n \pm \omega_k$ $\left(\begin{array}{lll} \text{lhs :} & + \text{ create} & - \text{ destroy} \\ \text{rhs :} & - \text{ create} & + \text{ destroy} \end{array}\right)$

Examples of $\chi^{(3)}$: SRS, SPM, supercontinuum generation

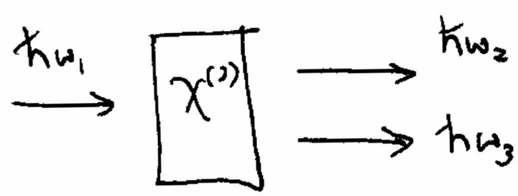
What about photons?

Energy is conserved, but not photon #

$$\hbar\omega_1 = \hbar\omega_2 + \hbar\omega_3$$



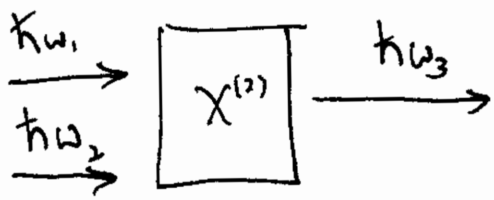
one high freq. photon splits into two lower frequency photons



(DFG)

new spectral components.

OR



$$\hbar\omega_3 = \hbar\omega_2 + \hbar\omega_1$$

(SFG)

Also momentum is conserved $\hbar k_1 = k_2\hbar + \hbar k_3$

photons $\Rightarrow$ semiclassical context

Important points on nonlinear oscillations

1) Frequency of oscillation depends on amplitude

2) Superposition principle does not hold

Nonlinear DE.

If $x(t)$ + $y(t)$ are a solution

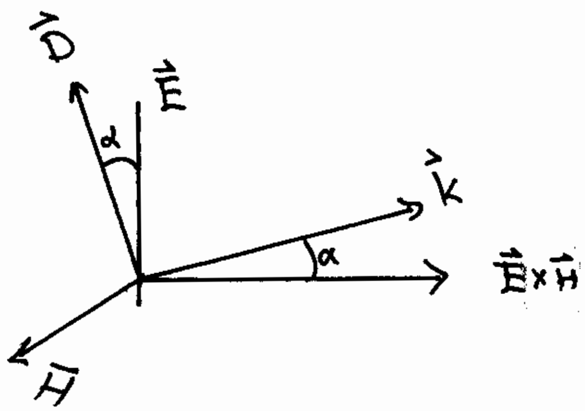
$a\,x(t) + b\,y(t)$ may <u>not</u> be a solution

Questions about Fields

1) is $\vec{D} \parallel \vec{E}$ in general? NO

2) is $\vec{S} = \vec{E} \times \vec{H} \parallel \vec{k}$ in general? NO

3) is $\vec{H} \perp \vec{E}$ and $\vec{H} \perp \vec{k}$ in general? YES



Maxwell's Eqs. inside matter

$$\boxed{\begin{array}{ll} \vec{\nabla} \times \vec{E} = -\partial_t \vec{B} & \vec{\nabla} \cdot \vec{D} = \rho_f \\ \vec{\nabla} \times \vec{H} = \vec{J}_f + \partial_t \vec{D} & \vec{\nabla} \cdot \vec{B} = 0 \end{array}}$$

From these we get

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \omega^2 \mu_0 \underbrace{\vec{D}}_{[\epsilon]\vec{E}}$$

matrix mult.

Review

<u>Maxwell's Wave Eq. For dielectric materials</u> ([illegible])

Start with

$$\vec{\nabla}\times\vec{E} = -\partial_t\vec{B} \qquad \vec{\nabla}\times\vec{H} = \vec{J} + \partial_t\vec{D}$$

$$\vec{\nabla}\cdot\vec{D} = \rho_v \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{D} = \epsilon_0\vec{E} + \vec{P} \qquad \vec{P} = \epsilon_0[\chi]\vec{E}$$

Assume sourceless, non magnetic ($\mu=\mu_0$) material with $\vec{J}=0$
($\rho_v = 0$)

Since $\vec{\nabla}\cdot\vec{D} = 0$ then $\vec{\nabla}\cdot\vec{E} = 0$, also $\vec{B} = \mu_0\vec{H}$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = -\mu_0\,\partial_t(\vec{\nabla}\times\vec{H})$$

OK for anisotropic

$$= -\mu_0\,\partial_t(\partial_t\vec{D}) \qquad \text{but } \vec{D} = \epsilon_0\vec{E} + \vec{P}$$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = -\mu_0\epsilon_0\,\partial_t^2\vec{E} - \mu_0\,\partial_t^2\vec{P} \qquad \left(\mu_0\epsilon_0 \equiv \frac{1}{c^2}\right)$$

But $\vec{\nabla}\times\vec{\nabla}\times\vec{E} = \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E}$ But $\vec{\nabla}\cdot\vec{E} = 0$

So

$$-\nabla^2\vec{E} = -\frac{1}{c^2}\partial_t^2 E - \mu_0\,\partial_t^2\vec{P}$$

$$\boxed{\nabla^2\vec{E} - \frac{1}{c^2}\partial_t^2\vec{E} = \underbrace{\mu_0\,\partial_t^2\vec{P}}_{\text{source.}}}$$

Valid for anisotropic

In Boyd

$$\nabla^2 \tilde{E} - \frac{n^2}{c^2} \partial_t^2 \tilde{E} = \frac{1}{\epsilon_0 c^2} \partial_t^2 \tilde{P}$$

Boyd Notation

$\tilde{E}(t)$ rapidly varying time dependent Field

$E(\omega)$ freq. , slowly varying

My notation

$\vec{\mathcal{E}}(t)$ Real instantaneous Field

$\vec{\mathbb{E}}(t)$ Complex instantaneous

$E(t)$, $\mathbb{E}(\omega)$ Amplitude

Intensity $\quad I = \dfrac{2n}{\eta} |E|^2 = 2n \sqrt{\dfrac{\epsilon_0}{\mu_0}} |E|^2$

[illegible]

$\eta = 377\,\Omega$

Weiding electric fields

Factors of 2

Boyd $\cancel{\tilde{E}(t) = E_1 e^{-i\omega_1 t} + E_2 e^{-i\omega_2 t} + c.c.}$

$$\tilde{E}(t) = E(\omega_1)\, e^{-i\omega_1 t} + c.c.$$

$$\tilde{P}^{(2)}(t) = \sum_n P(\omega_n)\, e^{-i\omega_n t}$$

other notation (which I like)

$$\tilde{E}(t) = \frac{1}{2}\left(E(\omega_1)\, e^{-i\omega_1 t} + c.c. \right)$$

$$\tilde{P}(t) = \frac{1}{2} \sum_n P(\omega_n)\, e^{i\omega_n t}$$

I will stick with Boyd's convention on $1/2$, but not with $\tilde{E}$

$$\vec{\mathcal{E}}(\vec{r},t) = \vec{E}(r)\, e^{-i\omega t} + \vec{E}^*(r)\, e^{i\omega t}$$

Real Fields

$$E(r) = A e^{i \vec{k}\cdot\vec{r}}, \quad A = u e^{i\phi}$$

slowly varying envelope

$$\vec{\mathcal{E}}(\vec{r},t) = \sum_{n=1}^{\infty}{}' \; \vec{\mathcal{E}}_n(\vec{r},t)$$

where $\vec{\mathcal{E}}_n = E_n(r)\, e^{-i\omega_n t} + c.c.$

complex conjugate

Factors of 1/2

$\boxed{\text{Boyd}}$ $\quad \vec{E}(\vec{r},t) = \sum_n{}' \tilde{\vec{E}}_n(\vec{r},t) \qquad (1.3.1)$

where $\tilde{\vec{E}}_n(\vec{r},t) = \vec{E}_n(r)\, e^{-i\omega_n t} + \text{c.c.} \qquad (1.3.2)$

$\sum'$: Sum over positive frequencies only

Also from Boyd

$$E_n = E(\omega_n) \qquad A_n = A(\omega_n)$$

where $E(-\omega_n) = E(\omega_n)^* \qquad A(-\omega_n) = A(\omega_n)^*$

$$\tilde{E}(\vec{r},t) = \sum_n E(\omega_n)\, e^{-i\omega_n t}$$ Sum over positive + negative frequencies

$$\tilde{E}(\bar{r},t) = \mathcal{E} \cos(\bar{k}\cdot\bar{r} - \omega t)$$

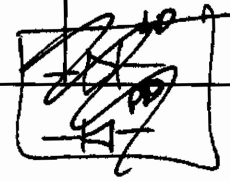
$$E(\omega) = \underbrace{\tfrac{1}{2}\mathcal{E}}_{A(\omega)} e^{i\bar{k}\cdot\bar{r}} \qquad E(-\omega) = \underbrace{\tfrac{1}{2}\mathcal{E}}_{A(-\omega)} e^{-i\bar{k}\cdot\bar{r}}$$

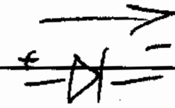
B+C

$$\vec{E}(t) = \frac{1}{2}\sum_{\omega'\geq 0} E_{\omega'} \exp(-i\omega' t) + E_{-\omega'} \exp(i\omega' t)]$$

$E(t)$ real so $\quad E_{-\omega'} = E_{\omega'}^*$

$$\vec{E}(\omega) = \frac{1}{2}\sum_{\omega'\geq 0}\left[E_{\omega'}\, \delta(\omega-\omega') + E_{-\omega'}\, \delta(\omega+\omega')\right]$$





K Factor

$$K(-\omega_\sigma; \omega_1, \omega_2 \cdots \omega_n) \qquad \omega_\sigma = \omega_1 + \omega_2 + \cdots \omega_n$$

$$= 2^{l+m-n} p$$

$p \equiv$ # of permutations of $\omega_1 \cdots \omega_n$

$n \equiv$ order of nonlinearity

$m \equiv$ set of n frequencies are zero

$l \equiv \quad l = 1$ if $\omega_\sigma = 0$ otherwise $l = 0$

$$\left(P^{(n)}_{\omega_\sigma}\right)_\mu = \epsilon_0 \sum_{\alpha_1 \cdots \alpha_n} \sum_{\omega} K(-\omega_\sigma; \omega_1 \cdots \omega_n) \times \chi^{(n)}_{\mu\alpha_1 \cdots \alpha_n}(-\omega_\sigma; \omega_1 \cdots \omega_n) \times (E_{\omega_1})_{\alpha_1} \cdots E(\omega_n)_{\alpha_n}$$

components

Sutherland

$$\vec{E}^{(\omega)}(\vec{r},t) = \hat{e}\, A(\vec{r},t) \exp(i kz - \omega t) + c.c.$$

include negative frequencies

negative → different direction

D notation

$$P_i(\omega_3) = \epsilon_0 D^{(2)} \sum_{jk} \chi^{(2)}_{ijk}(-\omega_3; \omega_1, \omega_2) E_j(\omega_1) E_k(\omega_2)$$

$$D^{(2)} = \begin{cases} 1 & \text{indistinguishable} \\ 2 & \text{distinguishable} \end{cases}$$

Lecture Two

Need to discuss seminal papers.

1) P.A. Franken et al 1962

2) Armstrong et al 1961

Seminal papers
Latin root $\Rightarrow$ seed

Lorentz Model

Review Wave Eq

$$\boxed{\vec{\nabla}^2 \vec{E} - \frac{1}{c^2}\partial_t^2 \vec{E} = \mu_0 \partial_t^2 \vec{P}}$$

For linear isotropic material $\vec{P} = \epsilon_0 \chi \vec{E}$ we can write

$$\boxed{\vec{\nabla}^2 \vec{E} - \frac{n^2}{c^2}\partial_t^2 \vec{E} = 0}$$

with $n^2 = 1 + \chi^2$

Index of refraction

$$v = \frac{c}{n(\omega)} \qquad c^2 = \frac{1}{\epsilon_0 \mu_0} \qquad k(\omega) = \frac{n(\omega)\,\omega}{c}$$

Back to wave eq. (1D)

$$\partial_x^2 \psi - \frac{1}{v^2}\partial_t^2 \psi = 0$$

Maxwell's wave eq.

$$\nabla^2 E - \frac{n^2}{c^2}\partial_t E = 0$$

Solution $\exp(i\vec{k}\cdot\vec{r} - i\omega t)$

Absorption

$$I_0 = I_{in}\, e^{-\alpha z} \qquad \text{Beer's Law}$$

$$\alpha = \alpha(\omega)$$

Lorentz Model For a linear dielectric medium

Collection of N electrons per unit volume. We want to Find the response of the medium to an applied electric Field $\vec{E}$

Induced polarization

$$\vec{P} = -Ne\vec{r} \qquad \text{(complex)} \;/$$

at equilibrium $\quad -e\vec{E} = k\vec{r}$

atomic restoring Force

$$\vec{r} = -\frac{e\vec{E}}{k}$$

So $$\vec{P} = \frac{Ne^2}{k}\vec{E}$$

Treat the medium as a collection of harmonic oscillators with $E = E(t)$

Damped driven oscillator

$$m\ddot{\vec{r}} + \underbrace{m\gamma\dot{\vec{r}}}_{\text{damping}} + \underbrace{k\vec{r}}_{\text{restoring Force}} = \underbrace{-e\vec{E}}_{\text{driving Force}}$$

Assume $E \sim e^{+i\omega t}$ so $r(t) \sim e^{i\omega t}$ in steady state

Sub into DE $\qquad \dot{r} \sim +i\omega e^{i\omega t} \qquad \ddot{r} \sim -\omega^2 e^{i\omega t}$

$$(-m\omega^2 + i\omega m\gamma + k)\vec{r} = -e\vec{E}$$

But $\vec{P} = -Ne\vec{r}$

$$\vec{r} = \frac{-e\vec{E}}{-m\omega^2 + i\omega m\gamma + k} = \frac{-eE/m}{(\omega_0^2 - \omega^2) + i\gamma\omega}$$

where $\omega_0^2 \equiv k/m$

So

$$\cancel{\vec{P} = \frac{Ne^2}{m\omega^2 - i\omega}}$$

$$\vec{P} = \frac{Ne^2/m}{(\omega_0^2 - \omega^2) + i\omega\gamma}\,\vec{E} = \epsilon_0 \chi_e^{(1)} \vec{E}$$

So $$\boxed{\chi_e^{(1)} = \frac{Ne^2/m\epsilon_0}{(\omega_0^2 - \omega^2) + i\omega\gamma}}$$ complex

Write $\chi_e^{(1)} = \chi_e' + i\,\chi_e''$ Express in real + imaginary parts

Use this polarization in the Maxwell's wave eq (source term)

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E} + \tfrac{1}{c^2}\partial_t^2\vec{E} = \mu_0 \partial_t^2 \vec{P}$$

source free medium so $\vec{\nabla}\cdot\vec{D} = 0$ so $\vec{\nabla}\cdot\vec{E} = 0$

$$\boxed{\nabla^2\vec{E} - \frac{1}{c^2}\partial_t^2\vec{E} = \mu_0 \partial_t^2\vec{P}}$$

Sub in $\vec{P}$ into wave eq.

$$\nabla^2 \vec{E} - \frac{1}{c^2}\partial_t^2 \vec{E} = \mu_0 \frac{Ne^2/m}{\omega_0^2 - \omega^2 + i\omega\gamma}\vec{E}$$

$$\nabla^2 \vec{E} = \frac{1}{c^2}\left(1 + \frac{\mu_0 Ne^2/m\, c^2}{\omega_0^2 - \omega^2 + i\gamma\omega}\right)\partial_t^2 \vec{E} \qquad c^2 = \frac{1}{\epsilon_0 \mu_0}$$

$$\nabla^2 E = \frac{1}{c^2}\left(\underbrace{1 + \frac{Ne^2}{m\epsilon_0}\frac{1}{\omega_0^2 - \omega^2 + i\gamma\omega}}_{\text{looks like index}^2}\right)\partial_t^2 E$$

$$\boxed{\bar{n}^2 = 1 + \frac{Ne^2}{m\epsilon_0}\left(\frac{1}{\omega_0^2 - \omega_0^2 + i\gamma\omega}\right)}$$

Complex index
(index + absorption)

Rewrite

$$\bar{n} = n + i\,\frac{c}{\omega}\,\frac{\alpha}{2} \qquad \left(\text{why } \tfrac{1}{2}?\right)$$

equate real + imaginary parts

$$\frac{1}{\omega_0^2 - \omega^2 + i\gamma\omega}\;\frac{(\omega_0^2 - \omega^2) - i\omega\gamma}{(\omega_0^2 - \omega^2) - i\omega\gamma} = \frac{1}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}\left(\omega_0^2 - \omega^2 - i\omega\gamma\right)$$

$$\text{So} \quad \bar{n}^2 = \left[1 + \frac{Ne^2}{m\epsilon_0}\left(\frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}\right)\right] + i\left[\frac{\omega\gamma}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}\right]$$

And $\quad \bar{n}^2 = \left(n^2 - \left(\frac{c\alpha}{2\omega}\right)^2\right) + i\left(\frac{2nc\alpha}{2\omega}\right)$

So

$$\boxed{\begin{aligned} n^2 - \frac{\alpha^2 c^2}{4\omega^2} &= 1 + \frac{Ne^2}{m\epsilon_0}\left(\frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}\right) \\ \frac{nc\alpha}{\omega} &= \frac{Ne^2}{m\epsilon_0}\left(\frac{\gamma\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}\right) \end{aligned}}$$

$\boxed{\text{solve for } n(\omega) \ + \ \alpha(\omega)}$

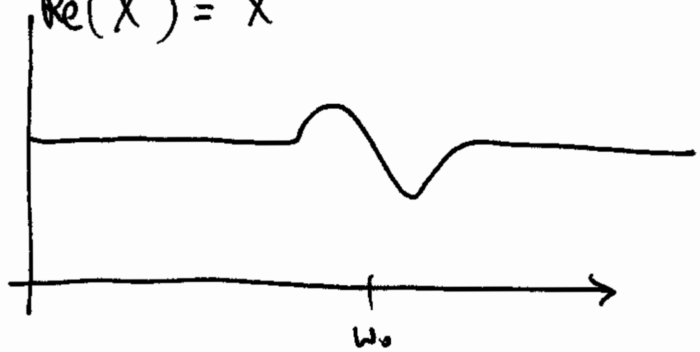
Define $\quad \chi_e = \chi_e' + i\chi_e'$

$$\chi_e'' = \frac{Ne^2}{m\epsilon_0}\left(\frac{\omega\gamma}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}\right)$$

$$\chi_e' = \frac{Ne^2}{m\epsilon_0}\left(\frac{(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}\right)$$

Look at susceptibility $\chi(\omega)$

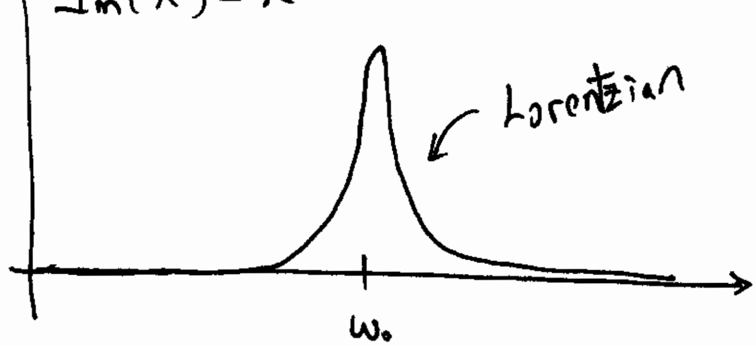
$Re(\chi) = \chi'$



Show plots

Kramer Kronig relations $\Rightarrow$

$Im(\chi) = \chi''$



Know χ' can get χ''

assuming causality

Time domain

$$\boxed{\vec{P} = \epsilon_0 \int_{-\infty}^{\infty} R(\tau)\, \mathcal{E}(t-\tau)\, d\tau}$$

Kramer's - Kronig relationship

Causality

$$\chi(\omega) = \int_0^{\infty} R(\tau)\, e^{i\omega\tau}\, d\tau$$

response, real function

where by Causality $R(\tau) = 0 \quad \tau < 0$
+ $R(\tau)$ is real $(P, \mathcal{E})$

(effect does not preceed the cause)

This implies $\chi(-\omega) = \chi^*(\omega)$

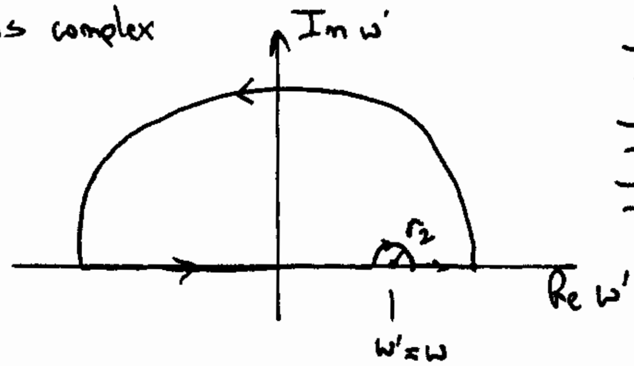
since $R(\tau)$ is real

$\chi(\omega)$ is analytic (single valued + possessing derivatives)
in upper half of complex plane $(\mathrm{Im}\,\omega > 0)$

$$\mathrm{Re}\{\chi(\omega)\} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{Im}\{\chi(\omega')\}\, d\omega'}{\omega' - \omega}$$

$$\mathrm{Im}\{\chi(\omega)\} = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{Re}\{\chi(\omega)\}\, d\omega'}{\omega' - \omega}$$

where ω' is complex



$\mathrm{In}(A) = 0$
$\mathrm{In}(C) = -\pi i\, \chi(\omega)$
$\mathrm{In}(B) = 0$

Fourier transform $\quad X(\omega) = \mathcal{F}\{R(t)\}$

If $R(t)$ is
- real
- $R(t) = 0$ for $t < 0$

Then $X(\omega) = \mathcal{F}\{R(t)\}$ will be ;

- $X(\omega)$ is complex $\quad X(\omega) = X'(\omega) + i\,X''(\omega)$
- $Re\{X(\omega)\}$ will be anti symmetric function $X'(-\omega) = -X'(\omega)$
- $Im\{X(\omega)\}$ will be symmetric

$$X''(-\omega) = X''(\omega)$$

Fourier transform

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\omega)\, e^{-i\omega t}\, d\omega$$

More on causality (from Arfken)

effect cannot preceed cause

convolution integral

$$H(t) = \int_{-\infty}^{\infty} R(t-t')\, G(t')\, dt' = R(t) \circledast G(t)$$

Causality demands

$$F(t-t') = 0 \quad \text{for} \quad t-t' < 0$$

Find

$$H(\omega) = \mathcal{F}\{H(t)\}$$

$$= \mathcal{F}\left\{ \int_{-\infty}^{\infty} R(t-t')\, G(t')\, dt' \right\}$$

$$= \mathcal{F}\{ R(t) \circledast G(t) \}$$

$$= \mathcal{F}\{R(t)\}\ \mathcal{F}\{G(t)\}$$

$$H(\omega) = R(\omega)\ G(\omega)$$

Titchmarsh Thm If $F(\omega)$ is square integrable over real ω axis

~~and F(ω) is zero for t<0~~

then one of the three statements is true

- $\mathcal{F}\{F(\omega)\}$ is zero for $t<0$

(See Jackson for more information)

- $F(\omega)$ is analytic in the complex plane for $\mathrm{Im}(\omega) > 0$

- The real + Imaginary parts of $F(\omega)$ are Hilbert transforms of each other

Special cases: $\chi_e'' \ll 1 + \chi_e'$

$$\bar{n} = \sqrt{1+\chi_e'} + i\left(\frac{\chi_e''/2}{\sqrt{1+\chi_e'}}\right)$$

index n — absorption

$n = \sqrt{1+\chi_e'}$

Weak absorption $\Rightarrow$ Far from resonance (ω_0)

Also

$$\chi_e' \sim \frac{Ne^2}{(2\pi c)^2 \epsilon_0 m} \frac{\lambda^2}{\lambda - \lambda_0^2}$$

OR $$n^2(\lambda) = 1 + \sum_i \left(\frac{A_i \lambda^2}{\lambda^2 - \lambda_{0i}^2}\right)$$ Sellmeier Eq

Optical Pulse : Group + Phase Velocities

Two plane waves of frequency ω_1 + ω_2 $\Rightarrow$ Mix to get beat frequency

optical pulse $\Rightarrow$ finite temporal duration

$\Rightarrow$ finite spectral bandwidth

Time + Frequency domains related by Fourier Transform

Phase + Group Velocity

Phase velocity $\Rightarrow$ velocity of one spectral component

$$\boxed{v_p = c/n(\omega)}$$

Remember

$$c = \lambda_0 f_0 = \frac{\lambda_0 \omega_0}{2\pi}$$

so $\lambda_0 = \frac{2\pi c}{\omega_0}$

Group velocity $\Rightarrow$ velocity of spectral packet about the carrier frequency ω_0

$$\boxed{v_g = \frac{d\omega}{d(\beta(\omega))}}$$

$\beta(\omega) \Rightarrow$ propagation along $\hat{z}$

$$\beta(\lambda) = n(\lambda)\frac{2\pi}{\lambda}$$

$$\beta(\omega) = \frac{n(\omega)\omega}{c}$$

Group Delay + Group Index

$$\tau_g = z\frac{1}{v_g} = z\frac{d\beta}{d\omega} = z\frac{d}{d\omega}(n k_0)$$

$$k_0 = \frac{2\pi}{\lambda_0}$$

$$= z\frac{d\lambda}{d\omega}\frac{d}{d\lambda}\left(n(\lambda)\frac{2\pi}{\lambda_0}\right)$$

$$= \frac{z}{c}\left(n - \lambda\, dn/d\lambda\right)$$

$$\boxed{N = n(\omega) + \omega\frac{dn}{d\omega}}$$

So $v_g = \frac{z}{\tau_g} = \frac{z}{z}\frac{c}{(n - \lambda\, dn/d\lambda)} \equiv \boxed{\frac{c}{N}}$

Where $\boxed{N = n - \lambda\frac{dn}{d\lambda}}$ Group Index

Group velocity

We can be more straight forward here.

$$v_g = \frac{d\omega}{d\beta(\omega)} = \left(\frac{d\beta}{d\omega}\right)^{-1} = \left(\frac{d}{d\omega}\left(\frac{n(\omega)\,\omega}{c}\right)\right)^{-1}$$

$$v_g = \left(\frac{1}{c}\left(\frac{dn}{d\omega}\omega + \frac{n(\omega)}{\cancel{c}}\right)\right)^{-1}$$

$$\boxed{v_g = \frac{c}{\left(n(\omega) + \omega\frac{dn(\omega)}{d\omega}\right)} = \frac{c}{N}}$$

Group index

$$\boxed{N = n + \frac{dn}{d\omega}\omega}$$

$$\lambda = \frac{2\pi c}{\omega}$$

In wavelength $\quad c = \lambda\omega/2\pi \quad \omega = \frac{2\pi c}{\lambda}$

$$N = n + \frac{dn}{d\lambda}\frac{d\lambda}{d\omega}\frac{2\pi c}{\lambda} = n \frac{dn}{d\lambda}\frac{2\pi c}{\lambda}\left(\frac{d\omega}{d\lambda}\right)^{-1}$$

where $\left(\frac{d\omega}{d\lambda}\right) = -\frac{2\pi c}{\lambda^2}$

$$n - \frac{dn}{d\lambda}\frac{\lambda^2}{2\pi c}\frac{2\pi c}{\lambda}$$

$$\boxed{N = n - \lambda\, dn/d\lambda}$$

Lecture 3

Review $\qquad \chi(\omega) = \chi'(\omega) + i\chi''(\omega)$

$$\chi'' \Rightarrow \text{absorption}$$

$$\chi' \Rightarrow \text{index}$$

Far from resonance $\qquad \chi'' \ll 1 + \chi'$

So $\qquad n = \sqrt{1 + \chi_e'}$

$$\alpha = \frac{2\omega}{c} \; \frac{\chi_e''(\omega)/2}{\sqrt{1 + \chi_e'}}$$

Express $n(\lambda)$ using Sellmeier eq.

$$n^2(\lambda) = 1 + \sum_i \left(\frac{A_i \lambda^2}{\lambda^2 - \lambda_i^2} \right)$$

Phase velocity $\qquad v_p = c/n$

Group velocity $\qquad v_g = c/N = \frac{d\omega}{d\beta} = c \, \frac{1}{\left(n + \omega \, dn/d\omega\right)} = \frac{c}{n + \lambda \frac{dn}{d\lambda}}$

where $\qquad \beta(\omega) = \frac{n(\omega)\omega}{c}$

If we express as taylor series about $\omega = \omega_0$

$$\beta(\omega) = \beta(\omega_0) + \left.\frac{\partial \beta}{\partial \omega}\right|_{\omega=\omega_0} (\omega - \omega_0) + \frac{1}{2} \left.\frac{\partial^2 \beta}{\partial \omega^2}\right|_{\omega=\omega_0} (\omega - \omega_0)^2 + \cdots$$

$$\beta(\omega) = \beta_0 + \beta_1 (\omega - \omega_0) + \frac{1}{2} \beta_2 (\omega - \omega_0)$$

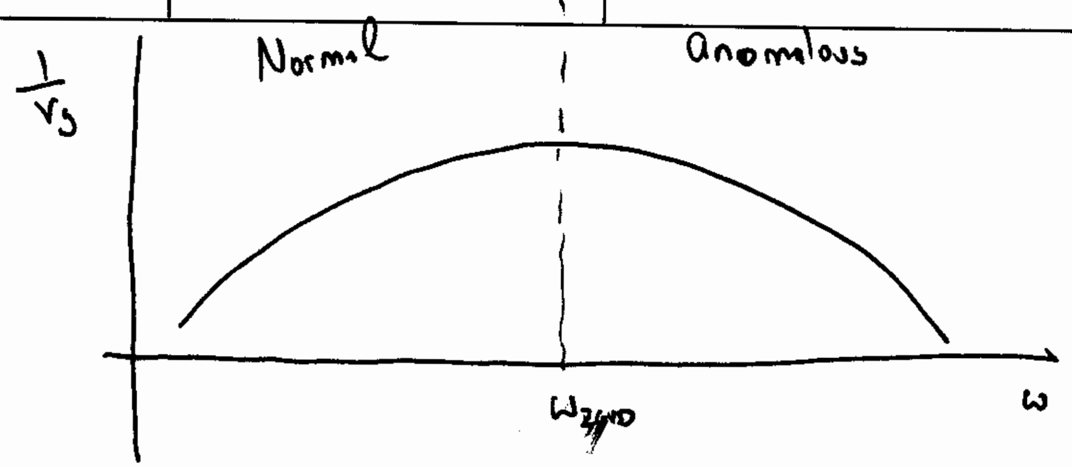
Group velocity dispersion GVD

$$\left.\frac{\partial^2 \beta}{\partial \omega^2}\right|_{\omega=\omega_0} = \frac{\partial}{\partial \omega}\left(\frac{1}{v_g}\right) = \left.\frac{\partial}{\partial \omega}(\beta_1)\right|_{\omega=\omega_0}$$

$$= c\,\frac{\partial}{\partial \omega}\left(\frac{1}{n + \omega\, dn/d\omega}\right)$$

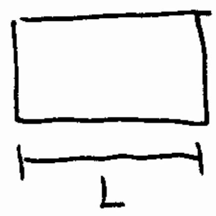
$$= c\,\frac{\partial \lambda}{\partial \omega}\,\frac{\partial}{\partial \lambda}\left(\frac{1}{n - \lambda\, dn/d\lambda}\right)$$

$$= \frac{\lambda^2}{2\pi}\,\frac{\partial}{\partial \lambda}\left(\frac{1}{n - \lambda\, dn/d\lambda}\right)$$



Group Velocity : Velocity of packet

Consider the propagation of a pulse thru a length L

$E_{in}(t) \rightarrow$  $E_{out}(t) \rightarrow$

$$E_{in}(t) = A_{in}(t)\, \exp(-i\omega_0 t)$$

$\uparrow$ envelope $\quad \uparrow$ carrier

$$\boxed{E_{out}(t) = \mathcal{F}^{-1}\left\{ \mathcal{F}\{E_{in}(t)\} \exp(i\phi(\omega)) \right\}}$$

where $\phi(\omega) = \beta(\omega) L$

$$E_{out}(t) = \frac{1}{\sqrt{2\pi}} \int E_{in}(\omega)\, e^{i\phi(\omega)} e^{-i\omega t}\, d\omega$$

where $E_{in}(\omega) = \mathcal{F}\{E_{in}(t)\}$

rewrite in reference frame moving with ω_0

$e^{-i(\omega-\omega_0)t}$

$$E_{out}(t) = \exp\left(+i(\beta_0 L - \omega_0 t)\right) \frac{1}{\sqrt{2\pi}} \int A_{in}(\omega-\omega_0) \exp\left(i(\beta(\omega) - \beta_0)\right) d(\omega-\omega_0)$$

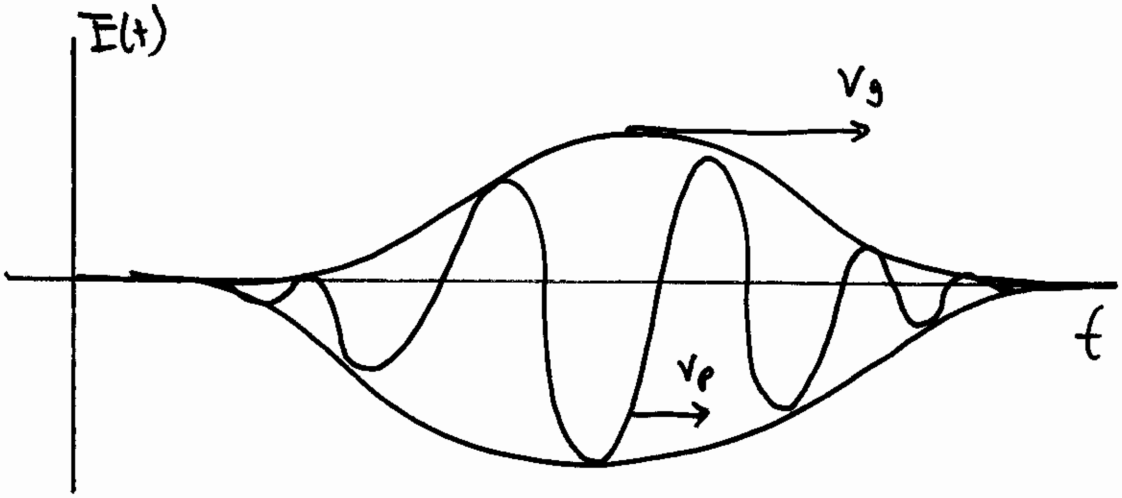
If $\beta(\omega) = \beta_0 + \beta_1(\omega - \omega_1)$

$$\boxed{E_{out}(t) = \underbrace{\exp\left(i(\beta_0 L - \omega_0 t)\right)}_{\text{phase shift with velocity}} \frac{1}{\sqrt{2\pi}} \int A_{in}(\omega-\omega_0) \exp\left(i\beta_1(\omega-\omega_0)L\right) \exp(-i(\omega-\omega_0)t)\, d(\omega-\omega_0)}$$

phase shift with velocity $v_p = \frac{\omega_0}{\beta_0} = \frac{\omega_0\, c}{n(\omega_0)\, \omega_0} = \frac{c}{n(\omega_0)}$

If $A_{out}(t) = \mathcal{F}^{-1}\{A_{out}(\omega)\} = \frac{1}{\sqrt{2\pi}} \int A_{in}(\omega-\omega_0) \exp\left(i\beta_1(\omega-\omega_0)L\right) \exp\left(-i(\omega-\omega_0)t\right) d(\omega-\omega_0)$

$$A_{out}(t) = A_{in}(t - \beta_1 L)$$



Anomalous + Normal Dispersion

Jackson ~~Anomalous~~ Normal n increases with increasing ω

Anomalous n decrease with increasing ω

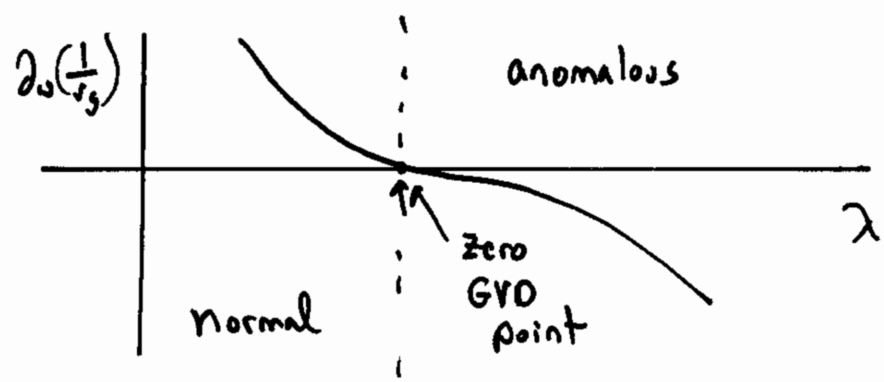
Define using group velocity

$\partial_\omega (1/v_g) > 0$ normal Group index increases with ω

$\partial_\omega (1/v_g) < 0$ anomalous Group index decreases with ω

$$c\frac{\partial}{\partial\omega}\left(\frac{1}{n-\lambda\, dn/d\lambda}\right) = c\,\frac{\partial\lambda}{\partial\omega}\frac{\partial}{\partial\lambda}\left(\frac{1}{n-\lambda\, dn/d\lambda}\right)$$

$$= \frac{\lambda^2}{2\pi}\frac{\partial}{\partial\lambda}\left(\frac{1}{n(\lambda)-\lambda\, dn/d\lambda}\right)$$



Fourier Transform

Forward transform

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{+i\omega t}\, dt$$

$$= \mathcal{F}\{ f(t) \}$$

Inverse transform

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\omega)\, e^{-i\omega t}\, d\omega$$

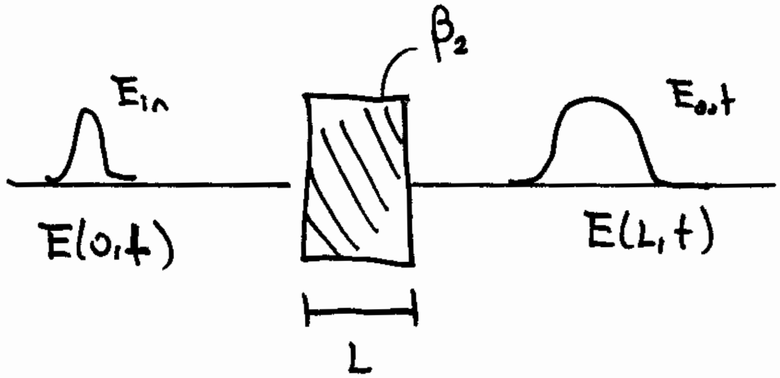
$$= \mathcal{F}^{-1}\{ f(\omega) \}$$

Pulse propagation

Dispersion induces a phase distortion on pulse $\phi(\omega)$

$$\beta(\omega) = n(\omega)\,\omega/c = \beta_0 + \beta_1(\omega-\omega_0)^2 + \tfrac{1}{2}\beta_2(\omega-\omega_0)^2 + \cdots$$

$$\phi(\omega) = \beta(\omega) L$$



$$E(0,t) = A(t)\exp(i\omega_0 t)$$

$\mathcal{F} \equiv$ Fourier Transform

$$E(L,t) = \mathcal{F}^{-1}\left\{ \mathcal{F}\left\{E(0,t)\right\} \exp(i\phi(\omega))\right\}$$

Each spectral component propagated over L, and "sees" dispersion $\beta(\omega)$

Each spectral component "sees" a delay with respect to the carrier frequency ω_0

Causes Frequency Chirp

Positive chirp : "redder" components travel faster than "bluer"

Negative chirp : "bluer" components travel faster than "redder"

$$\beta_1 = \frac{1}{v_g} = \frac{1}{c}\left(n + \omega\frac{dn}{d\omega}\right) \qquad \beta_3 =$$

$$\beta_2 = \frac{1}{c}\left(2\left(\frac{dn}{d\omega}\right) + \omega\frac{d^2n}{d\omega^2}\right) \simeq \frac{\lambda^3}{2\pi c^2}\frac{d^2n}{d\lambda^2}$$

$\beta_2 > 0$ Normal

$\beta_2 < 0$ Anamolous

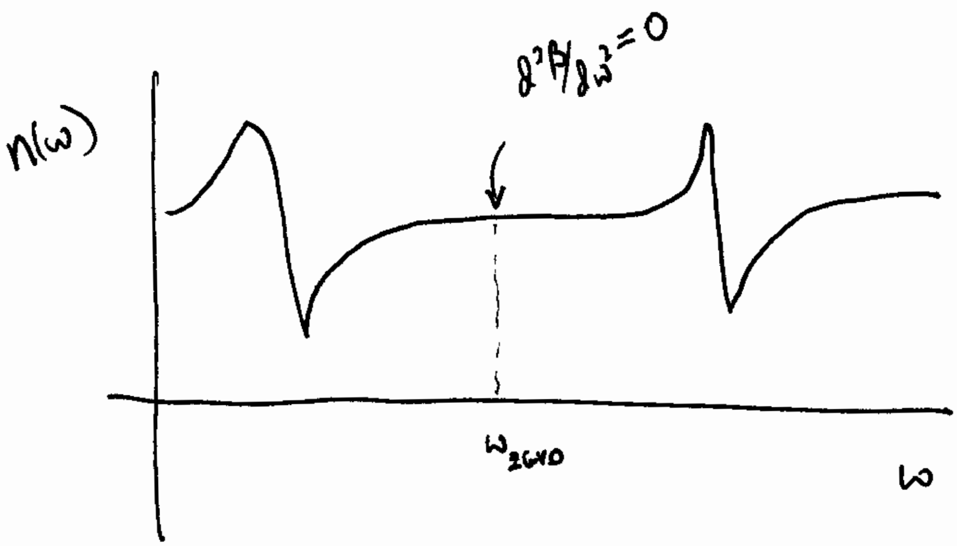
Zero GVD

$$\left.\frac{\partial^2 \beta(\omega)}{\partial \omega^2}\right|_{\omega=\omega_0} = 0$$

Second derivative of $\beta(\omega)$ is zero

$$\beta_2 = 0$$

$$\beta(\omega) = \frac{n(\omega)\,\omega}{c}$$



$$\boxed{\text{For Fused silica} \quad 1272\,\text{nm}}$$

$$\boxed{\text{Normal} \quad \beta_2 > 0 \qquad \text{Anomalous} \quad \beta_2 < 0}$$

D parameter

$$D(\lambda) = \frac{1}{c}\,\frac{\partial N(\lambda)}{\partial \lambda} = -\frac{2\pi c}{\lambda_0^2}\,\beta_2$$

D + β_2 have opposite signs!

Chirp

$$\Delta\omega = -d\varphi/dt \qquad \text{temporal chirp}$$

$$\Delta\omega = -d\varphi/d\omega \qquad \text{spectral chirp}$$

(moving frame)

Intensity & phase

$$E(t) = \sqrt{I(t)}\, \exp(-i\,\varphi(t))$$

$$\beta(\omega) = \beta_0 + \beta_1(\omega-\omega_0) + \frac{1}{2}\beta_2(\omega-\omega_0)^2 + \frac{1}{6}\beta_3(\omega-\omega_0)^3$$

$$\beta_1 = \frac{1}{v_g} = \frac{N}{c} = \frac{1}{c}\left(n + \omega\frac{dn}{d\omega}\right)$$

$$\beta_2 = \frac{1}{c}\left(2\frac{dn}{d\omega} + \omega\frac{d^2n}{d\omega^2}\right) \simeq \frac{\lambda^2}{2\pi c^2}\frac{dn^2}{d\lambda^2}$$

$$\beta_3 = \frac{1}{c}\left(2\frac{d^2n}{d\omega^2} + \frac{d^2n}{d\omega^2} + \omega\frac{d^3n}{d\omega^3}\right)$$

$$\beta_2 = \frac{1}{c}\left(2\frac{dn}{d\omega} + \omega\frac{d^2n}{d\omega^2}\right) \quad \left\{\begin{array}{l} \omega = \frac{2\pi c}{\lambda} \\ \frac{d\omega}{d\lambda} = -\frac{2\pi c}{\lambda^2} \end{array}\right.$$

$$= \frac{1}{c}\left(2\frac{dn}{d\lambda}\frac{d\lambda}{d\omega} + \frac{2\pi c}{\lambda}\frac{d}{d\omega}\left(\frac{dn}{d\lambda}\frac{d\lambda}{d\omega}\right)\right)$$

$$= \frac{1}{c}\left[2\frac{dn}{d\lambda}\left(-\frac{\lambda^2}{2\pi c}\right) + \frac{\cancel{2\pi c}}{\lambda}\frac{d}{d\omega}\left(\frac{dn}{d\lambda}\,\frac{-\lambda^2}{\cancel{2\pi c}}\right)\right.$$

$$= \frac{1}{c}\left[-\frac{\lambda^2}{\pi c}\frac{dn}{d\lambda} - \frac{1}{\lambda}\frac{d\lambda}{d\omega}\frac{d}{d\lambda}\left(\frac{dn}{d\lambda}\,\lambda\right)\right.$$

$$= \frac{1}{c}\left[\right.$$

Material Dispersion for Fused Silica

This notebook determines the wavelength index of refraction, group index, group velocity dispersion, quadratic and cubic dispersion coefficients for bulk fused silica.

■ Intial Definitions

Use c as the speed of light (in nm/fs).

```
c = 299.792458 ;
```

■ Determine Sellmeier equations and the material dispersion for bulk fused silica

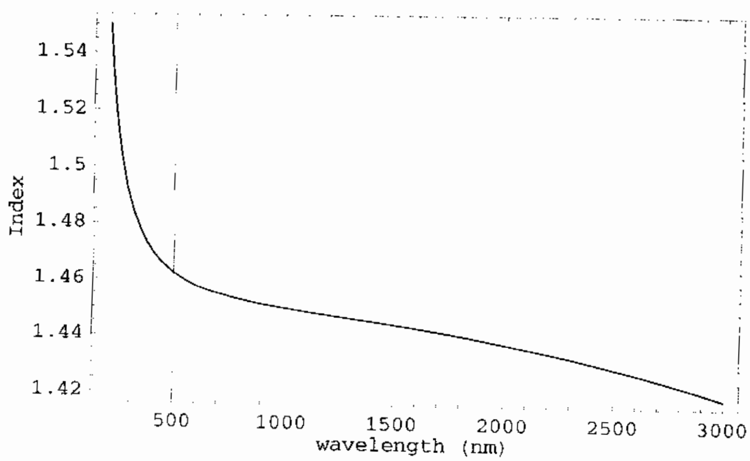
Define the Sellmeier equation and coefficients for fused silica, values taken from "Fundamentals of Optical Fibers", J.A. Buck., pg 127. The equation is good for wavelengths in nanometers.

```
B1 = 0.6961663; B2 = 0.4076426; B3 = 0.8974794;
C1 = 0.0684043; C2 = 0.1162412; C3 = 9.896161;
```

$$n_o[\lambda\_] = \sqrt{\frac{B1\,(\lambda/1000)^2}{(\lambda/1000)^2 - C1^2} + \frac{B2\,(\lambda/1000)^2}{(\lambda/1000)^2 - C2^2} + \frac{B3\,(\lambda/1000)^2}{(\lambda/1000)^2 - C3^2} + 1} \,;$$

Plot the index as a function of wavelength

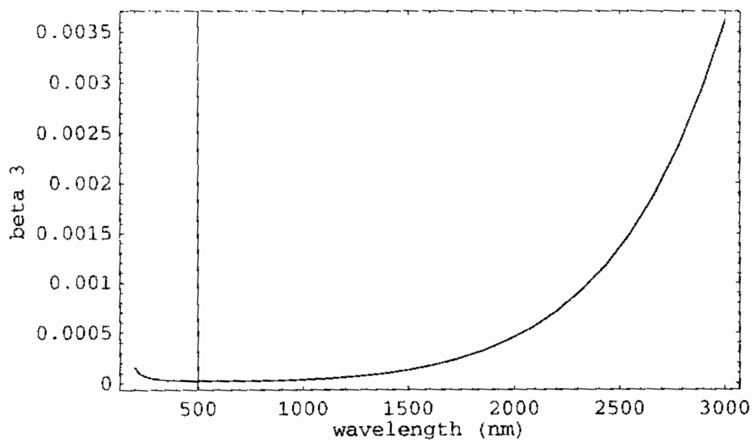
```
Plot[nₒ[λ], {λ, 200, 3000}, PlotRange -> {All, All}, Frame -> True,
    FrameLabel -> {"wavelength (nm)", "Index"}];
```



Deterimine the group index N_g using the expresion we derived in class.

$$N_g[\lambda\_] = n_o[\lambda] - \lambda\,\partial_\lambda\,(n_o[\lambda])\,;$$

```
Plot[β3[λ], {λ, 200, 3000}, PlotRange -> {All, All}, Frame -> True,
    FrameLabel -> {"wavelength (nm)", "beta 3"}];
```



■ Regions of normal and anomalous dispersion in fused silica

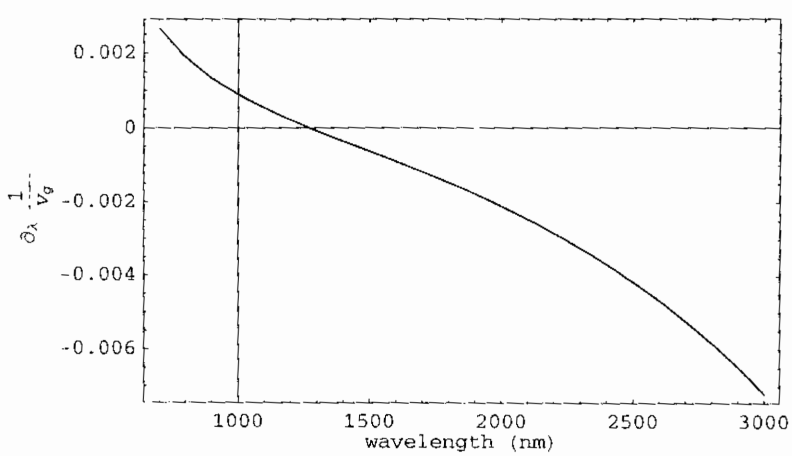
To find the region of normal and anomalous dispersion, we need to find the derivative of $1/v_g$, which is related to the group velocity dispersion.

The normal dispersion region is where $\partial_\lambda \frac{1}{v_g} > 0$

The anamolous dispersion region is where $\partial_\lambda \frac{1}{v_g} < 0$

$$\text{dvgd}\lambda[\lambda\_] = \frac{\lambda}{2\pi}\,\partial_\lambda \frac{1}{N_g[\lambda]};$$

$$\text{Plot}\left[\text{dvgd}\lambda[\lambda],\ \{\lambda, 700, 3000\},\ \text{PlotRange} \to \{\text{All}, \text{All}\},\ \text{Frame} \to \text{True},\ \text{FrameLabel} \to \left\{\text{"wavelength (nm)"},\ \text{"}\partial_\lambda \frac{1}{v_g}\text{"}\right\}\right];$$



From the graph we find that the zero group velocity dispersion wavelength is 1272 nm.

Dispersion + Pulse Broadening

Use Sellmeier Equation to describe $n(\lambda)$

$$n^2 - 1 = \sum_j \left(\frac{A_j \lambda^2}{\lambda^2 - \lambda_j^2} \right)$$

Pulse Broadening in Bulk material

Mode propagation constant $\beta(\omega)$

component of wavevector along propagation direction

Describe $\beta(\omega)$ as a Taylor Series

$$\beta(\omega) = n(\omega)\, \omega/c = \beta_0 + \beta_1 (\omega - \omega_0) + \frac{1}{2} \beta_2 (\omega - \omega_0)^2 + \cdots$$

$\omega_0 \equiv$ Carrier frequency

$$\beta_1 = 1/v_g \quad \text{group velocity}$$

$$\beta_n = \left. \frac{d^n \beta}{d\omega^n} \right|_{\omega_0}$$

$$v_g = \frac{c}{N}$$

Group Index $\boxed{N = n - \lambda\, dn/d\lambda}$

Need to determine β_n from $n(\lambda)$

$$\beta_1 = \frac{1}{c}\left(n + \omega \frac{dn}{d\omega}\right)$$

$$\beta_2 = \frac{1}{c}\left(2 \frac{dn}{d\omega} + \omega \frac{d^2n}{d\omega}\right) \simeq \frac{\lambda^3}{2\pi c^2} \frac{d^2n}{d\lambda^2}$$

$$\beta_3 \cong -\frac{\lambda^4}{2\pi^2 c^2}\left(3 \frac{d^2n}{d\lambda^2} + \lambda \frac{d^3n}{d\lambda^3}\right)$$

Sign of Group velocity dispersion $\Rightarrow$ Sign of β_2

$\beta_2 > 0$	Positive, Normal
$\beta_2 < 0$	Negative, Anomalous

Nonlinear susceptibility : anharmonic oscillator

Restoring force $\Rightarrow$ Beyond Hooke's Law

Two types of media

Noncentrosymmetric $\Rightarrow$ Lacks inversion symmetry

Centrosymmetric $\Rightarrow$ Inversion center as symmetry

Lacks Inversion symmetry $\Rightarrow$ Special properties

Inversion symmetry / center $\Rightarrow$ reflection about point brings compound back it itself

Crystal KDP $\Rightarrow$ Noncentrosymmetric (lacks inversion symmetry)
Fused silica SiO_2 $\Rightarrow$ Centrosymmetric since it is a symmetric molecule.

* Media that lack inversion symmetry have nonzero $\chi^{(2n)}$ (even orders) *

Noncentrosymmetric materials

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x + a x^2 = -eE(t)/m$$

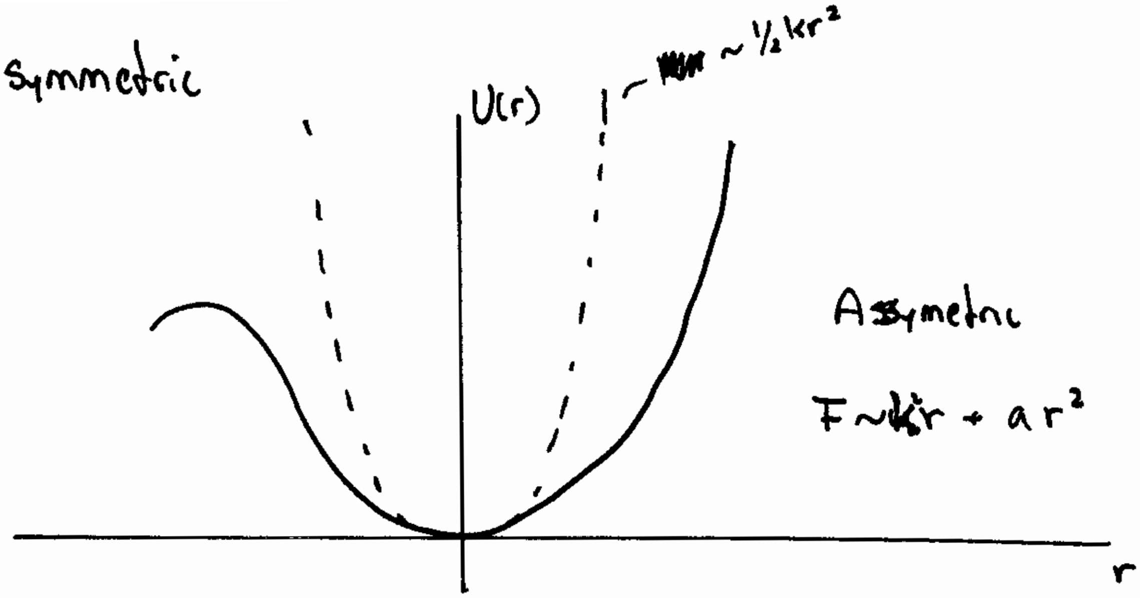
Restoring force $\vec{F} = -m\omega_0^2 x - m a x^2$

How to solve this eq $\Rightarrow$ method of successive approximations

Perturbative series

Non centrosymmetric

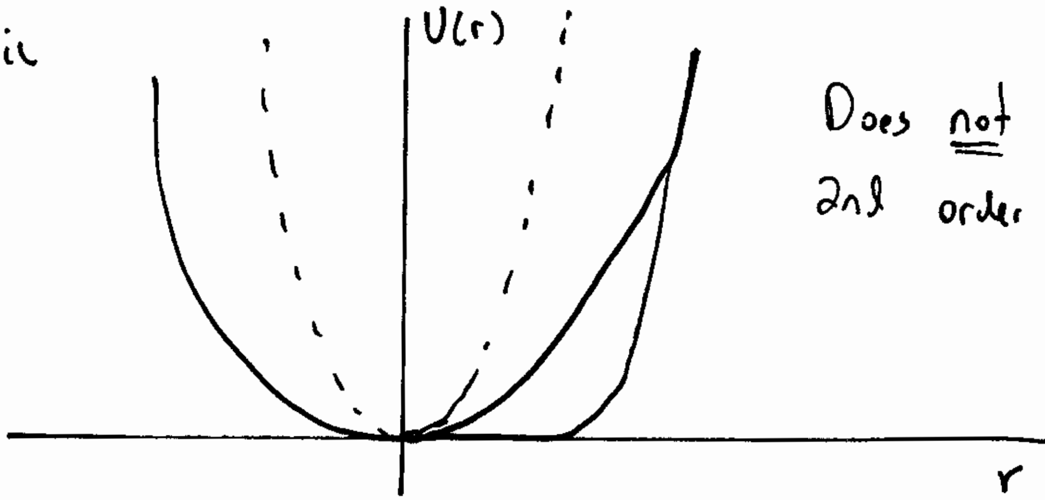
Has 2nd order terms



Assymetric

$F \sim kr + ar^2$

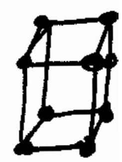
Centrosymmetric



Does <u>not</u> have 2nd order terms

Symmetric

KDP Non centrosymmetric { tetragonal crystal group



fused silica Centrosymmetric

{ Amorphous
no crystal symmetric
Symmetric

Summarize anharmonic oscillator

Hookes law   nonlinear

Noncentrosymmetric, $\boxed{F_{restoring} = -m\omega_0^2 x - m a x^2}$

lacks inversion symmetry

asymmetric potential

$\omega_0^2 = k/m$

extend Lorentz model

Polarization exhibits oscillations at

Nonlinear: $\pm 2\omega_1 \quad \pm 2\omega_2 \quad \pm(\omega_1+\omega_2), \quad \pm(\omega_1-\omega_2), \quad 0$

Linear: $\omega_1 \quad + \quad \omega_2$

so $x(t)$ exhibits oscillation at these frequencies

Examples of noncentrosymmetric material ($\chi^{(2)}$ materials)

Crystals: KDP, BBO Lithium Niobate

Centrosymmetric materials $\quad F_{restoring} = -m\omega_0^2 x + m b x^3$

- Has inversion symmetry
- symmetric potential, but <u>not</u> parabolic
- Polarization exhibits oscillations at (3rd order terms)
  $3\omega_1 \quad 3\omega_2 \quad 3\omega_3, \quad \pm(\omega_1 \pm \omega_2 \pm \omega_3) \ldots\ldots$
- Polarization <u>does</u> <u>not</u> exhibit 2nd order terms
  $2\omega_1 \quad 2\omega_2 \ldots$
- Polarization exhibits linear terms

Examples of ~~non~~centrosymmetric materials: glasses, plasmas, gases.

Solve for case without driving force

Example $$\boxed{\ddot{x} + \omega_0^2 x^2 = -\alpha x^2 - \beta x^3}$$

Solution
$$x = x_0 + \cancel{x_0} + \cancel{x_2} + \cdot$$
$$\omega = \omega_0 + \omega_1 + \omega_2 - -$$

$$x_0 = a\cos(\omega t)$$

Sub in DE $$x(t) = x_1 + x_0 = x_1 + a\cos(\omega t)$$
$$= x_1 + a\cos((\omega_0+\omega_1)t)$$

$$\dot{x}(t) = \dot{x}_1 - a\sin((\omega_0+\omega_1)t)(\omega_0+\omega_1)$$
$$\ddot{x}(t) = \ddot{x}_1 - a(\omega_0+\omega_1)^2\cos((\omega_0+\omega_1)t)$$

$$\ddot{x}_1 + \ddot{x}_0 + \omega_0^2(x_1+x_0) = -\alpha(x_1+x_0)^2 - \beta(x_0+x_1)^2$$
$$\left(\ddot{x}_1 - a\omega^2\cos(\omega t) + \omega_0^2 x_1\right) + \omega_0^2 a\cos\omega t = -\alpha(x_1+x_0)^2 - \beta(\ \ )^3$$
$$\ddot{x}_1 - a(\omega_1+\omega_0)^2\cos(\omega t) + \omega_0^2 x_1 + \omega_0^2 a\cos(\omega t) = \text{''} \quad \text{''}$$
$$\ddot{x}_1 - a\omega_1^2\overset{\to\text{small}}{\cos(\omega t)} - a\omega_0^2\cancel{\cos(\omega t)} - 2\omega_0\omega_1\cos(\omega t) + \omega_0^2 x_1$$
$$+\ \omega_0^2\cancel{a}\cancel{\cos\omega t} = \text{''} \quad \text{''}$$

$$\ddot{x}_1 + \omega_0 x_1 \simeq 2a\omega_1\omega_0\cos\omega t - \alpha a^2\cos^2(\omega t)$$

$\omega_1 \simeq 0$ no resonant term with ω

~~Then~~ $\cancel{x = x_0 + x_1 + x_2}$ ~~& Solve for~~ $\cancel{x_2}$ $\cancel{\omega_2}$

$\cancel{\omega_0 = \omega_0 + 0 + \omega_2}$

Solve for ω_1 + x_1

$$\boxed{\begin{aligned} \omega_1 &= 0 \\ x_1 &= \frac{\alpha a^2}{2\omega_0^2} + \frac{\alpha a^2}{6\omega_0^2}\cos(2\omega t) \end{aligned}}$$

Next Solution $\quad x = x_0 + x_1 + x_2$

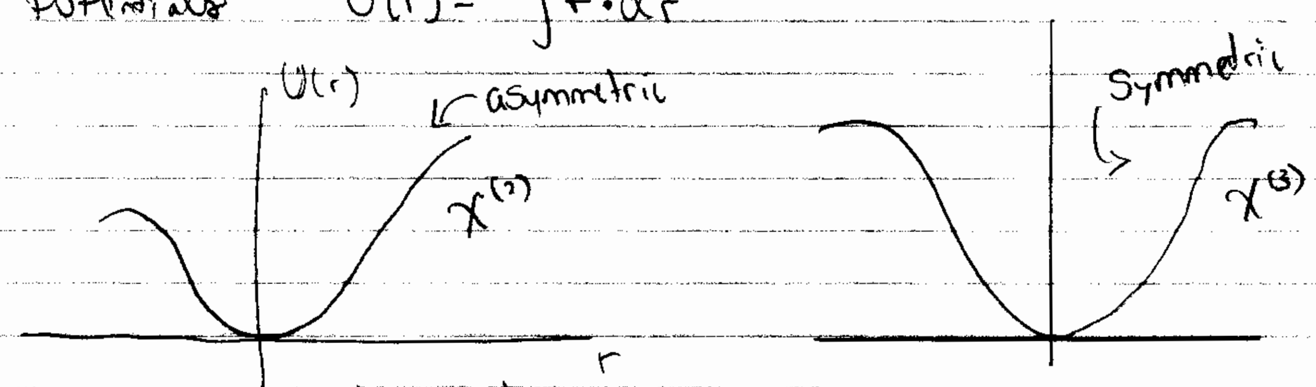
$$\omega = \omega_0 + \omega_1 + \omega_2$$

$$\boxed{\begin{aligned} \omega_2 &= \frac{3\gamma}{8\omega_0} - \frac{5\alpha^2}{12\,\omega_0^2}\,a^2 \\ x_2 &= \frac{a^3}{16\omega_0}\left(\frac{\alpha^2}{3\omega_0^2} - \frac{1}{2}\beta\right)\cos(3\omega t) \end{aligned}}$$

$\Rightarrow$ Notice if $\alpha = 0$ then no 2ω term!

$\alpha = 0 \Rightarrow$ Centrosymmetric medium

Potentials $\quad U(r) = -\int \vec{F}\cdot d\vec{r}$



Noncentrosymmetric $\qquad$ Centrosymmetric

Look at solving (with a driving force)

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x + ax^2 = -eE(t)/m$$

Form of Electric field

$$E(t) = \left(E_1 e^{-i\omega_1 t} + E_2 e^{-i\omega_2 t}\right) + \text{c.c.}$$

Use perturbative solution

Replace $E(t) \rightarrow \lambda E(t)$

Solution in power series expansion

$$x(t) = \lambda x^{(1)} + \lambda^2 x^{(2)} + \lambda^3 x^{(3)}$$

Terms of λ^n satisfy sides of equation

$$\dot{x}(t) = \lambda \dot{x}_1^{(1)} + \lambda^2 \dot{x}^{(2)} + \lambda^3 \dot{x}^{(3)}$$
$$\ddot{x}(t) = \lambda \ddot{x}_1^{(1)} + \lambda^2 \ddot{x}^{(2)} + \lambda^{(3)} \ddot{x}^{(3)}$$

Sub into DE

Lorentz model $\Rightarrow$ $\lambda\left[\ddot{x}^{(1)} + 2\gamma\dot{x}^{(1)} + \omega_0^2 x^{(1)}\right] = -eE(t)/m \; \lambda \quad (1)$

$$\lambda^2\left[\ddot{x}^{(2)} + 2\gamma\dot{x}^{(2)} + \omega_0^2 x^{(2)} + a\left(x^{(1)}\right)^2\right] = 0\lambda^2 \quad (2)$$

$$\lambda^3\left[\ddot{x}^{(3)} + 2\gamma x^{(3)} + \omega_0^2 x^{(3)} + 2a x^{(1)} x^{(2)}\right] = 0\lambda^3 \quad (3)$$

Steady state solution for $x^{(1)}$

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{-e}{m}\left(E_1 e^{-i\omega_1 t} + E_2 e^{-i\omega_2 t}\right)$$

Steady state solution $\Rightarrow$ ignore transient solution

$$\boxed{x^{(1)}(t) = \frac{-eE_1}{m D(\omega_1)} e^{-i\omega_1 t} + \frac{-eE_2}{m D(\omega_2)} e^{-i\omega_2 t} + \text{c.c.}}$$

where $D(\omega_j) = \omega_0^2 - \omega_j^2 - 2i\omega_j\gamma$ (Eq 1.4.8 wrong?)

Square $x^{(1)}$ and substitute into (2)
The square contains terms

$$\begin{cases} \pm 2\omega_1 \quad \pm 2\omega_2 \quad \pm(\omega_1+\omega_2) \quad \pm(\omega_1-\omega_2) \\ \text{and} \quad 0 \end{cases}$$

$$\left(x^{(1)}(t)\right)^2 = \frac{+e^2}{m^2}\left[\frac{E_1}{(\omega_0^2-\omega_1^2-2i\omega_1\gamma)} e^{-i\omega_2 t} + \frac{E_2}{(\omega_0^2-\omega_2^2-2i\omega_2\gamma)} e^{-i\omega_1 t} + \frac{E_1}{(\omega_0^2-\omega_1^2+2i\omega_1\gamma)} e^{+i\omega_2 t} + \frac{E_2}{(\omega_0^2-\omega_2^2+2i\omega_2\gamma)} e^{+i\omega_1 t}\right]^2$$

$$= \frac{e^2}{m^2}\left[\frac{E_1^2}{(\omega_0^2-\omega_1-2i\omega\gamma)^2} e^{-i2\omega_2 t} + \frac{E_2^2}{(\omega_0^2-\omega_2^2-2i\omega_2\gamma)^2} e^{-i2\omega_1 t}\right.$$

$$+ \frac{E_1^2}{(\omega_0^2-\omega_1^2+2i\omega_1\gamma)^2} e^{i2\omega_1 t} + \frac{E_2^2}{(\omega_0^2-\omega_2^2+2i\omega_2\gamma)^2} + \frac{E_1 E_2}{(\ )(\ )} e^{-i(\omega_1+\omega_2)}$$

$$+ \frac{E_1^2}{(\ )(\ )} e^{0} + E$$

Solve for Linear Case : Lorentz model

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{-e}{m}\mathcal{E}(t)$$

$$\mathcal{E}(t) = E_1 e^{-i\omega_1 t} + E_2 e^{-i\omega_2 t} + E_1^* e^{+i\omega_1 t} + E_2^* e^{i\omega_2 t}$$

Say • $x \sim e^{i\omega_1 t} + e^{i\omega_2 t}$

For one frequency ω_1

$$x(t)\left(\omega_1^2 - \omega_0^2 - 2i\omega_1\gamma\right) = -\,e/m\,\mathcal{E}$$

so $$x(t) = \frac{-e/m}{(\omega_0^2 - \omega_j^2 - 2i\omega_j\gamma)}\, e/m\,\mathcal{E}(t)$$

For two frequencies $\omega_1 + \omega_2$

$$x^{(1)}(t) = x^{(1)}(\omega_1)e^{-i\omega_1 t} + x^{(1)}(\omega_2)\,e^{-i\omega_2 t} + \text{c.c.}$$

where $$x^{(1)}(\omega_j) = -e/m\; E_j / D(\omega_j)$$

$$D(\omega_j) \equiv \omega_0^2 - \omega_j - 2i\omega_j\gamma$$

Look at $+2\omega_1$ term

$$\ddot{x}^{(2)} + 2\gamma\dot{x}^{(2)} + \omega_0^2 x^{(2)} = \frac{a\,eE_1^2/m^2}{D^2(\omega_1)} e^{-2i\omega_1 t}$$

Look for Solution $\boxed{x^{(2)}(t) = x^{(2)}(2\omega_1) e^{-2i\omega_1 t}}$

Find $\boxed{x^{(2)}(2\omega_1) = \frac{-a(e/m)^2 E_1^2}{D(2\omega_1) D^2(\omega_1)}}$

Now Find $\chi^{(2)}$

$$P^{(1)}(\omega_j) = \epsilon_0 \chi^{(1)}(\omega_j) E(\omega_j)$$

From Before $P^{(1)}(\omega_j) = -Ne\, x^{(1)}(\omega_j)$

$\left\{ x^{(1)} = \frac{P}{-Ne} \quad \chi = \frac{P}{\epsilon_0 E} \right.$

So $\boxed{\chi^{(1)}(\omega_j) = \frac{Ne^2/m\epsilon_0}{D(\omega_j)}}$ Solve for $\chi^{(1)}$

For $\chi^{(2)}$

$$P^{(2)}(2\omega_1) = \chi^{(2)}(2\omega_1, \omega_1, \omega_1) E^2(\omega_1)$$
$$P^{(2)}(2\omega_1) = -Ne\, x^{(2)}(2\omega_1)$$

So $\boxed{\chi^{(2)}(2\omega_1; \omega_1, \omega_1) = \frac{N(e^3/m)a}{D(2\omega_1) D^2(\omega_1)}}$

or in terms of $\chi^{(1)}$

$$\boxed{\chi^{(2)} = \frac{ma}{N^2 e^3} \chi^{(1)}(2\omega_1) \left(\chi^{(1)}(\omega_1)\right)^2}$$

Centro Symmetric materials

$$F = -m\omega_0^2 x + mbx^3$$

Symmetric potential

Find $\chi^{(2)}$ + $\chi^{(3)}$ using same procedure

$$E(t) = E_1 e^{-i\omega_1 t} + E_2 e^{-i\omega_2 t} + E_3 e^{-i\omega_3 t} + c.c.$$

Find $r^{(2)} = 0$ So $\chi^{(2)} = 0$

$$\boxed{\begin{aligned} &\ddot{r}^{(1)} + 2\gamma \dot{r}^{(1)} + \omega_0^2 r^{(1)} = -e\bar{E}(t)/m \\ &\ddot{r}^{(2)} + 2\gamma \dot{r}^{(1)} + \omega_0^2 r^{(2)} = 0 \Leftarrow \text{no driving term} \\ &\ddot{r}^{(3)} + 2\gamma r^{(3)} + \omega_0^2 r^{(3)} - b\left(\bar{r}^{(1)} \cdot \bar{r}^{(1)}\right)\bar{r}^{(1)} = 0 \end{aligned}}$$

damped eq

$$r^{(3)}(\omega_q) = -\sum_{(mnp)} \frac{be^3\left(\bar{E}(\omega_m)\cdot\bar{E}(\omega_n)\right) E(\omega_p)}{m^3\, D(\omega_q)\, D(\omega_m)\, D(\omega_n)\, D(\omega_p)}$$

Find $\chi^{(3)}$ where

$$P_i^{(3)}(\omega_q) = \sum_{jkl}\sum_{(mnp)} \chi^{(3)}_{ijkl}(\omega_q\, \omega_m\, \omega_n\, \omega_p)\, E_j(\omega_m)\, E_k(\omega_n)\, E_l(\omega_p)$$

So

$$\boxed{\chi_{ijkl}(\omega_q\, \omega_m\, \omega_n\, \omega_p) = \frac{Nb'e^4\, \delta_{jk}\, \delta_{il}}{m^3\, D(\omega_q)\, D(\omega_m)\, D(\omega_n)\, D(\omega_p)}}$$

Lecture 4
Fourier Transforms

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{+i\omega t}\, dt$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int f(\omega)\, e^{-i\omega t}\, d\omega$$

Properties

1) If $f(t)$ is real

then $f(-\omega) = f^*(\omega)$

2) Time Scaling

If $h(t) = f(t-\tau)$ then

$$h(\omega) = f(\omega)\, e^{-i\omega\tau}$$

Shift in time is phase modulation

3) Frequency offset

If $h(t) = f(t)\, e^{i\omega_0 t}$

then $h(\omega) = F(\omega - \omega_0)$

4) Convolution Th$^{\underline{m}}$

$$f(t) \circledast g(t) = \int f(t')\, g(t-t')\, dt'$$

Then $\mathcal{F}\{ f(t) \circledast g(t) \} = f(\omega)\, g(\omega)$

5) Parseval's Th$^{\underline{m}}$

$$\int f(t)\, f^*(t)\, dt = \frac{1}{2\pi} \int F(\omega)\, f^*(\omega)\, d\omega$$

Fast Fourier Transforms

Numerical FT

Need to have

2^n data pts $\quad N = 2^n$

Set time axis

· Temporal Range $\quad T$

$$\boxed{\delta t = \frac{T}{2^n}} = \frac{T}{N}$$

Then for frequency axis

$$\boxed{\delta\omega = \frac{2\pi}{2^n \delta t}} \quad \frac{2\pi}{N \delta t}$$

Can create time + Frequency arrays

For $j=1$ to 2^n

time $\quad \left(j - \frac{2^n}{2}\right)\delta t$

frequency $\quad \left(j - \frac{2^n}{2}\right)\delta\omega \quad \leftarrow$ relative frequency

Absolute frequency $\quad$ freq + ω_0

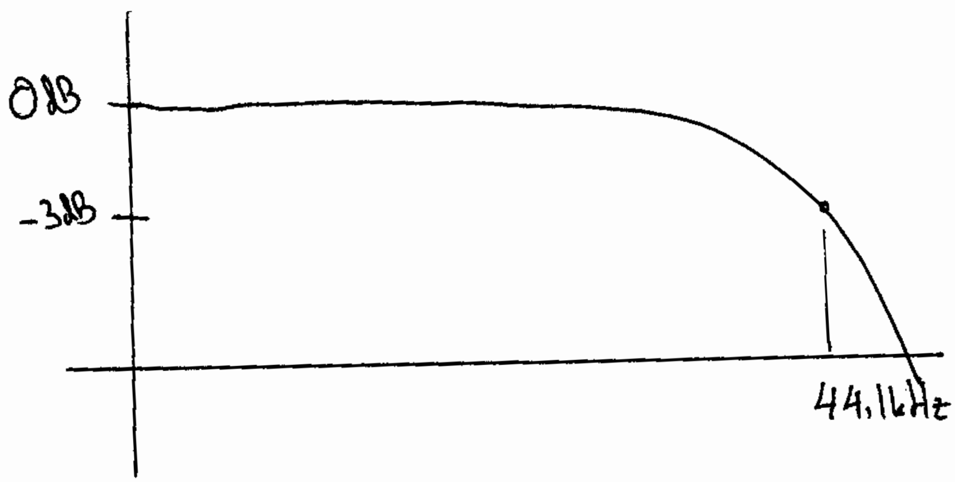
<u>Nyquist thm</u>

To prevent false spectral components

If f is the highest Frequency component you want to sample at $f_N \doteq 2f$

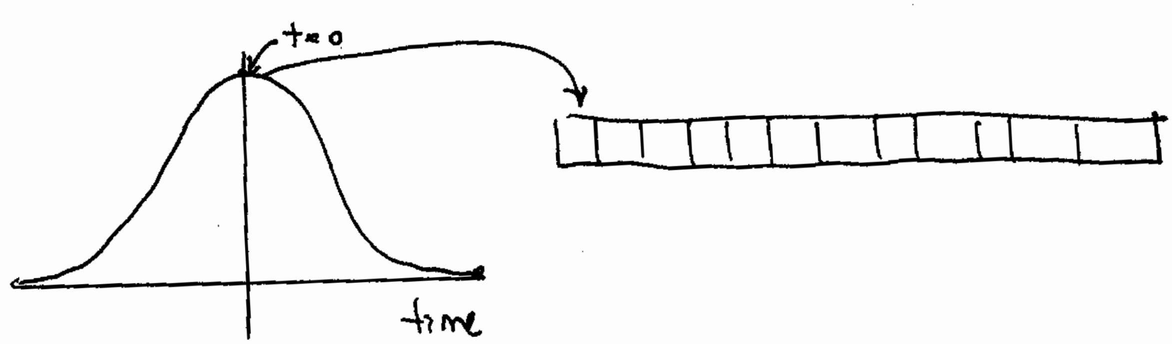
Audio (CD's)

Sample @ 44.1 kHz



Proper ~~[illegible]~~ positioning of Data

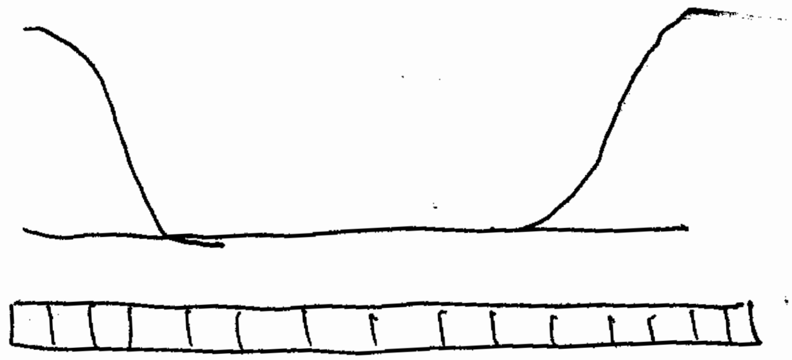
$t=0$ element must be in 1st array element



<u>To do this shift the data</u>

Data

array



<u>If you dont do this.</u>

your data $f(t)$ is now $f(t + T/2)$

$$f\left(t + \frac{N\delta t}{2}\right)$$

The FFT gives

$$f(\omega)e^{i\omega T/2} \approx f_n(\omega)(-1)^n$$

T is large so $e^{i\omega T/2}$ will osillate very fast

It will look like for the n^{th} point

$$f_n(\omega)(-1)^n$$

Formal Definition of Nonlinear Susceptibility (Boyd)

Write electric fields as superposition of freq. components

$$\vec{\mathcal{E}}(\vec r,t) = {\sum_{n=1}^{\infty}}' \vec{\mathcal{E}}_n(\vec r,t) \qquad \left(\textstyle\sum' \text{ positive frequencies}\right)$$

where $\vec{\mathcal{E}}_n = \vec{E}_n(\vec r)\, e^{-i\omega_n t} + \vec{E}_n^{\,*}(\vec r)\, e^{+i\omega_n t}$

$$\vec{E}_n(r) = \vec{A}_n e^{i \vec k_n \cdot \vec r} \qquad \vec{A}_n \equiv \text{complex}$$

Another notation (over positive & negative frequencies)

$$\vec{\mathcal{E}}(\vec r,t) = \sum_n E(\omega_n) e^{-i\omega_n t} = \sum_n A(\omega_n) e^{i(\vec k_n\cdot\vec r - \omega_n t)}$$

$$\vec{\mathcal{P}}(\vec r,t) = \sum_n \vec{P}(\omega_n) e^{-i\omega_n t}$$

For the nonlinear polarization consider a $\chi^{(2)}$ process to give $\omega_n + \omega_m$

$$\boxed{\vec{P}(\omega_n+\omega_m) = \epsilon_0 \sum_{i=1}^{3} \hat{x}_i \sum_{jk}^{3} \sum_{(nm)} \chi^{(2)}_{ijk}(\omega_n+\omega_m; \omega_n, \omega_m)\, E_j(\omega_n) E_k(\omega_m)}$$

The $\sum_{(nm)}$ implies sum over n & m such that $\omega_n + \omega_m$ is fixed

Example SFG $\omega_3 = \omega_1 + \omega_2$

$$P_i(\omega_3) = \epsilon_0 \sum_{jk} \Big[\chi^{(3)}_{ijk}(\omega_3; \omega_1, \omega_2) E_j(\omega_1) E_k(\omega_2)$$

$$+ \chi^{(2)}_{ijk}(\omega_3; \omega_2, \omega_1) E_j(\omega_2) E_k(\omega_1) \Big]$$

Using symmetries intrinsic permutation symmetry we can switch the indices

$$\chi_{ijk}(\omega_3; \omega_1, \omega_2) = \chi_{ikj}(\omega_3; \omega_2, \omega_1)$$

So
$$\boxed{P_i(\omega_3) = 2\epsilon_0 \sum_{jk} \chi^{(2)}_{ijk}(\omega_3; \omega_1, \omega_2) E_j(\omega_1) E_k(\omega_2)}$$

However, consider $P_i(\omega_2) \Rightarrow \omega_2 = \omega_3 - \omega_1$

$$P_i(\omega_2) = \epsilon_0 \sum_{jk} \Big[\chi^{(2)}_{ijk}(\omega_2; \omega_3, -\omega_1) E_j(\omega_3) E_k(-\omega_1)$$

$$+ \chi^{(2)}_{ijk}(\omega_2; -\omega_1, \omega_3) E_j(-\omega_1) E_k(+\omega_3) \Big]$$

But the fields are real so $E_k(-\omega_1) = E_k^*(\omega_1)$

And using permutation symmetry we can write

$$\boxed{P_i(\omega_2) = 2\epsilon_0 \sum_{jk} \chi^{(2)}_{ijk}(\omega_2; \omega_3, -\omega_1) E_j(\omega_3) E_k^*(\omega_1)}$$

- Notice we get a 2 if the fields are distinguishable. This 2 does not show up for SHG (see book).
- Also notice for $-\omega_n$ we get conjugate fields

$$\chi^{(2)}(\omega_n+\omega_n;\ \omega_n,\ \omega_m)$$

↑ resultant Field

Degeneracy Factor

For $\chi^{(2)}$ we can write, using a degeneracy factor D

$$P_i(\omega_n+\omega_m) = \epsilon_0 D \sum_{jk} \chi_{ijk}^{(2)}(\omega_n+\omega_m;\ \omega_n,\omega_m)\, E_j(\omega_n)\, E_k(\omega_m)$$

Where $D = \begin{cases} 1 & \text{indistinguishable fields (SHG, optical rec.)} \\ 2 & \text{distinguishable Fields (SFG, DF)} \end{cases}$

For $\chi^{(3)}$

$$P_i(\omega_o+\omega_n+\omega_m) = \epsilon_0 D \sum_{jkl} \chi_{ijkl}^{(3)}(\omega_o+\omega_n+\omega_m;\ \omega_o,\omega_n,\omega_m) * E_j(\omega_o)\, E_k(\omega_m)\, E_l(\omega_m)$$

Where $D = \begin{cases} 1 & \text{all fields indistinguishable} \\ 3 & \text{two field indistinguishable} \\ 6 & \text{all fields distinguishable} \end{cases}$

Degeneracy comes from # of terms to ~~pos~~ represent possible interaction

$3\omega_1$ — <u>one</u> term — $E(\omega_1)E(\omega_1)E(\omega_1)$

$2\omega_1-\omega_2$ — <u>three</u> terms —

$E(\omega_1)E(\omega_1)E^*(\omega_2)$

$E(\omega_1)E^*(\omega_2)E(\omega_1)$

$E^*(\omega_2)E(\omega_1)E(\omega_1)$

Lecture 5 Properties of the nonlinear susceptibility

Cover: 2nd order process
Symmetries to reduce complexity of $\chi^{(2)}$
Voit notation

Consider a 2nd order process

$$P_i(\omega_n+\omega_m;\omega_n,\omega_m) = \epsilon_0 \sum_{jk}\sum_{(nm)} \chi^{(2)}_{ijk}(\omega_n+\omega_m;\omega_n,\omega_m) E_j(\omega_n) E_k(\omega_m)$$

Mutual interaction of three waves at ω_1 ω_2 & ω_3

For different combinations we will need to know tensors

$$\chi_{ijk}(\omega_1;\omega_3,-\omega_2) \qquad \chi_{ijk}(\omega_1,-\omega_2,\omega_3)$$

~~Note?~~

$$\chi_{ijk}(\omega_2;\omega_3,-\omega_1) \qquad \chi_{ijk}(\omega_2;-\omega_1,\omega_3)$$

$$\chi_{ijk}(\omega_3;\omega_1,\omega_2) \qquad \chi_{ijk}(\omega_3;\omega_2,\omega_1)$$

Another Six (over neg. freq)

$$\chi_{ijk}(-\omega_1;-\omega_3,\omega_2) \quad \text{etc.} \ldots$$

Picture of Nonlinear process

$$\chi^{(2)}(\omega_n + \omega_m ; \omega_n , \omega_n)$$

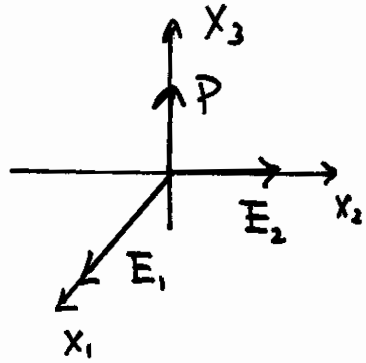
Specifically $\omega_1 , \omega_2 \Rightarrow \omega_3 = \omega_1 + \omega_2$

$\omega_1 = \omega_3 - \omega_2$

$\omega_2 = \omega_3 - \omega_1$

$$P_i(\omega_n + \omega_m) = \sum_{jk}\sum_{(nm)} \chi^{(2)}_{ijk}(\omega_n + \omega_m ; \omega_n, \omega_m) E_j(\omega_n) E_k(\omega_m)$$

One case, might look like:



$E_1 \sim e^{-i\omega_1 t}$

$E_2 \sim e^{-i\omega_2 t}$

of course there are many more possible situations

The number of tensors to describe all interactions

$3! \times 2 = 12$ tensors

$\times\ 27$ component

324 complex numbers!

$\cancel{\chi_{ijk}(\pm\omega_1, \pm\omega_2, \pm\omega_3)}$

$\cancel{\chi_{ijk}(\pm\omega_2, \pm\omega_1, \pm\omega_3)}$

$\cancel{\chi_{ijk}(\pm\omega_3, \pm\omega_2 \pm\omega_1)}$

⋮

$\chi^{(3)} \Rightarrow$ Four fields $\chi^{(3)}(\omega_n ; \omega_1\ \omega_2\ \omega_3)$

$4! * 2 = 24$

81

1944 Complex numbers

Review of Symmetries

1) Reality of Fields $\quad \bar{P} = P_i(\omega_n+\omega_m)e^{-i(\omega_n+\omega_m)t} + P_i(-\omega_n-\omega_m)\,e^{+i(\omega_n+\omega_m)t}$

$$\chi^{(2)}_{ijk}(-\omega_n-\omega_m; -\omega_n, -\omega_m) = \left(\chi^{(2)}_{ijk}(\omega_n+\omega_m; \omega_n, \omega_m)\right)^*$$

since negative + positive frequency components of P

$$P_i(-\omega_n-\omega_m) = P_i(\omega_n+\omega_m)^*$$

2) Intrinsic permutation symmetry (matter of convienence "forced" symmetry)

Require the susceptibility to be unchanged by simultaneous interchange of last two frequency arguments.

$$\chi^{(2)}_{ijk}(\omega_n+\omega_m; \omega_n, \omega_m) = \chi^{(2)}_{ikj}(\omega_n+\omega_m; \omega_m, \omega_n)$$

3) Lossless Media

Far from resonance $\quad \chi^{(2)}_{ijk}(\omega_n+\omega_m; \omega_n, \omega_m)$ is <u>REAL</u>!

4) Full Permutation Symmetry (Lossless media)

All frequency components of nonlinear susceptibility can be changed as long as corresponding Cartesian indices are changed simultaneously.

$$\chi^{(2)}_{ijk}(\omega_3=\omega_1+\omega_2) = \chi^{(2)}_{jki}(-\omega_1=\omega_2-\omega_3) \qquad \left\{\begin{array}{l} i\to\omega_3 \\ j\to\omega_1 \\ k\to\omega_2 \end{array}\right.$$

Due reality of fields

$$\cancel{\chi^{(2)}_{ijk}(\omega_3=\omega_1+\omega_2) = \mid\chi\mid}$$

$$\chi_{jki}^{(2)}(-\omega_1 = \omega_2 - \omega_3) = \chi_{jki}^{(2)}(\omega_1 = -\omega_2 + \omega_3)^*$$

which due to the reality of $\chi^{(2)}$ is equal to $\chi_{jki}^{(2)}(\omega_1 = -\omega_2 + \omega_3)$

$$\text{So} \Rightarrow \quad \chi_{ijk}^{(2)}(\omega_3 = \omega_1 + \omega_2) = \chi_{jki}^{(2)}(\omega_1 = -\omega_2 + \omega_3)$$

5) Kleinman Symmetry

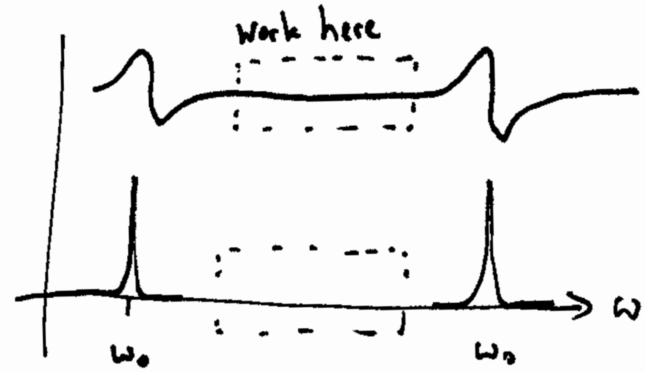
Assume: 1) frequencies are smaller than resonant

$\omega < \omega_0$

$\Rightarrow$ $\chi^{(2)}$ is frequency independent

Just like for χ'

- index monotonically increases
- absorption is small



$\Rightarrow$ Response of $\chi^{(2)}$ is nearly "instantaneous"

$$\vec{P}(t) = \epsilon_0 \chi^{(2)} \mathcal{E}^2(t)$$

instead of

$$\vec{P}(t) = \epsilon_0 \iint \chi^{(2)}(t-t', t-t'') \mathcal{E}(t-t') \mathcal{E}(t-t'')\, dt'\, dt''$$

More on Kleinman Symmetry

Material is lossless $\Rightarrow$ Full permutation symmetry

Implies indices can be permuted as long as frequencies are permuted.

Example
$$\chi^{(2)}_{ijk}(\omega_3 = \omega_1 + \omega_2) = \chi^{(2)}_{jki}(\omega_1 = -\omega_2 + \omega_3)$$
$$= \chi^{(2)}_{kij}(\omega_2 = \omega_3 - \omega_1) \qquad \left\{ \begin{array}{l} i \to \omega_3 \\ j \to \omega_1 \\ k \to \omega_2 \end{array} \right.$$

Assume $\chi^{(2)}$ does not depend on frequency

$\Rightarrow$ permute indices <u>without</u> permuting frequencies

So
$$\chi^{(2)}_{ijk}(\omega_3 = \omega_1 + \omega_2) = \chi_{jki}(\omega_3 = \omega_1 + \omega_2)$$
$$= \chi_{kij}(\omega_3 = \omega_1 + \omega_2)$$
$$\vdots$$

Ignore nonlinear dispersion

Contracted Notation : Voit notation

$$\boxed{d_{ijk} \equiv \frac{1}{2} \chi^{(2)}_{ijk}}$$

useful for Kleinman symmetry ⇒ • Symmetric in last two indices
• Symmetric where $\omega_n = \omega_m$ (SHG)

jk :	11	22	33	23, 32	31, 13	12, 21
l :	1	2	3	4	5	6

Susceptibility is a 3x6 matrix

$$d_{il}$$

Kleinman Symmetry any indice d_{ijk} can be freely permuted

$$d_{12} = d_{122} = d_{212} = d_{26}$$

$$d_{14} = d_{123} = d_{213} = d_{25}$$

Relate back to nonlinear polarization

$$\begin{pmatrix} P_x(2\omega) \\ P_y(2\omega) \\ P_z(2\omega) \end{pmatrix} = 2 \begin{bmatrix} & d_{il} & \end{bmatrix} \begin{bmatrix} E_x^2(\omega) \\ E_y^2(\omega) \\ E_z^2(\omega) \\ 2E_y E_z \\ 2E_x E_z \\ 2E_x E_y \end{bmatrix}$$

d_{eff}

Fixed geometry → polarization
→ propagation direction

SHG

$$P(2\omega) = 2 d_{eff} E^2(\omega)$$

Number of Independent Elements for $\chi^{(2)}_{ijk}(\omega_3, \omega_2, \omega_1)$

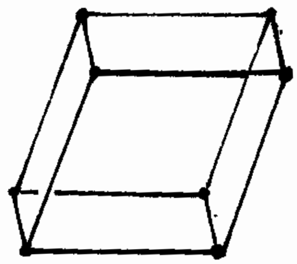
324 complex quantities

- Reality of field $\Rightarrow$ ½ are independent
- Intrinsic Permutation Symmetry $\Rightarrow$ 81 Independent
- Lossless medium $\Rightarrow$ All real
- Full permutation $\Rightarrow$ 27 Independent
- SHG + voit notation $\Rightarrow$ 18 Independent
- Kleinman Symmetry $\Rightarrow$ 10 Indepent
- Crystalline Symmetry $\Rightarrow$ Further reduction

Lecture 6 Crystal Structure & Nonlinear Optics

- Crystal Symmetry further reduces the number of independent elements of $\chi^{(2)}_{ijk}$

32 crystal classes
7 Crystal systems



Quartz

trigonal 32 (D_3)
Positive Uniaxial

Linear Properties

$$\chi^{(1)} \Rightarrow \begin{pmatrix} X & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & Z \end{pmatrix}$$

Nonlinear properties

$xxx = -xyy = -yyx = -yxy$

$xyz = -yxz$

$xzy = -yzx$

$zxy = -zyx$

Everything else zero

& Kleiman Symmetry

$xxx \Rightarrow 11 \qquad xyy \Rightarrow 12 \qquad yyx \Rightarrow 26$

$xyz \Rightarrow 14 \qquad yxz \Rightarrow 25$

$xzy \Rightarrow 14 \qquad -yzx \Rightarrow 25$

$zxy \Rightarrow 36 \qquad 3$

So
$d_{11} = -d_{12} = -d_{26}$
$d_{14} = -d_{25}$

Class 32

Pictorial

Lithium Niobate (Elvis)

Class 3m (Trigonal)

Uniaxial crystal

KDP

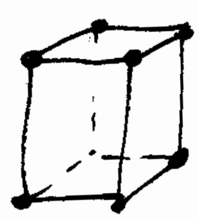
Class $\bar{4}2m$ (Tetragonal)

Uniaxial Crystal

Potassium Niobate

Class mm2 (orthorhombic)

Biaxial Crystal

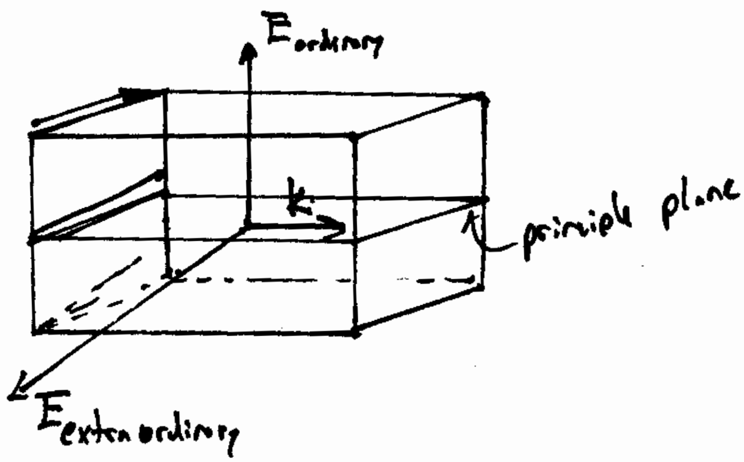


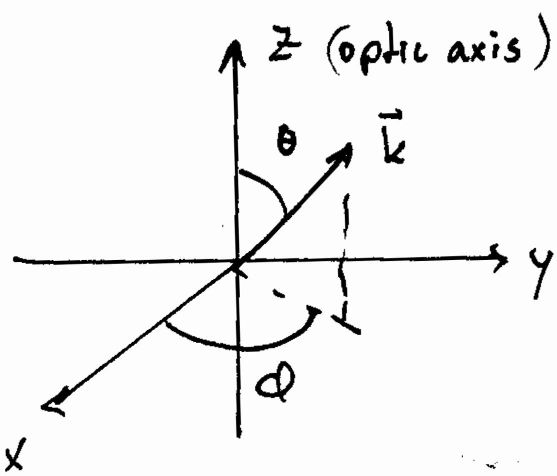
Catagories of birefringent crystals

__Uniaxial__ one optic axis $\Rightarrow$ the orientation where there is __no__ birefringence

plane containing k + optic axis $\Rightarrow$ principle plane

Ordinary beam $\Rightarrow$ polarization $\perp$ to principle plane
extraordinary beam $\Rightarrow$ polarization $\parallel$ to principle plane



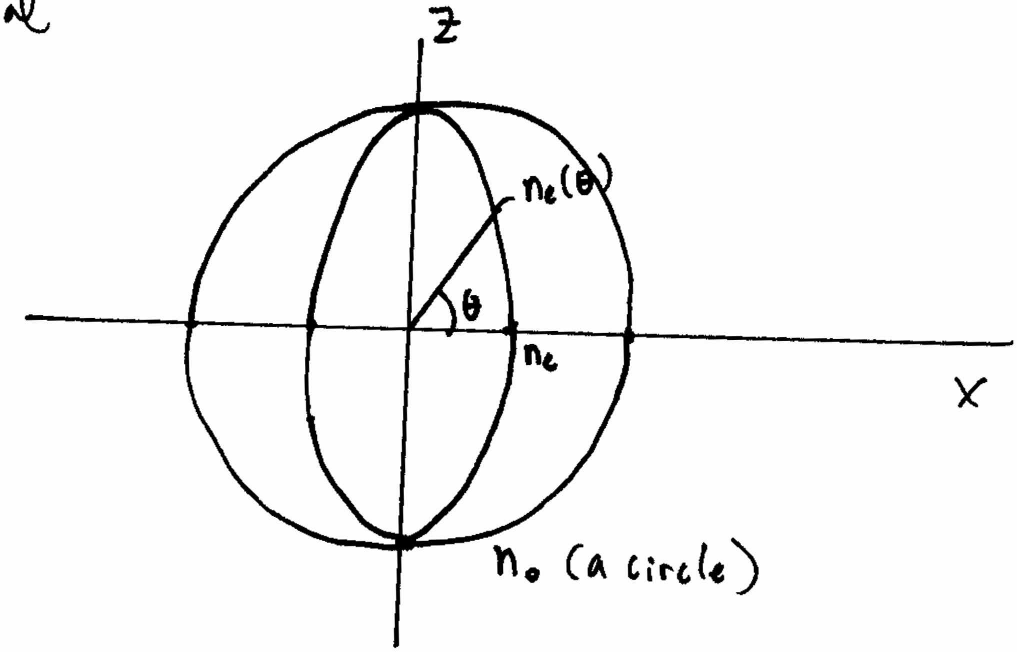


Draw index ellipsoids

$n_o < n_e$ positive uniaxial
$n_o > n_e$ negative uniaxial

$n_o > n_e$

negative uniaxial



n_o does not depend on θ

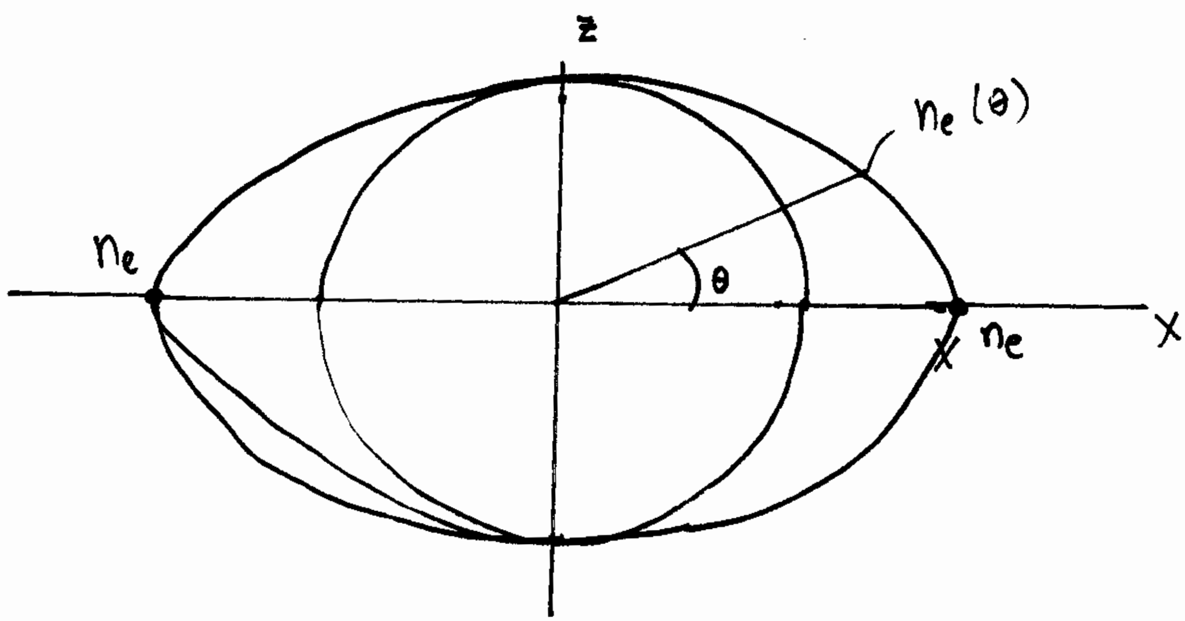
$$\boxed{\frac{1}{n_e^2(\theta)} = \frac{\sin^2\theta}{n_e^2} + \frac{\cos^2\theta}{n_o^2}}$$

OR

$$\boxed{n_e(\theta) = n_o \left[\frac{1 + \tan^2\theta}{1 + (n_o/n_e)^2 \tan^2\theta} \right]^{1/2}}$$

$n_o < n_e$

positive uniaxial



Extraordinary + Ordinary indices

$$\frac{1}{n_e^2(\theta)} = \frac{\sin^2\theta}{n_e^2} + \frac{\cos^2\theta}{n_o^2}$$

$$n_o(\theta) = n_o$$

$$n_e(0) = n_o$$

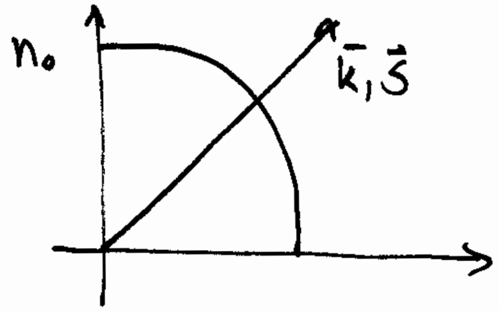
$$n_e(90°) = n_e$$

$$\Delta n\ (0) = 0$$

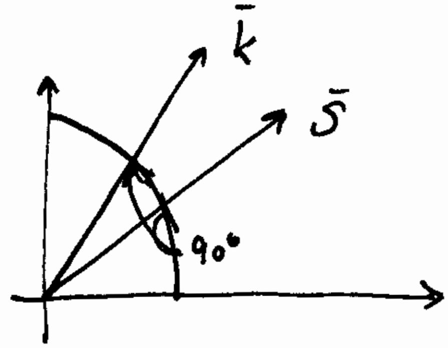
$$\Delta n\ (90) = n_e - n_o$$

Walk off in ~~Birefringent~~ Uniaxial Crystals

The direction of $\vec{k}$ does not correspond to $\vec{S}$ in the crystal



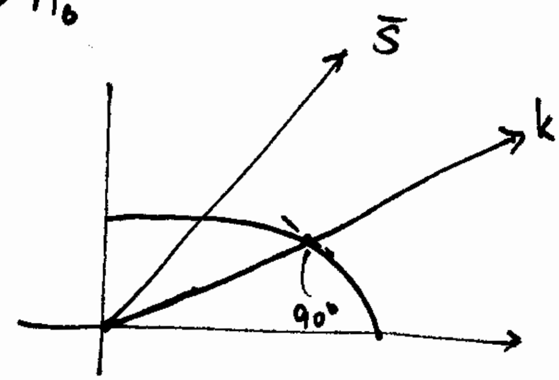
$n_o > n_e$



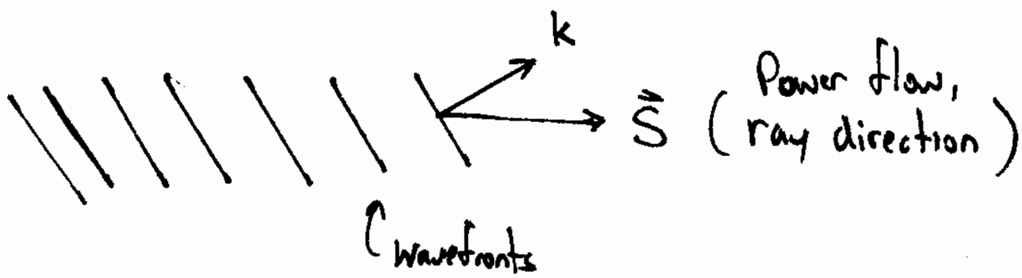
$n_e > n_o$

$\vec{k}$ is $\perp$ to the surface of $n_e(\theta)$

$\bar{S}$ does not need to be $\perp$



What does a wave look like where $\vec{S} \nparallel \vec{k}$!?

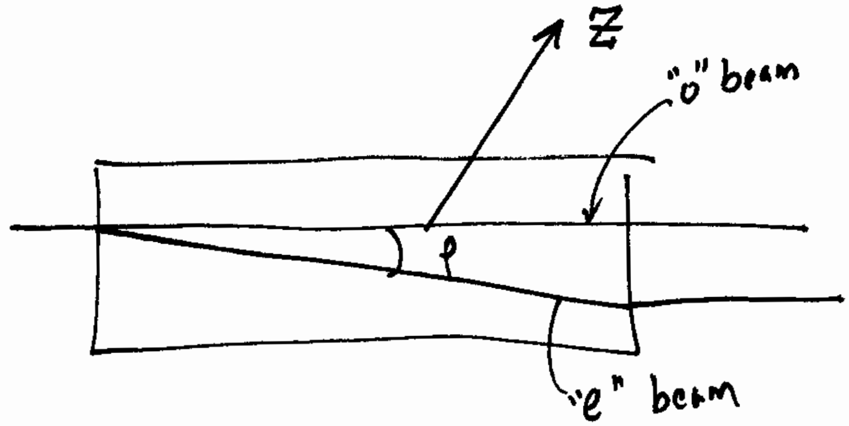


In the crystal the ordinary + extra ordinary beams separate or walk off

* k is $\perp$ to wavefronts *

Walk off angle $\boxed{\rho = \pm \tan^{-1}\left(\left(\frac{n_o^2}{n_e^2}\right) \tan\theta \right) \mp \theta}$ $\left(\begin{matrix} - & \text{neg} \\ + & \text{pos} \end{matrix}\right)$

Walk off

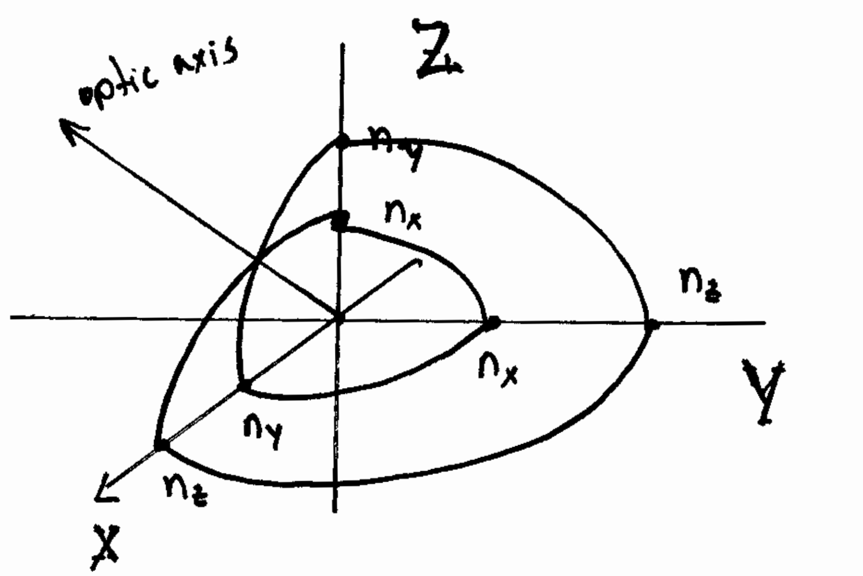


~~Birefr~~ Biaxial Crystals

three separate indices n_x n_y n_z

index surfaces

Principle planes XY YZ XZ



~~There will be two~~

Treat in plane ⇒ polarizion as circle + ellipse

General Nonlinear Wave equation

From before we derived

$$\vec{\nabla}^2 \vec{E} - \frac{1}{c^2}\partial_t^2 \vec{E} = \mu_0 \partial_t^2 \vec{P} \qquad \mu_0\epsilon_0 = \frac{1}{c^2}$$

(2.1.17 in Boyd)

We wish to derive a nonlinear wave equation. Write the equation using our real notation

$$\boxed{\vec{\nabla}^2 \vec{\mathcal{E}} - \frac{1}{c^2}\partial_t^2 \vec{\mathcal{E}} = \mu_0 \partial_t^2 \vec{\mathcal{P}}^{NL}}$$

Break up $\vec{\mathcal{P}}$

$$\vec{\mathcal{P}} = \underbrace{\vec{\mathcal{P}}^{(1)}}_{\text{linear}} + \underbrace{\vec{\mathcal{P}}^{NL}}_{\text{nonlinear}}$$

$$\nabla^2 \vec{\mathcal{E}} - \frac{1}{c^2}\partial_t^2 \vec{\mathcal{E}} = \mu_0 \partial_t^2 \vec{\mathcal{P}}^{(1)} + \mu_0 \partial_t^2 \vec{\mathcal{P}}^{NL}$$

$$\vec{\nabla}^2 \vec{\mathcal{E}} - \frac{1}{c^2}\partial_t^2 \vec{\mathcal{E}} = \epsilon_0\mu_0[\chi^{(1)}]\,\partial_t \vec{\mathcal{E}} + \mu_0 \partial_t^2 \vec{\mathcal{P}}^{NL}$$

using $\vec{\mathcal{P}} = \epsilon_0[\chi^{(1)}]\vec{\mathcal{E}}$

~~But~~ Define $[\epsilon] = 1 + [\chi^{(1)}]$

$$\boxed{\vec{\nabla}^2 \vec{\mathcal{E}} - \frac{[\epsilon]}{c^2}\partial_t^2 \vec{\mathcal{E}} = \mu_0 \partial_t^2 \vec{\mathcal{P}}^{NL}} \qquad \text{2.1.17 in Boyd}$$

We wish to write this eq. for the n^{th} frequency component of the respective fields. Write a Fourier superposition of each field.

$$\vec{\mathcal{E}}(\vec{r},t) = \sum_{n=1}^{\infty} \vec{\mathcal{E}}_n(\vec{r},t)$$

$$\vec{\mathcal{P}}^{NL}(\vec{r},t) = \sum_{n=1}^{\infty} \vec{\mathcal{P}}^{NL}_n(\vec{r},t)$$

where

$$\vec{\mathcal{E}}_n(\vec{r},t) = E_n(\vec{r})\, e^{-i\omega_n t} + c.c.$$

$$\vec{\mathcal{P}}^{NL}_n(\vec{r},t) = P^{NL}_n(\vec{r})\, e^{-i\omega_n t} + c.c.$$

For each freq. we can write

$$\boxed{\cancel{\nabla^2 \vec{E}_n - \frac{[\epsilon(\omega_n)]}{c^2}\partial_t^2 \vec{E}_n = \mu_0 \partial_t^2 \vec{P}^{NL}_n}}$$

$$\boxed{\nabla^2 \vec{\mathcal{E}}_n - \frac{[\epsilon(\omega_n)]}{c^2}\partial_t^2 \vec{\mathcal{E}}_n = \mu_0 \partial_t^2 \vec{\mathcal{P}}^{NL}_n} \quad \begin{pmatrix}\text{Boyd}\\ 2.1.21\end{pmatrix}$$

using the time dependence of $\vec{\mathcal{E}}_n \Rightarrow$

$$\partial_t^2 \vec{\mathcal{E}}_n = -E_n \omega^2 e^{-i\omega t} + E_n^* \omega^2 e^{+i\omega_n t}$$

$$= -\omega^2 \vec{\mathcal{E}}_n$$

and equating common $e^{i\omega t}$ terms

$$\left(\nabla^2 E_n(r) + \frac{\omega_n^2}{c^2}[\epsilon(\omega_n)]\, \vec{E}_n(r) + \mu_0 \omega_n^2 P_n^{NL}(r)\right) e^{-i\omega t}$$

$$= \left(\nabla^2 E_n^*(r) + \frac{\omega_n^2}{c^2}[\epsilon(\omega_n)]\, E_n^*(r) + \mu_0 \omega_n^2 P_n^{*NL}\right) e^{i\omega t}$$

OR

$$\boxed{\nabla^2 \vec{E}_n(\vec{r}) + \frac{\omega_n^2}{c^2}[\epsilon(\omega_n)]\, \vec{E}_n(\vec{r}) = -\mu_0 \omega_n^2 \vec{P}^{NL}_n} \quad \begin{matrix}\text{Boyd}\\ 2.1.23\end{matrix}$$

General Nonlinear wave equation (from Shen Chpt. 3) (B1)

(Boyd Section 2.1)

From before we derived

(Wrong notation?)

$$\nabla^2 \vec{E} - \frac{1}{c^2}\partial_t^2 \vec{E} = \mu_0 \partial_t^2 \vec{P}$$

OR

$$\boxed{\left[\vec{\nabla}\times(\vec{\nabla}\times) + \mu_0\epsilon_0 \partial_t^2\right]\vec{E} = \mu_0 \partial_t^2 \vec{P}} \quad \Big\} \mu_0\epsilon_0 = \frac{1}{c^2}$$

We wish to rewrite this for nonlinear propagation. Break the electric field into Fourier components

$$\vec{E}(\vec{r},t) = \sum_{l=1}^{\infty} \vec{E}_l(\vec{k}_l . \omega_l) \qquad \text{infinite series}$$

$$= \sum_l A_l \exp(i(\vec{k}_l\cdot\vec{r} - \omega_l t))$$

and the polarization

$$\vec{P}(\vec{r},t) = \vec{P}^{(1)}(\vec{r},t) + \vec{P}^{NL}(\vec{r},t)$$

where $\vec{P}^{(1)} = \sum_l [\chi^{(1)}(\omega_l)] \vec{E}_l(\vec{k}_l,\omega_l)$ ↑ matrix mult.

$$\vec{P}^{NL} = \sum_{m=1}^{\infty} \vec{P}^{NL}(\vec{k}_m, \omega_m)$$

- Define $[\epsilon] = 1 + [\chi^{(1)}]$

- Now $\partial_t^2 \vec{E}(t) = \partial_t^2 \sum_{l=1}^{\infty} E_l(\vec{k}_l, \omega_l) = -\omega^2 \vec{E}(\vec{r},t)$

(same for P)

so rewrite wave eq in Freq. domain

$$\left(\vec{\nabla}\times(\vec{\nabla}\times \cancel{E} - \frac{1}{c^2}\omega^2\right) E(\vec{k},\omega) = \mu_0 \omega^2 \vec{P}(k,\omega)$$

$$\left[\vec{\nabla}\times(\nabla\times) - \frac{\omega^2}{c^2}\right]\vec{E}(\bar{k},\omega) = \mu_0\omega^2\left(P^{(1)} + P^{NL}\right)$$

$$\vec{P}^{(1)} = \epsilon_0[\chi^{(1)}]\vec{E}$$

$$\left[\vec{\nabla}\times(\vec{\nabla}\times) - \frac{\omega^2}{c^2}\right]\vec{E} = \mu_0\omega^2\epsilon_0[\chi^{(1)}]E + \omega^2\vec{P}^{NL}\mu_0$$

$$\left[\vec{\nabla}\times(\vec{\nabla}\times) - \frac{\omega^2}{c^2}\left(1 + [\chi^{(1)}]\right)\right]\vec{E} = \mu_0\omega_0^2\vec{P}^{NL}$$

$$\boxed{\left[\vec{\nabla}\times(\vec{\nabla}\times) - \frac{\omega_l^2}{c^2}[\epsilon]\right]\vec{E}(\bar{k}_l,\omega_l) = \omega_m^2\mu_0\vec{P}^{NL}(\bar{k}_m,\omega_m)}$$

OR

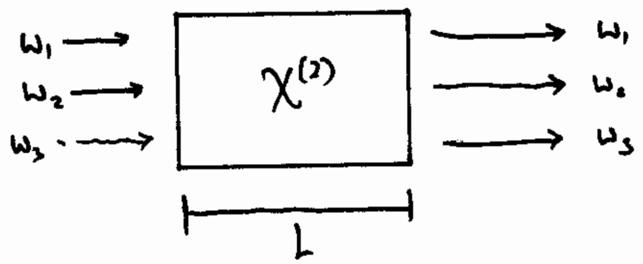
$$\boxed{\vec{\nabla}^2\vec{E}_l + \frac{\omega^2}{c^2}[\epsilon]\vec{E}_l = \omega_m^2\mu_0\vec{P}^{NL}_m}$$ (Boyd Eq. 2.1.23)

In time domain

$$\boxed{\vec{\nabla}^2\vec{E}_n - \frac{[\epsilon]}{c^2}\partial_t^2\vec{E}_n = \mu_0\partial_t^2\vec{P}_n^{NL}}$$ (Boyd Eq. 2.1.21)

Coupled Wave Eqs for Sum Frequency Generation

Set up



$\omega_3 = \omega_1 + \omega_2$

$\omega_s = \omega_{p1} + \omega_{p2}$

The wave eq. must hold for each field at ω_1, ω_2 + ω_3.

$$\vec{\nabla}\,\vec{\mathcal{E}}_n - \frac{[\epsilon_o(\omega_n)]}{c^2}\,\partial_t^2\vec{\mathcal{E}}_n = \mu_0\,\partial_t^2\vec{\mathcal{P}}_n^{\omega_n} \qquad n = 1, 2, 3$$

This will give us three coupled wave eqs. The nonlinear susceptibilities will serve as coupling coefficients.

In detail let $\omega_3 = \omega_1 + \omega_2$, $\bar{k}_3 = \bar{k}_1 + \bar{k}_2$, the three eqs. will be (using 2.1.23)

(1) $$\nabla^2\vec{E}_1(\bar{k}_1,\omega_1) + \frac{\omega_1^2}{c^2}[\epsilon(\omega_1)]\,\vec{E}_1(\bar{k}_1,\omega_1) = -\mu_0\,\epsilon_0\,\chi^{(2)}(\omega_1 = -\omega_2+\omega_3; -\omega_2, \omega_3) : \vec{E}_2^*(\omega_2)\,\vec{E}_3(\omega_3)$$

$$= -\epsilon_0\mu\,2\sum_{ijk}^{3}\hat{x}_i\,\chi_{ijk}^{(2)}(\omega_1 = -\omega_2+\omega_3; -\omega_2, \omega_3)\,E_{2j}^*(\omega_2)\,E_{3k}(\omega_3)$$

(2) $$\nabla^2\vec{E}_2(\omega_2) + \frac{\omega_2^2}{c^2}[\epsilon(\omega_2)]\,\vec{E}_2(\omega_2) = -\mu_0\,\epsilon_0\,\chi^{(2)}(\omega_2 = \omega_3-\omega_1; \omega_3, -\omega_1) : E_3(\omega_3)\,\vec{E}_1^*(\omega_1)$$

$$= -\mu_0\,\epsilon_0\,2\sum_{ijk}\hat{x}_i\,\chi_{ijk}^{(2)}(\omega_2 = \omega_3-\omega_1; \omega_3, -\omega_1)\,E_{3j}(\omega_3)\,\vec{E}_{1k}^*(\omega_1)$$

(3) $$\nabla^2\vec{E}_3(\omega_3) + \frac{\omega_3^2}{c^2}[\epsilon(\omega_3)]\,\vec{E}_3(\omega_3) = -\mu_0\,\epsilon_0\,\chi^{(2)}(\omega_3 = \omega_1+\omega_2; \omega_1, \omega_2) : E_1(\omega_1)\,E_2(\omega_2)$$

$$= -\mu_0\,\epsilon_0\,2\sum_{ijk}\hat{x}_i\,\chi_{ijk}^{(2)}(\omega_3 = \omega_1+\omega_2; \omega_1, \omega_2)\,E_{1j}(\omega_1)\,E_{2k}(\omega_2)$$

Where do the complex conjugates (+ factor of two) come from?

From before (1.3.12)

$$P_i(\omega_n+\omega_m) = \epsilon_0 \sum_{jk} \sum_{(nm)} \chi^{(2)}_{ijk}(\omega_n+\omega_m; \omega_n, \omega_m)\, E_j(\omega_n)\, E_k(\omega_m)$$

Sum over cartesian coordinates

Sum over $n+m$ such that $\omega_n+\omega_m$ is fixed

We have $\omega_3 = \omega_1+\omega_2$ or $\omega_1 = \omega_3-\omega_2$ or $\omega_2 = \omega_3-\omega_1$

For ω_2

$$\vec{P}(\omega_2) = \epsilon_0 \sum_i \hat{x}_i \sum_{jk} \Big[\chi^{(2)}_{ijk}(\omega_2=\omega_3-\omega_1; \omega_3, -\omega_1)\, E_j(\omega_3)\, E_k(-\omega_1)$$

$$+ \chi^{(2)}_{ijk}(\omega_2=\omega_3-\omega_1; -\omega_1, \omega_3)\, E_j(-\omega_1)\, E_k(\omega_3) \Big]$$

Can simplify this 1) Fields are real: so $E_j(-\omega_n) = E_j^*(\omega_n)$

2) Intrinsic permutation symmetry: $\chi^{(2)}_{ijk}(\omega_2; \omega_3, -\omega_1)$

$= \chi_{ikj}(\omega_2; -\omega_1, \omega_3)$

So

$$\vec{P}(\omega_2) = 2\epsilon_0 \sum_i \hat{x}_i \sum_{jk} \chi^{(2)}_{ijk}(\omega_2=\omega_3-\omega_1; \omega_3, -\omega_1)\, E_j(\omega_3)\, E_k^*(\omega_1)$$

Similary

$$\vec{P}(\omega_1) = 2\epsilon_0 \sum_i \hat{x}_i \sum_{jk} \chi^{(2)}_{ijk}(\omega_1=\omega_3-\omega_2; \omega_3, -\omega_2)\, E_j(\omega_3)\, E_k^*(\omega_2)$$

These vector equations are very ugly, so I will try to write the eq's in a ~~samplifier~~ simplier form. Following Boyd

Our coupled Eq's are

$$\nabla^2 \vec{\mathcal{E}}_n - \frac{[\epsilon_0(\omega_n)]}{c^2} \partial_t^2 \vec{\mathcal{E}}_n = \mu_0 \partial_t^2 \vec{\mathcal{P}}_n^{NL}$$

Start with getting an eq for ~~III~~ $\vec{\mathcal{E}}_3$

Let $\vec{\mathcal{E}}_3 = A_3 e^{i(k_3 z - \omega_3 t)} + c.c.$ $\left\{ \begin{array}{l} k_3 = \frac{n_3 \omega_3}{c} \\ n_3^2 = \epsilon^{(1)}(\omega_3) \end{array} \right.$

write $\vec{\mathcal{P}}_3' = P_3 e^{-i\omega_3 t} + c.c.$

where $P_3 = 4\epsilon_0 d_{eff} E_1 E_2$ $\Big\{$ Def of d_{eff}

$$\vec{\mathcal{E}}_i(z,t) = E_i e^{-i\omega_i t} + c.c. \qquad E_i = A_i e^{ik_i z}$$

So $P_3 = 4\epsilon_0 d_{eff} A_1 A_2 e^{i(k_1+k_2)z}$ $\Big\{$

Put all this into the wave eq. for 1D $\Big\{$ (No ϵ_0 because $\epsilon_0\mu_0 = \frac{1}{c^2}$)

$$\left[\frac{d^2 A_3}{dz^2} + 2ik_3 \frac{dA_3}{dz} - k_3^2 A_3 \overset{\text{cancel}}{+} \frac{\epsilon(\omega_3)\omega_3^2 A_3}{c^2}\right] \exp(ik_3 z - i\omega_3 t) + c.c.$$

$$= -\frac{4 d_{eff} \omega_3^2}{c^2} A_1 A_2 \exp\left(i((k_1+k_2)z - \omega_3 t)\right) + c.c.$$

Can cancel 3rd + 4th terms since $k_3^2 = \frac{\omega_3^2 \epsilon(\omega_3)}{c^2}$

In detail with c.c. terms, but subtract lhs 3rd + 4th terms

$$\left(\frac{d^2A_3}{dz^2} + 2ik_3\frac{dA_3}{dz}\right)\exp\left(i(k_3z-\omega_3t)\right)$$

$$+\left(\frac{d^2A_3^*}{dz^2} - 2ik_3\frac{dA_3^*}{dz}\right)\exp\left(-i(k_3z-\omega_3t)\right)$$

$$= -\frac{4d_{eff}\,\omega_3^2}{c^2}A_1A_2\exp\left(i((k_1+k_2)z-\omega_3t)\right)$$

$$-\frac{4d_{eff}}{c^2}\omega_3^2A_1^*A_2^*\exp\left(-i((k_1+k_2)z-\omega_3t)\right)$$

group terms in $\exp(+i\omega_3t)$ and $\exp(-i\omega_3t)$

$$\left[\left(\frac{d^2A_3}{dz^2} + 2ik_3\frac{dA_3}{dz}\right)\exp(ik_3z) + \frac{4d_{eff}}{c^2}A_1A_2\exp\left(i(k_1+k_2)\right)\right]e^{-i\omega_3t}$$

$$= \left[\left(\frac{d^2A_3^*}{dz^2} - 2ik_3\frac{dA_3^*}{dz}\right)\exp(-ik_3z) + \frac{4d_{eff}}{c^2}A_1^*A_2^*\exp(-i(k_1+k_2)z)\right]e^{i\omega_3t}$$

Each [ ] individually must be zero for all time. Divide by $\exp(ik_3z)$ and we get.

$$\frac{d^2A_3}{dz^2} + 2ik_3\frac{dA_3}{dz} = -\frac{4d_{eff}\,\omega_3^2}{c^2}A_1A_2\exp\left(i(k_1+k_2-k_3)z\right)$$

Impose the SVEA $\left|\frac{d^2A_3}{dz^2}\right| \ll \left|k_3\frac{dA_3}{dz}\right|$ more about this later...

We get

$$(1)\quad \boxed{\frac{dA_3}{dz} = \frac{2i\,d_{eff}\,\omega_3^2}{k_3c^2}A_1A_2\exp(i\Delta kz)}$$

where $\Delta k = k_1+k_2-k_3$

$k_3 = \frac{n_3\omega_3}{c}$

$n_3 = n(\omega_3)$

OK we got two more eq's to go

For $\vec{\mathcal{E}}_2$

Write the fields in a similar manner. However the nonlinear polarization will be different. Specifically

$$P_2 = 4\epsilon_0 d_{eff} E_3 E_1^*$$

with $E_3 = A_3 e^{ik_3 z}$

$E_1^* = A_1^* e^{-ik_1 z}$

E_1^* is from $\omega_3 = \omega_1 + \omega_2$ or $\omega_2 = \omega_3 - \omega_1$ and our discussion before on $\vec{P}^{NL}$

So into wave eq

$$\left[\frac{\partial^2 A_2}{\partial z^2} + 2ik_2 \frac{\partial A_2}{\partial z} - k_2^2 A_2 + \frac{\epsilon(\omega_2)\omega_2^2 A_2}{c^2}\right] \exp(ik_2 z - i\omega_2 t) + c.c.$$

$$= -\frac{4 d_{eff}\omega_2^2}{c^2} A_3 A_1^* \exp(i(k_3 - k)z - \omega_2 t) + c.c.$$

We can simplify as before by 1) Using SVEA 2) $k_2^2 = \frac{\epsilon(\omega_2)\omega_2^2}{c^2}$

3) c.c. are each equal to zero. Divide by $\exp(ik_2 z)$ and we have

$$\frac{\partial A_2}{\partial z} = \frac{2i\, d_{eff}\omega_2^2}{k_2 c^2} A_3 A_1^* \exp(-i\overbrace{(k_1 + k_2 - k_3)}^{\Delta k} z)$$

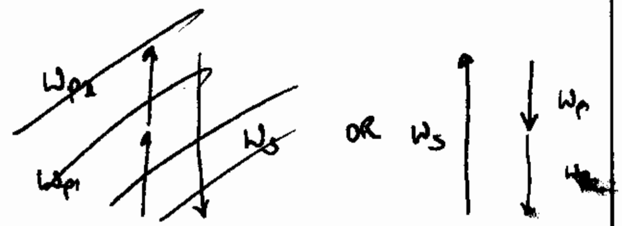
Using same procedure we will get

$$\frac{\partial A_1}{\partial z} = \frac{2i\, d_{eff}\omega_1^2}{k_1 c^2} A_3 A_2^* \exp(-i\Delta k z)$$

Manley-Rowe relations

Consider SFG

$$\omega_s = \omega_{p1} + \omega_{p2}$$



Two photons destroyed to make a photon at ω_s

$$\frac{I_{p1}}{\omega_{p1}} \equiv \text{photon flux of pump 1}$$

$$I_{p1} = \frac{P}{A} = \frac{1}{A}\frac{E}{\Delta t} = \frac{N\hbar\omega_{p1}}{\Delta t\, A}$$

$\uparrow$ Area

$$\text{So } \frac{I_{p1}}{\omega_p} = \frac{N\hbar}{\Delta t\, A} \sim \frac{\#\text{ of photons}}{\text{time (Area)}}$$

Some define photon flux as $P\lambda$, $P \equiv$ power

$$P\lambda = \frac{E}{\Delta t}\lambda = N\frac{hc}{\lambda}\frac{1}{\Delta t} = \frac{N}{\Delta t} \sim \frac{\#\text{photons}}{\text{time}}$$

OR $\boxed{P\lambda = \frac{I}{\omega}A}$

The Manley-Rowe relationship is a relation between photon flux

For $\omega_3 = \omega_2 + \omega_1$ we can write

$$\frac{d}{dz}\left(\frac{I_2}{\omega_2} + \frac{I_3}{\omega_3}\right) = 0 \qquad \frac{d}{dz}\left(\frac{I_1}{\omega_1} + \frac{I_3}{\omega_3}\right) = 0$$

$$\frac{d}{dz}\left(\frac{I_1}{\omega_1} - \frac{I_2}{\omega_2}\right) = 0$$

$$M_1 = \frac{I_2}{\omega_2} + \frac{I_3}{\omega_3} \equiv \text{const. of motion}$$

SFG: ω_3 created by the annihilation of ω_1 & ω_2

DFG: ω_2 created by the annihilation of ω_3 and creation of ω_1

$(\omega_2 = \omega_3 - \omega_1)$

$$\frac{d}{dz}\left(\frac{I_{m+1}}{\omega_{m+1}} \pm \frac{I_\mu}{\omega_\mu}\right) = 0 \qquad \omega_\mu \begin{cases} \mu & (+\ \text{destroyed}) \\ \mu & (-\ \text{created}) \end{cases}$$

ω_{m+1} is created if ω_μ is

Coupled Equations to solve

SFG (Boyd) $\omega_3 = \omega_{p1} + \omega_{p2}$

$$\frac{dA_3}{dz} = \frac{2i\, d_{eff}\, \omega_3^2}{k_3 c^2} A_1 A_2 \exp(i\Delta k z)$$

$$\frac{dA_2}{dz} = \frac{2i\, d_{eff}\, \omega_2^2}{k_2 c^2} A_3 A_1^* \exp(-i\Delta k z)$$

$$\frac{dA_1}{dz} = \frac{2i\, d_{eff}\, \omega_1^2}{k_1 c^2} A_3 A_2^* \exp(-i\Delta k z)$$

(use $k_n = \frac{n(\omega_n)\omega_n}{c}$) $\longrightarrow$

Different notation (Sutherland)

$$\frac{dA_s}{dz} = \frac{2i\, d_{eff}\, \omega_s}{n_s c} A_{p1} A_{p2} \exp(i\Delta k z)$$

$$\frac{dA_{p1}}{dz} = \frac{2i\, d_{eff}\, \omega_{p1}}{n_{p1} c} A_s A_{p2}^* \exp(-i\Delta k z)$$

$$\frac{dA_{p2}}{dz} = \frac{2i\, d_{eff}\, \omega_{p2}}{n_{p2} c} A_s A_{p1}^* \exp(-i\Delta k z)$$

$\Delta k = k_{p1} + k_{p2} - k_s$

$\omega_s = \omega_{p1} + \omega_{p2}$

$\left\{ \begin{array}{l} \omega_s : \text{signal} \\ \omega_1 : \text{pump 1} \quad \omega_2 : \text{pump 2} \end{array} \right.$

SHG $2\omega = \omega$

$$\frac{dA_{2\omega}}{dz} = \frac{2i\, d_{eff}\, \omega}{n_{2\omega} c} A_\omega^2 \exp(i\Delta k z)$$

$\Delta k = 2k_\omega - k_{2\omega}$

$$\frac{dA_\omega}{dz} = \frac{2i\, d_{eff}\, \omega}{n_\omega c} A_\omega^* A_{2\omega} \exp(-i\Delta k z)$$

DFG $\omega_i = \omega_{p1} - \omega_{p2}$

$\omega_{p1} \equiv$ pump
$\omega_i \equiv$ idler $\quad \omega_{p2} \equiv$ signal $\quad \Delta k = k_{p1} - k_{p2} - k_i$

$$\frac{dA_i}{dz} = \frac{2i\, d_{eff}\, \omega_i}{n_i c} A_{p1} A_{p2}^* \exp(+i\Delta k z) \quad \text{(idler)}$$

$$\frac{dA_{p1}}{dz} = \frac{2i\, d_{eff}\, \omega_{p1}}{n_{p1} c} A_i A_{p2} \exp(-i\Delta k z) \quad \text{(pump)}$$

$$\frac{dA_{p2}}{dz} = \frac{2i\, d_{eff}\, \omega_{p2}}{n_{p2} c} A_i^* A_{p1} \exp(+i\Delta k z) \quad \text{(signal)}$$

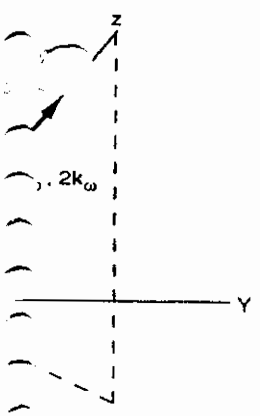
(B&C off by 1/2)

...g. 2, which illustrates SHG ...d harmonic wave polarized ...resent the laboratory coordinate ...present the principal axes of the ...$_{\mathrm{ff}} = d_{xyy}$. However, in general ... of the tensor components in the ...les θ and φ as well. For most ...or related to other components. ...s and for various optical wave

...Waves

...ne nonlinear polarization. The form of these equations, for a ...its within a constant K, where

(28)



... coordinate system (x, y, z) and

From "Handbook of Nonlinear Optics" Sutherland

SHG.

$$\frac{dA_{2\omega}}{dz} = iK\frac{2\omega}{n_{2\omega}c}d_{\mathrm{eff}}A_{\omega}^2\exp(i\Delta kz) \tag{29}$$

$$\frac{dA_{\omega}}{dz} = iK\frac{2\omega}{n_{\omega}c}d_{\mathrm{eff}}A_{\omega}^*A_{2\omega}\exp(-i\Delta kz) \tag{30}$$

SFG.

$$\frac{dA_{\mathrm{s}}}{dz} = iK\frac{2\omega_{\mathrm{s}}}{n_{\mathrm{s}}c}d_{\mathrm{eff}}A_{\mathrm{p1}}A_{\mathrm{p2}}\exp(i\Delta kz) \tag{31}$$

$$\frac{dA_{\mathrm{p1}}}{dz} = iK\frac{2\omega_{\mathrm{p1}}}{n_{\mathrm{p1}}c}d_{\mathrm{eff}}A_{\mathrm{s}}A_{\mathrm{p2}}^*\exp(-i\Delta kz) \tag{32}$$

$$\frac{dA_{\mathrm{p2}}}{dz} = iK\frac{2\omega_{\mathrm{p2}}}{n_{\mathrm{p2}}c}d_{\mathrm{eff}}A_{\mathrm{s}}A_{\mathrm{p1}}^*\exp(-i\Delta kz) \tag{33}$$

DFG.

$$\frac{dA_{\mathrm{d}}}{dz} = iK\frac{2\omega_{\mathrm{d}}}{n_{\mathrm{d}}c}d_{\mathrm{eff}}A_{\mathrm{p1}}A_{\mathrm{p2}}^*\exp(i\Delta kz) \tag{34}$$

$$\frac{dA_{\mathrm{p1}}}{dz} = iK\frac{2\omega_{\mathrm{p1}}}{n_{\mathrm{p1}}c}d_{\mathrm{eff}}A_{\mathrm{d}}A_{\mathrm{p2}}\exp(-i\Delta kz) \tag{35}$$

$$K = \begin{cases} 1 & (\mathrm{SI}) \\ 4\pi & (\mathrm{cgs}) \end{cases}$$

$$\frac{dA_{\mathrm{p2}}}{dz} = iK\frac{2\omega_{\mathrm{p2}}}{n_{\mathrm{p2}}c}d_{\mathrm{eff}}A_{\mathrm{d}}^*A_{\mathrm{p1}}\exp(i\Delta kz) \tag{36}$$

These equations were first solved by Armstrong et al. [3]. In general, both the modulus and phase of the complex field amplitudes are computed. However, to compute the output intensities of the generated waves, only the modulus is used. The intensity of a wave at some position z is given by

$$I_{\alpha} = 2\varepsilon_0 n_{\alpha}c|A_{\alpha}|^2 \tag{37}$$

in SI units, and

$$I_{\alpha} = \frac{n_{\alpha}c}{2\pi}|A_{\alpha}|^2 \tag{38}$$

in cgs units. The optical power of a given wave is computed from

$$\mathcal{P} = \int_A I\,dA \tag{39}$$

<u>General solution to coupled eqs.</u> (Armstrong) (SFG) $\omega_3 = \omega_1 + \omega_2$

Write the complex amplitudes in terms of an ~~[illegible]~~ magnitude + phase

$$A_j = \sqrt{\frac{I_j}{2 n_j \epsilon_0 c}}\; u_j\, e^{i\Phi_j} \qquad \begin{cases} u_j = u_j(\zeta) = u_j(z) \\ \Phi_j = \Phi_j(\zeta) \end{cases}$$

and define a normalized length

$$\zeta \equiv z/L_{NL} \qquad L_{NL} = \frac{1}{4\pi d_{eff}} \sqrt{\frac{2\epsilon_0 n_1 n_2 n_3 c \lambda_2 \lambda_3}{I_1(0)}}$$

Sub. this into the three coupled eqs. We will get four coupled eqs., 3 in magnitude and one in phase θ

where $\quad \theta \equiv \Delta k z + \Phi_3 - \Phi_2 - \Phi_1$

3 magnitudes
$$\frac{du_1}{d\zeta} = -u_2 u_3 \sin\theta \qquad \frac{du_3}{d\zeta} = u_1 u_2 \sin\theta$$
$$\frac{du_2}{d\zeta} = -u_3 u_1 \sin\theta$$

phase
$$\frac{d\theta}{d\zeta} = \Delta k L_{NL} + \cot\theta \, \frac{d}{d\zeta} \ln(u_1 u_2 u_3)$$

Constants of motion are given by 1) Energy conservation
2) Manley-Rowe relations

1) $\omega_1 u_1^2 + \omega_2 u_2^2 + \omega_3 u_3^2 = 1$

2) $m_1 = u_2^2 + u_3^2$, $m_2 = u_3^2 + u_1^2$, $m_3 = u_1^2 - u_2^2$

Useing the $d\theta/ds$, it can be integrated to get

$$\cos\theta = \left(\Gamma + \frac{1}{2}\Delta k L_{NL}\right) / u_1 u_2 u_3$$

This allows us to eliminate $\sin\theta$ and using the Manley Rowe relations we can write

$$y = \pm\frac{1}{2}\int_{u_3^2(0)}^{u_3^2(3)} \frac{d(u_3^2)}{\left[u_3^2\,(m_3 - u_3^2)(m_1 - u_3^2) - \left(\Gamma + \frac{1}{2}\Delta k L_{NL} u_3^2\right)^2\right]^{1/2}}$$

This is an elliptic integral with $y = sn^{-2}(\ \ )$

General solutions for fields

$$A_n \sim sn\left[z/L_{NL} \, , \, \gamma \right] \qquad \text{(more later)}$$

Jacobi elliptic function

We will solve these equations for different cases

1) Perfect phase matching $\Delta k = 0$

2) No pump depletion $\dfrac{dA_3}{dz} = 0$

(This is done in Boyd)

Table on next page ~~summarizes~~ summerizes solution under these cases.

$\Rightarrow$ See armstrong for how to solve general eqs.

Solutions as a function of length (z/L_{NL})

* The general solution will be elliptic sine/cosine functions *

Define power efficiencies

$$\eta \equiv \frac{P_{out}(z)}{P_{in}} \quad \text{e.g.} \quad \eta_{2\omega} = \frac{P_{2\omega}(z)}{P_\omega(0)}$$

	SFG	SHG	DFG
$dA_\omega/dz = 0$ $\Delta k = 0$	⊛ $\eta_s \sim d_{eff}^2 L^2 I_{p_1}$	⊛ $\eta_{2\omega} \sim d_{eff}^2 L^2 I_\omega$	⊛ $\eta_d \sim d_{eff}^2 L^2 I_{p_1}$
$dA_\omega/dz \neq 0$ $\Delta k = 0$	$\eta_s \approx sn^2[z/L_{NL}, \gamma]$ $\gamma^2 \equiv \frac{\lambda_{p2} P_{p2}(0)}{\lambda_{p1} P_{p1}(0)}$	⊛ $\eta_{2\omega} \sim \tanh^2(z/L_{NL})$ $\eta_\omega \sim \operatorname{sech}^2(\ )$	$\eta_d \sim sn^2[i\, z/L_{NL}, i\gamma]$ $\gamma^2 \equiv \frac{\lambda_{p2} P_{p2}(0)}{\lambda_{p1} P_{p1}(0)}$
$dA_\omega/dz \neq 0$ $\Delta k \neq 0$	$\eta_s \sim sn^2[\ ,\ ]$	$\eta_{2\omega} \sim sn^2[\ ,\ ]$	$\eta_d \sim sn^2[\ ,\ ]$
$dA_\omega/dz = 0$ $\Delta k \neq 0$	⊛ $\eta_s \sim \frac{\sin^2(\frac{\Delta k L}{2})}{(\frac{\Delta k L}{2})^2}$ $A_1(z) \sim \cos(\ )$ $A_3(z) \sim \sin(\ )$	⊛ $\eta_{2\omega} \sim \frac{\sin^2(\ )}{(\ )^2}$ $A_\omega(z) \sim \cos(\ )$ $A_{2\omega}(z) \sim \sin(\ )$	⊛ $\eta_d \sim \frac{\sin^2(\ )}{(\ )^2}$ $A_s \sim \cosh(\ )$ signal $A_d \sim \sinh(\ )$ idler

⊛ In Boyd

Jacobi Elliptic functions $\quad sn(z,k) \quad cn(z,k), \ dn(z,k)$

Functions of elliptic integrals (theta Functions)

$$\theta_1(z,q) = 2\sum_{n=0}^{\infty} (-1)^n \, q^{(n+1/2)^2} \sin((2n+1)z)$$

$$\theta_2(z,q) = 2\sum_{n=0}^{\infty} q^{(n+1/2)^2} \cos((2n+1)z)$$

Generalized sin() + sinh()

$sn(z, k)$ — z: ~~complex~~ argument possibly complex; k: modulus

$$\left\{ \begin{array}{l} sn(z,k) \equiv \dfrac{\theta_3(0,q)\,\theta_1(\zeta,q)}{\theta_2(0,q)\,\theta_4(\zeta,q)} \\[2ex] K(k) = \dfrac{\pi}{2}\theta_3^2(0,q), \quad \zeta = \dfrac{\pi z}{2K(k)} \end{array} \right.$$

Limiting cases

$sn(z,k) \rightarrow \sin z \qquad k \rightarrow 0$

$cn(z,k) \rightarrow \cos z \qquad k \rightarrow 0 \qquad$ (Small 2nd pump large pump 1)

$sn(z,k) \rightarrow \tanh z \qquad k \rightarrow 1 \qquad$ (Equal pumps)

$cn(z,k) \rightarrow \mathrm{sech}\, z \qquad k \rightarrow 1$

Imaginary argument

$sn(ix, k) = i\, sc(x, k') \qquad k^2 + k'^2 = 1$

$cn(ix, k) = nc(x, k')$

Purely real moduli

~~Real moduli~~

$$\mathrm{sn}(z, 1/k) = k\,\mathrm{sn}(z/k, k)$$

$$\mathrm{cn}(z, 1/k) = \mathrm{dn}(z/k, k)$$

Purely imaginary moduli

$$\mathrm{sn}(z, ik) = k_1\,\mathrm{sd}(z/k_1', k_1)$$

where $k_1 \equiv \dfrac{k}{\sqrt{1+k^2}}$ & $k_1 k_1' = \dfrac{k}{1+k^2}$

$$\mathrm{cn}(z, ik) = \mathrm{cd}(z/k_1', k_1)$$

~~Integral Definitions / Inverse functions~~

Representations as elliptic integrals

$$\mathrm{arcsn}(x,k) = \int_0^x \frac{dt}{(1-t^2)(1-k^2t^2)}$$

$$\mathrm{arccn}(x,k) = \int_0^x \frac{dt}{(1-t^2)(k'^2+k^2t^2)}$$

General solution	$sn^2(z/L_{NL}, \gamma)$

Lets get a physical feel for the Jacobi elliptic Functions

Example

SFG with $\Delta k = 0$

$$\eta_3 \sim sn^2\left(z/L_{NL}, \gamma\right)$$

With

$$L_{NL} \equiv \frac{1}{4\pi d_{eff}} \sqrt{\frac{2\varepsilon_0\, n_{p1} n_{p2} n_3\, c\, \lambda_{p1} \lambda_{p2}}{I_{p1}(0)}}$$

$$\gamma^2 = \frac{\lambda_{p2} P_{p2}(0)}{\lambda_{p1} P_{p1}(0)}$$

$$\gamma \sim 0 \qquad sn(u,\gamma) \simeq \sin(u)$$

$$\gamma \to 1 \qquad sn(u,\gamma) \to \tanh(u)$$

Case d

Physical parameters

$L_{NL} \equiv$ Characteristic nonlinear length

$\rightarrow$ 58% conversion effi. for SHG

$\gamma^2 \equiv$ ratio of photon Flux

λP proportional to photon Flux

$$\lambda P = \lambda \frac{N\hbar\omega}{\Delta t} = \lambda \frac{Nhc}{\lambda} \frac{1}{\Delta t} = hc \frac{N}{\Delta t}$$

$$\omega_s = \omega_{p1} + \omega_{p2}$$

So

$$\gamma^2 = \frac{\lambda_{p2} P_{p2}(0)}{\lambda_{p1} P_{p1}(0)}$$

$\gamma^2 \simeq 0$ small flux of 2nd pump

pump 2 becomes depleted and converts back to pump 1

$\gamma^2 = 1$ Equal Flux of pump 1 + pump 2 photons

Both pumps are depleted asymptotically to zero.

Maximum sum Frequency power is

$$\lambda_{p2}/\lambda_s \, P_{p2}(0)$$

Maximum sum freq Flux = photon Flux from pump 2

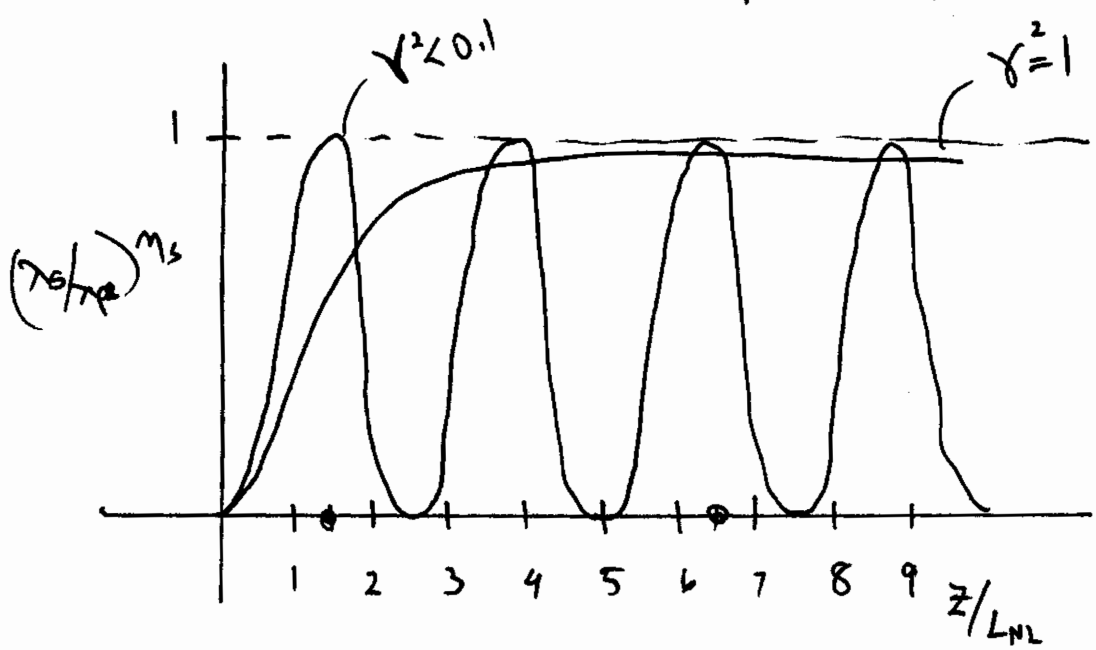


Table 3 Conversion Efficiency Formulas in the Infinite Plane Wave, Nondepleted Pump Approximation

SHG ($2\omega = \omega + \omega$)

$$\eta_{2\omega} = \frac{\mathcal{P}_{2\omega}}{\mathcal{P}_{\omega}} = \eta_{2\omega}^{0}\frac{\sin^2(\Delta kL/2)}{(\Delta kL/2)^2}$$

$$\eta_{2\omega}^{0} = \frac{8\pi^2 d_{\mathrm{eff}}^2 L^2 I_{\omega}}{\varepsilon_0 n_{\omega}^2 n_{2\omega} c\lambda_{\omega}^2} \quad \text{(SI)} \qquad \eta_{2\omega}^{0} = \frac{512\pi^5 d_{\mathrm{eff}}^2 L^2 I_{\omega}}{n_{\omega}^2 n_{2\omega} c\lambda_{\omega}^2} \quad \text{(cgs)}$$

SFG ($\omega_s = \omega_{p1} + \omega_{p2}$)

$$\eta_s = \frac{\mathcal{P}_s}{\mathcal{P}_{p2}} = \eta_s^0\frac{\sin^2(\Delta kL/2)}{(\Delta kL/2)^2}$$

$$\eta_s^0 = \frac{8\pi^2 d_{\mathrm{eff}}^2 L^2 I_{p1}}{\varepsilon_0 n_{p1} n_{p2} n_s c\lambda_s^2} \quad \text{(SI)} \qquad \eta_s^0 = \frac{512\pi^5 d_{\mathrm{eff}}^2 L^2 I_{p1}}{n_{p1} n_{p2} n_s c\lambda_s^2} \quad \text{(cgs)}$$

No pump depletion

$\Delta k \neq 0$

DFG ($\omega_d = \omega_{p1} - \omega_{p2}$)

$$\eta_d = \frac{\mathcal{P}_d}{\mathcal{P}_{p2}} = \eta_d^0\frac{\sin^2(\Delta kL/2)}{(\Delta kL/2)^2}$$

$$\eta_d^0 = \frac{8\pi^2 d_{\mathrm{eff}}^2 L^2 I_{p1}}{\varepsilon_0 n_{p1} n_{p2} n_d c\lambda_d^2} \quad \text{(SI)} \qquad \eta_d^0 = \frac{512\pi^5 d_{\mathrm{eff}}^2 L^2 I_{p1}}{n_{p1} n_{p2} n_d c\lambda_d^2} \quad \text{(cgs)}$$

Table 4 Frequency Conversion Efficiency Formulas in the Infinite Plane Wave Approximation, Including Pump Depletion

SHG $\quad \eta_{2\omega} = \tanh^2(L/L_{NL})$

$$L_{NL} = \frac{1}{4\pi d_{\mathrm{eff}}}\sqrt{\frac{2\varepsilon_0 n_{\omega}^2 n_{2\omega} c\lambda_{\omega}^2}{I_{\omega}(0)}} \quad \text{(SI)} \qquad L_{NL} = \frac{1}{16\pi^2 d_{\mathrm{eff}}}\sqrt{\frac{n_{\omega}^2 n_{2\omega} c\lambda_{\omega}^2}{2\pi I_{\omega}(0)}} \quad \text{(cgs)}$$

Pump Depletion

SFG $\quad \eta_s = \dfrac{\lambda_{p2}}{\lambda_s}\,\mathrm{sn}^2[(L/L_{NL}), \gamma] \qquad \gamma^2 = \dfrac{\lambda_{p2}\mathcal{P}_{p2}(0)}{\lambda_{p1}\mathcal{P}_{p1}(0)}$

$$L_{NL} = \frac{1}{4\pi d_{\mathrm{eff}}}\sqrt{\frac{2\varepsilon_0 n_{p1} n_{p2} n_s c\lambda_{p2}\lambda_s}{I_{p1}(0)}} \quad \text{(SI)} \qquad L_{NL} = \frac{1}{16\pi^2 d_{\mathrm{eff}}}\sqrt{\frac{n_{p1} n_{p2} n_s c\lambda_{p2}\lambda_s}{2\pi I_{p1}(0)}} \quad \text{(cgs)}$$

$\Delta k = 0$

DFG $\quad \eta_d = -\dfrac{\lambda_{p2}}{\lambda_d}\,\mathrm{sn}^2[i(L/L_{NL}), i\gamma] \qquad \gamma^2 = \dfrac{\lambda_{p2}\mathcal{P}_{p2}(0)}{\lambda_{p1}\mathcal{P}_{p1}(0)}$

$$L_{NL} = \frac{1}{4\pi d_{\mathrm{eff}}}\sqrt{\frac{2\varepsilon_0 n_{p1} n_{p2} n_d c\lambda_{p2}\lambda_d}{I_{p1}(0)}} \quad \text{(SI)} \qquad L_{NL} = \frac{1}{16\pi^2 d_{\mathrm{eff}}}\sqrt{\frac{n_{p1} n_{p2} n_d c\lambda_{p2}\lambda_d}{2\pi I_{p1}(0)}} \quad \text{(cgs)}$$

Table 5 Limiting Forms of SFG Efficiency Formulas in the Infinite Plane Wave Approximation, Including Pump Depletion

SFG ($\gamma \ll 1$) $\quad \eta_s = \dfrac{\lambda_{p2}}{\lambda_s}\sin^2\dfrac{L}{L_{NL}}$

Pump Depletion

SFG ($\gamma = 1$) $\quad \eta_s = \dfrac{\lambda_{p2}}{\lambda_s}\tanh^2\dfrac{L}{L_{NL}}$

$\Delta k = 0$

$$L_{NL} = \frac{1}{4\pi d_{\mathrm{eff}}}\sqrt{\frac{2\varepsilon_0 n_{p1} n_{p2} n_s c\lambda_{p2}\lambda_s}{I_{p1}(0)}} \quad \text{(SI)} \qquad L_{NL} = \frac{1}{16\pi^2 d_{\mathrm{eff}}}\sqrt{\frac{n_{p1} n_{p2} n_s c\lambda_{p2}\lambda_s}{2\pi I_{p1}(0)}} \quad \text{(cgs)}$$

Table 6 Limiting Forms of the DFG Efficiency in the Infinite Plane Wave Approximation, Including Pump Depletion

DFG ($\gamma \ll 1$) $\quad \eta_d = \dfrac{\lambda_{p2}}{\lambda_d}\sinh^2\dfrac{L}{L_{NL}}$

Pump Depletion

DFG ($\gamma = 1$) $\quad \eta_d = \dfrac{\lambda_{p2}}{\lambda_d}\dfrac{\mathrm{sn}^2[\sqrt{2}(L/L_{NL}), 1/\sqrt{2}]}{2 - \mathrm{sn}^2[\sqrt{2}(L/L_{NL}), 1/\sqrt{2}]}$

$\Delta k = 0$

$$L_{NL} = \frac{1}{4\pi d_{\mathrm{eff}}}\sqrt{\frac{2\varepsilon_0 n_{p1} n_{p2} n_d c\lambda_{p2}\lambda_d}{I_{p1}(0)}} \quad \text{(SI)} \qquad L_{NL} = \frac{1}{16\pi^2 d_{\mathrm{eff}}}\sqrt{\frac{n_{p1} n_{p2} n_d c\lambda_{p2}\lambda_d}{2\pi I_{p1}(0)}} \quad \text{(cgs)}$$

Table 7 Frequency Conversion Efficiency Formulas in the Infinite Plane Wave Approximation, Including Pump Depletion and the Effects of Phase Matching

SHG	$\eta_{2\omega} = \gamma\,\mathrm{sn}^2\{[\sqrt{1+(\Delta kL/4)^2(L_{NL}/L)^2} + (\Delta kL/4)(L_{NL}/L)](L/L_{NL}), \gamma\}$ $\gamma = [\sqrt{1+(\Delta kL/4)^2(L_{NL}/L)^2} - (\Delta kL/4)(L_{NL}/L)]^2$
SFG	$\eta_s = \frac{\lambda_{p2}}{\lambda_s}\frac{(1+\gamma_0^{-2})}{2}p_-\,\mathrm{sn}^2[\sqrt{\frac{1}{2}(1+\gamma_0^2)p_+}(L/L_{NL}), \gamma]$ $\gamma^2 = \frac{p_-}{p_+} \qquad \gamma_0^2 = \frac{\lambda_{p2}\mathcal{P}_{p2}(0)}{\lambda_{p1}\mathcal{P}_{p1}(0)}$ $p_\pm = 1 + \frac{(\Delta kL/2)^2(L_{NL}/L)^2}{1+\gamma_0^2} \pm \sqrt{\left[1+\frac{(\Delta kL/2)^2(L_{NL}/L)^2}{1+\gamma_0^2}\right]^2 - \left(\frac{2\gamma_0}{1+\gamma_0^2}\right)^2}$
DFG	$\eta_d = -\frac{\lambda_{p2}}{\lambda_d}\frac{(1-\gamma_0^{-2})}{2}p_-\,\mathrm{sn}^2[i\sqrt{\frac{1}{2}(1-\gamma_0^2)p_+}(L/L_{NL}), i\gamma]$ $\gamma^2 = -\frac{p_-}{p_+} \qquad \gamma_0^2 = \frac{\lambda_{p2}\mathcal{P}_{p2}(0)}{\lambda_{p1}\mathcal{P}_{p1}(0)}$ $p_\pm = 1 - \frac{(\Delta kL/2)^2(L_{NL}/L)^2}{1-\gamma_0^2} \pm \sqrt{\left[1-\frac{(\Delta kL/2)^2(L_{NL}/L)^2}{1-\gamma_0^2}\right]^2 + \left(\frac{2\gamma_0}{1-\gamma_0^2}\right)^2}$
$(\gamma \ll 1)$	$\eta_d = \frac{\lambda_{p2}}{\lambda_d}\frac{1}{1-(\Delta kL/2)^2(L_{NL}/L)^2}\sinh^2[\sqrt{1-(\Delta kL/2)^2(L_{NL}/L)^2}\quad(L/L_{NL})]$

Pump Depletion
$\Delta k \neq 0$

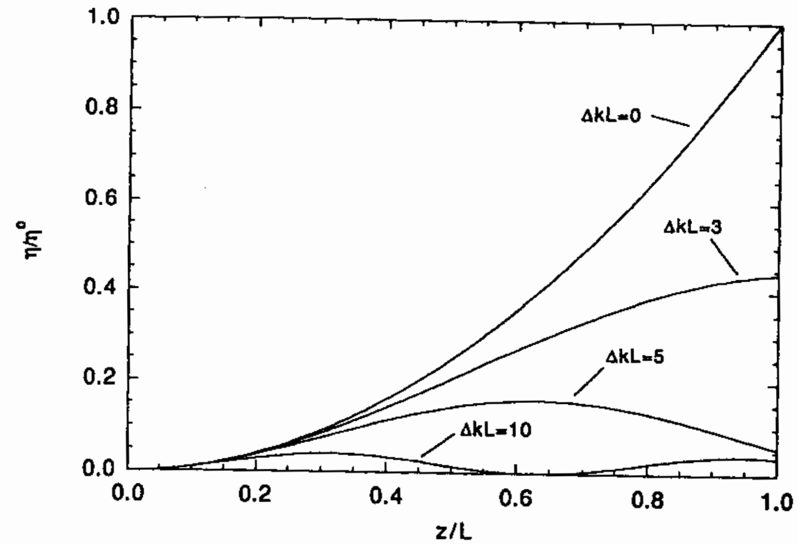
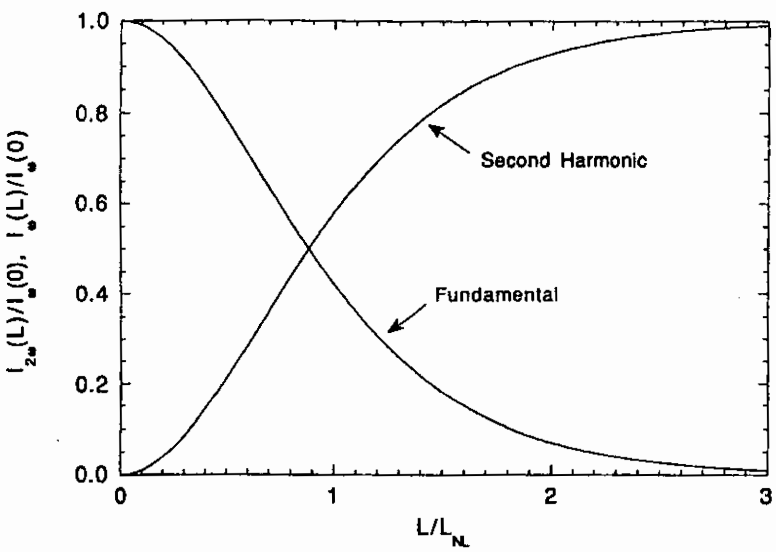


Figure 6 Normalized conversion efficiency as a function of position in a nonlinear medium for various values of phase mismatch for SHG, SFG, and DFG.

SHG, SFG, DFG

No pump depletion

Δk varies



igure 7 Second harmonic and fundamental intensities as functions of crystal length ıd nonlinear interaction length for phase matched SHG including pump depletion.

SHG with pump depletion and $\Delta k = 0$

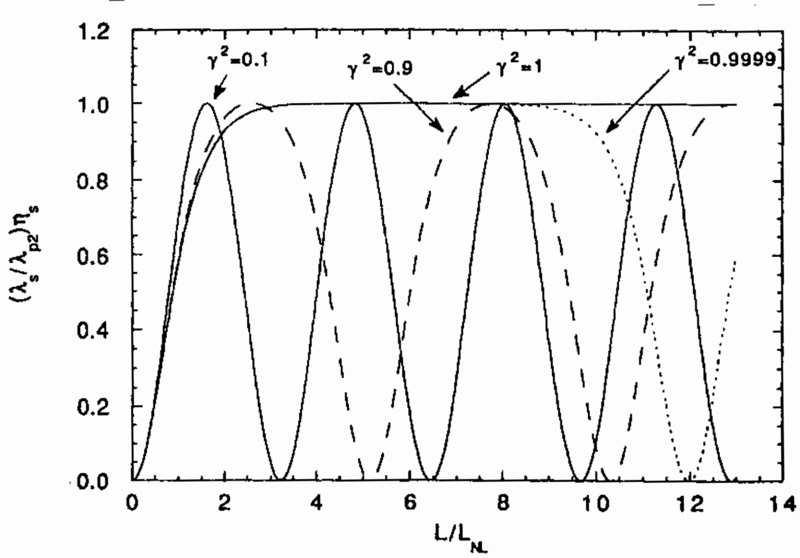


Figure 8 Sum-frequency conversion efficiency as a function of crystal length and nonlinear interaction length with several values of the modulus for phase matched SFG including pump depletion.

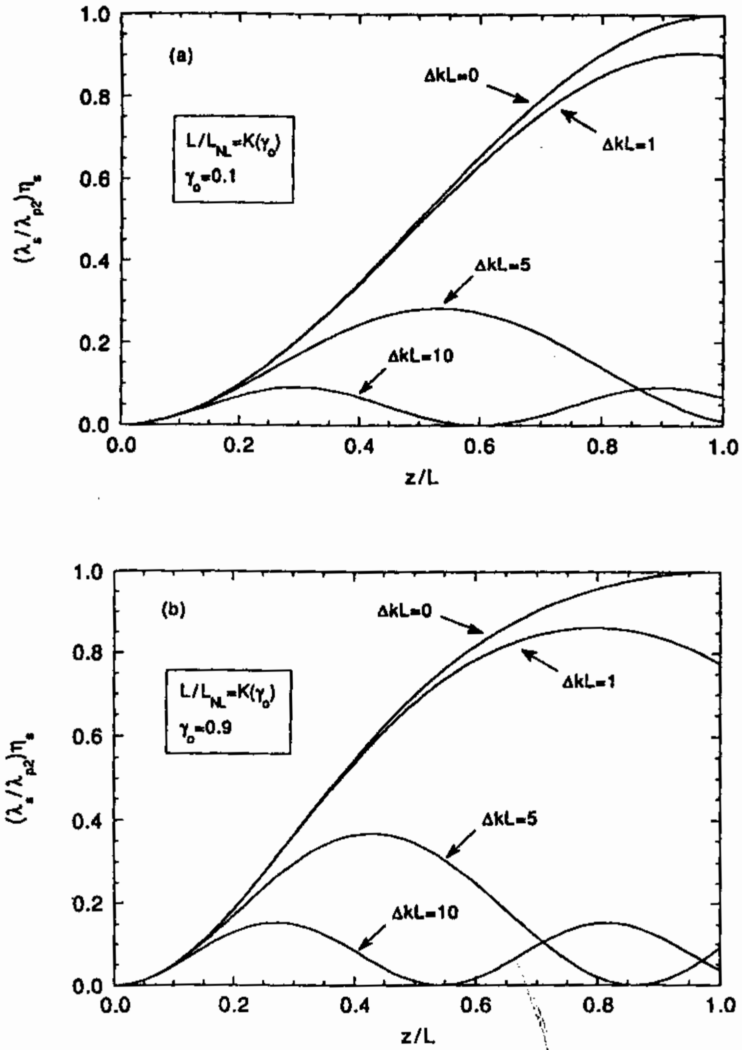
SFG

include pump depletion

for different

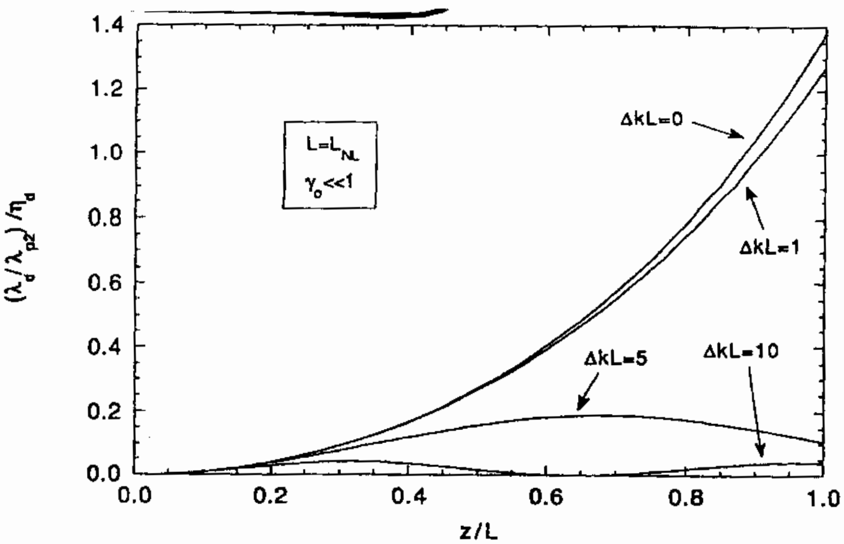
$$\gamma^2 = \frac{\lambda_{p2} P_{p2}(0)}{\lambda_{p1} P_{p1}(0)}$$

properties of SnI, J



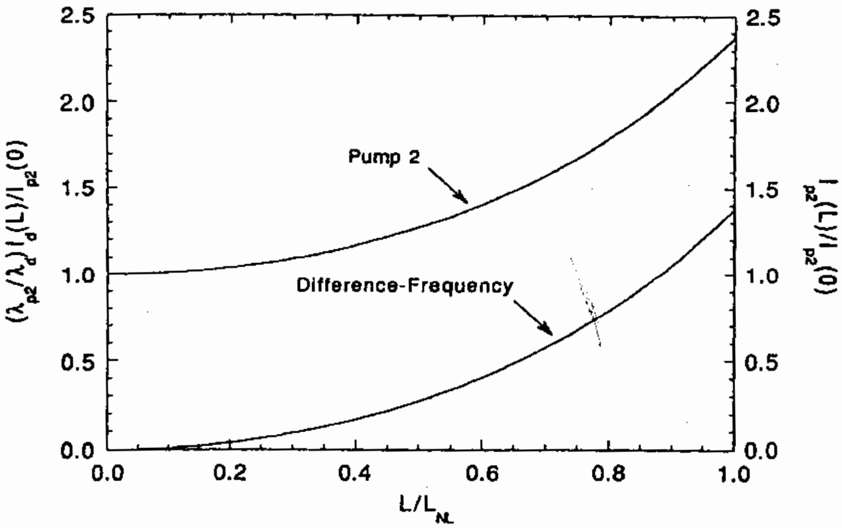
SFG with pump depletion + phase mismatch

Figure 12 Sum-frequency conversion efficiency as a function of position in a nonlinear medium including pump depletion and the effects of phase mismatch. (a) $\gamma_0 = 0.1$. (b) $\gamma_0 = 0.9$.



DFG with pump depletion + phase mismatch for strong pump and weak signal

Figure 13 Difference-frequency conversion efficiency as a function of position in a nonlinear medium including pump depletion and the effects of phase mismatch for the condition $\lambda_{p2}\mathcal{P}_{p2}(0) \ll \lambda_{p1}\mathcal{P}_{p1}(0)$.



igure 9 Phase matched difference-frequency and pump 2 intensities as functions of rystal length and nonlinear interaction length for the condition $\lambda_{p2}\mathcal{P}_{p2}(0) \ll \lambda_{p1}\mathcal{P}_{p1}(0)$.

DFG for weak signal input (like no pump depletion)

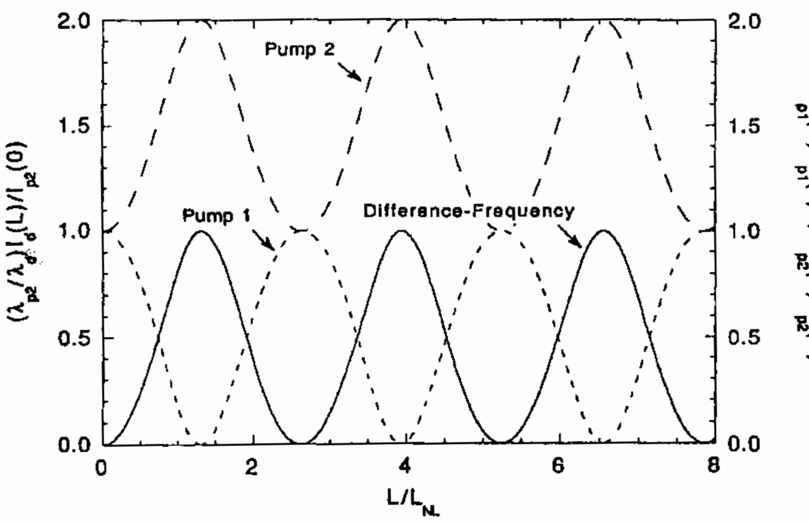


Figure 10 Phase matched difference-frequency, pump 1, and pump 2 intensities as functions of crystal length and nonlinear interaction length for the condition $\lambda_{p2}\mathcal{P}_{p2} = \lambda_{p1}\mathcal{P}_{p1}$.

DFG
For compariable pump and signal

$\lambda_{p2} P_{p2} = \lambda_{p1} P_{p1}$

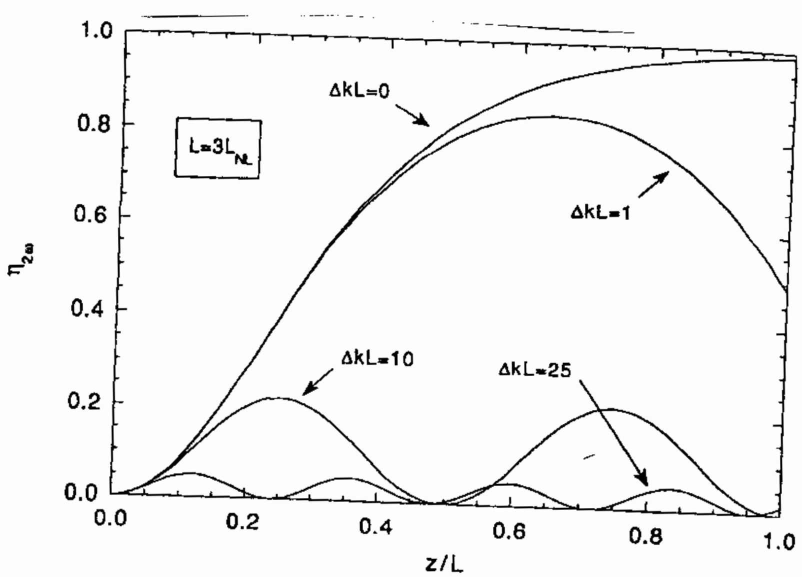


Figure 11 Second harmonic conversion efficiency as a function of position in a nonlinear medium including pump depletion and the effects of phase mismatch.

SHG including phase matching and pump depletion

sn[,] behavior

Sum Frequency Generation for no pump depletion

Assume $\frac{dA_1}{dz} = 0$ but $\Delta k \neq 0$

$$\frac{dA_2}{dz} = 0$$

$$\frac{dA_3}{dz} = \frac{2 i d_{eff} \omega_3^2}{k_3 c^2} A_1 A_2 e^{i \Delta k z}$$

use $k_3 = \frac{n_3 \omega_3}{c}$ $\quad c^2 = \frac{1}{\mu_0 \epsilon_0}$

$$\boxed{\frac{dA_3}{dz} = \frac{2 i d_{eff} \omega_3}{n_3 c} A_1 A_2 e^{i \Delta k z}}$$

Integrate DE

$$A_3(L) = \int_0^L \frac{2 i d_{eff} \omega_3}{n_3 c} A_1 A_2 \exp(i \Delta k z)\, dz$$

$$= \frac{2 i d_{eff} \omega_3}{n_3 c} A_1 A_2 \left(\frac{\exp(i \Delta k L) - 1}{i \Delta k} \right)^2$$

$$= \frac{2 i d_{eff} \omega_3}{n_3 c} A_1 A_2 L^2 \frac{\sin^2(\Delta k L)}{(\Delta k L)^2}$$

use
$e^{ix} = \cos x + i \sin x$

$\operatorname{sinc}(\Delta k L) = \frac{\sin \Delta k L}{\Delta k L}$

Find intensity $I_j = 2 \epsilon_0 n_j c |A_j|^2$

$$I_3 = \frac{8 n_3 \epsilon_0 c d_{eff}^2 |A_1|^2 |A_2|^2}{n_3^2 c^2} L^2 \operatorname{sinc}^2(\Delta k L)$$

But $|A_1|^2 = \frac{I_1}{2 \epsilon_0 n_1 c}$ + $|A_2|^2 = \frac{I_2}{2 \epsilon_0 n_2 c}$ $\quad \omega_3 = \frac{2\pi c}{\lambda_3}$

So

$$I_3 = c \, \frac{8 \, d_{eff}^2 \, \varepsilon_0 \, n_3 \, c \, \omega_3^2}{n_3^2 \, c^2} \; \frac{I_1}{2 \varepsilon_0 n_1 c} \; \frac{I_2}{2 \varepsilon_0 n_2 c} \; L^2 \, \mathrm{sinc}^2(\Delta k L)$$

$$I_3 = \frac{8 \, d_{eff}^2}{\varepsilon_0 \, n_1 n_2 n_3 \, c^3} \left(\frac{4 \pi^2 c^2}{\lambda_3^2} \right) I_1 I_2 \, L^2 \, \mathrm{sinc}^2(\Delta k L)$$

$$\boxed{I_3 = \frac{8 \pi^2 \, d_{eff}^2}{\varepsilon_0 \, n_1 n_2 n_3 \, c \, \lambda_3^2} \; I_1 I_2 \, L^2 \, \mathrm{sinc}^2(\Delta k L)}$$

(Boyd 2.2.19)

where $\mathrm{sinc}\, x \equiv \frac{\sin x}{x}$

<u>Main Points</u> :

1) $I_3 \sim L^2$

2) Efficiency depends on phase mismatch

3) d_{eff}^2 dependence

4) $1/\lambda_3^2$ dependence (or ω_3^2 dependence)

Slowly varying amplitude approximation

Sometimes called slowly varying envelope approx. (SVEA)

Energy transfer between waves in a nonlinear medium is significant only if the wave has traveled a distance longer than its wave length. Thus

$$\left|\frac{d^2A}{dz^2}\right| \ll \left| k \frac{dA}{dz} \right| \qquad k \equiv \frac{2\pi}{\lambda} \quad \text{is valid}$$

The amplitude A does not change much over propagation $\Delta z = \lambda$.

Shen states the real significance of this approximation is that we can neglect the oppositely propagating wave produced by P^{NL}

(Shen pg 47-49)
(Butcher pg 216)

From SVEA $\quad \dfrac{dA_1}{dz} = \dfrac{i\omega^2}{kc^2} P_1^{NL}(\omega,z) \exp(-i(kz-\omega t))$

From solving propagation ~~eqn~~ with $E(\omega,z) = A_F \underset{\text{Forward}}{\exp(i(kz-\omega t))}$
$+ A_B \exp(i(-kz-\omega t))$
backward

$$\frac{dA_F}{dz} = \frac{i\omega^2}{kc^2} P^{NL}(\omega,z) \exp(-i(kz-\omega t))$$ SVEA implies this

$$\frac{dA_B}{kz} = -\frac{i\omega^2}{kc^2} P^{NL}(\omega,z) \exp(i(kz-\omega t))$$ Not this

The slowly varying amplitude approx in time

$$\left|\frac{\partial^2 A}{\partial t^2}\right| \ll \left|\omega \frac{\partial A}{\partial t}\right|$$

Phase matching and coherence length

Define coherence length $\boxed{L_c = \frac{\pi}{\Delta k}}$

The coherence length is the distance over which the desired freq. of radiation is generated.

Energy Th$^{\underline{m}}$ (Poynting Th^m)

Stored energy in an E/m field can do work on dipoles

$$-\underbrace{\frac{\partial}{\partial t}\int_V u_{em}\, dV}_{\text{total power}} = \underbrace{\oint_S I\cos\theta\, dS}_{\substack{\text{(energy that flow out of}\\ S\text{)}}} + \underbrace{\int_V \left\langle \vec{E}\cdot\frac{\partial \vec{P}}{\partial t}\right\rangle dV}_{\substack{\text{Rate at which work is done by}\\ \text{electric field on the medium}}}$$

Note Sometimes this is written as

$$-\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\epsilon E^2 + \frac{B^2}{2\mu}\right) dV = \underbrace{\int_V \vec{J}_f\cdot\vec{E}\, dV}_{\substack{\text{Heat}\\ \text{(Joule heating)}\\ \text{for metals}}} + \underbrace{\oint_S (\vec{E}\times\vec{H})\cdot d\vec{a}}_{\text{Energy Flux}}$$

Consider the phase difference between the polarization and electric field: $\Delta\phi$

$$\Delta\phi \equiv \phi_{pol} - \phi_{field}$$

By performing the time average we get

$$2\,\mathrm{Re}\{\vec{E}\cdot\vec{P}^*\} \Rightarrow \text{Energy to medium}$$

If the dielectric const. is complex we get $0 < \Delta\phi < \pi$

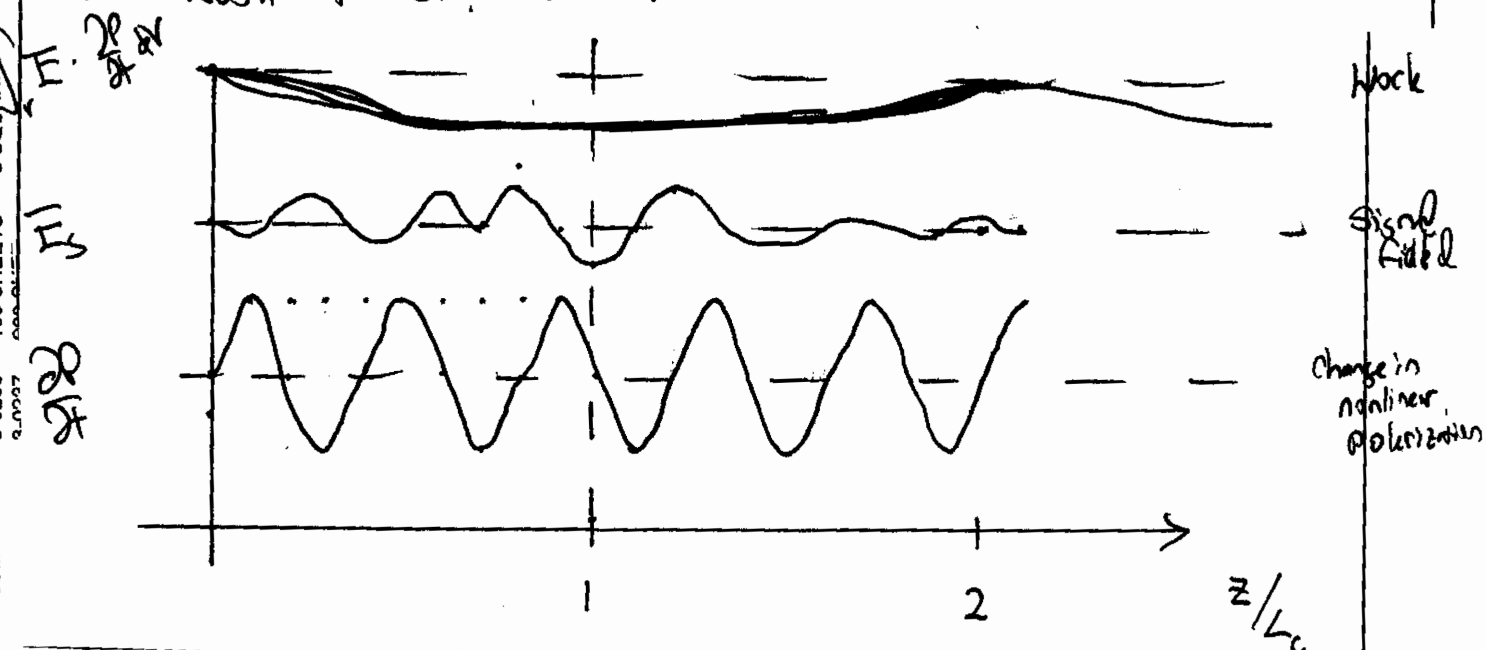
- Field does positive work on the polarization
- Field amplitude decreases (absorption)

If $-\pi < \Delta\phi < 0$

- Field does negative work on polarization
- Field amplitude increases (gain)
-

Back to coherence length

Look at SHG for L_c



- Front $z=0$: π phase shift between E_s + $\partial_t P$
  negative work done by pump field
  Energy $\Rightarrow$ pump $\rightarrow$ signal
- At $z=L_c$: $\pi/2$ phase shift between E_s + $\partial_t \vec{P}$
  No time average work done
- Beyond $z=L_c$: $\Delta\phi < \pi/2$
  positive work done by field onto medium
  Energy $\Rightarrow$ signal $\rightarrow$ pump

Sign of Work

Field does negative work on polarization $\Rightarrow$ Field amplitude increases

Field does positive work on polarization $\Rightarrow$ Field amplitude decreases

Review:

Derived three coupled equations to describe A_1 A_2 + A_3

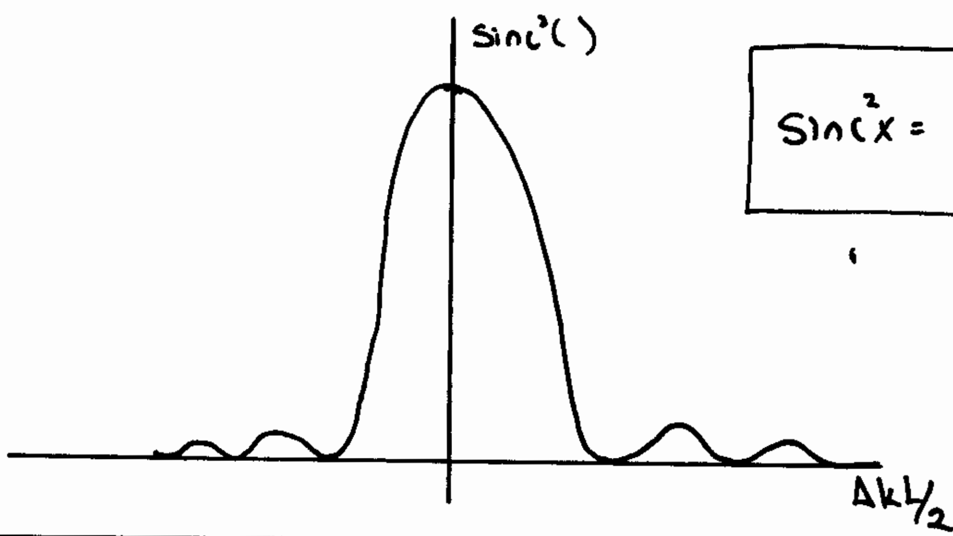
~~assuming constant "pumps" A_1 + A_2~~ Assume SVEA

$$\boxed{\begin{aligned}
\frac{dA_1}{dz} &= \frac{2i\omega_1}{n_1 c} d_{eff}\, A_3 A_2^* \exp(-i\Delta k z) \\
\frac{dA_2}{dz} &= \frac{2i\omega_2}{n_2 c} d_{eff}\, A_1^* A_3 \exp(-i\Delta k z) \\
\frac{dA_3}{dz} &= \frac{2i\omega_3}{n_3 c} d_{eff}\, A_1 A_2 \exp(+i\Delta k z)
\end{aligned}}$$

By assuming constant pumps A_1 + A_2 we integrate the 1st equation and find the intensity

$$\boxed{A_3 = \frac{2i\, d_{eff}\, \omega_3}{n_3 c} A_1 A_2 \left(\frac{\exp(i\Delta k L) - 1}{i\Delta L}\right) L}$$

Find I_3 $\Rightarrow$ $I_3 = 2\varepsilon_0 c n_1 A_3 A_3^* = \dfrac{8\pi^2 d_{eff}^2}{\varepsilon_0 n_1 n_2 n_3 c \lambda_3^2} I_1 I_2 L^2 \left(\dfrac{\sin(\Delta k L/2)}{\Delta k L/2}\right)^2$



$$\boxed{\operatorname{sinc}^2 x = \frac{\sin^2 x}{x^2}}$$

Lecture 9 Analytic results for SHG + SFG

Wish to look at two cases

1) SFG : For ~~so~~ ~~undepleted~~ depleted pumps where $I_2 \gg I_1$, $\Delta k \neq 0$

2) SHG : For depleted pumps + $\Delta k \neq 0$.

Can always solve coupled differential equations numerically.

Case 1) Rewrite out coupled DE's

$$\boxed{\frac{dA_1}{dz} = K_1 A_3 \exp(-i\Delta k z) \quad (1) \qquad \frac{dA_3}{dz} = K_3 A_1 \exp(+i\Delta k z) \quad (2) \qquad \frac{dA_2}{dz} = 0}$$

Where

$$K_1 \equiv \frac{2i\omega_1 d_{eff}}{n_1 c} A_2^* \qquad K_3 \equiv \frac{2i\omega_3 d_{eff}}{n_3 c} A_2$$

Seek solutions of the form

$$A_1(z) = \left[A_{1+} \exp(igz) + A_{1-} \exp(-igz)\right] \exp(-i\Delta k z/2)$$

$$A_3(z) = \left[A_{3+} \exp(igz) + A_{3-} \exp(-igz)\right] \exp(-i\Delta k z/2)$$

$g \equiv$ rate of spatial variation of A_1 + A_3

Same rate since A_1 + A_3 are coupled via energy conservation

Substitute into $\partial A_1/\partial z$ (1)

derivative

$$\frac{\partial A_1}{\partial z} = \left(iA_{1+}\, g \exp(igz) - iA_{1-}\, g \exp(-igz)\right)\exp(-i\Delta k z/2)$$

$$+ A_1(z)\; i\,\Delta k/2$$

Sub In

$$\left(igA_{1+} \exp(igz) - igA_{1-} \exp(-igz)\right)\exp(-i\Delta k/2\, z)$$

$$- \left(A_{1+} \exp(igz) + A_{1-} \exp(-igz)\right) i\,\Delta k/2\, \exp(-i\Delta k/2\, z)$$

$$= K_1\left[A_{3+} \exp(igz) + A_{3-} \exp(-igz)\right] \exp(-i\Delta k z/2\, z)$$

Equation must hold for all z, $\exp(igz)$ + $\exp(-igz)$ must maintain equality separately. Separate these terms:

$$\boxed{\begin{aligned} A_{1+}\left(ig - \tfrac{1}{2} i\Delta k\right) &= K_1 A_{3+} \\ -A_{1-}\left(ig + \tfrac{1}{2} i\Delta k\right) &= K_1 A_{3-} \end{aligned}} \quad \begin{aligned} &(3) \\ &(4) \end{aligned}$$

Now substitute solutions in $\partial A_2/\partial z$ + get similar eqs.

$$\boxed{\begin{aligned} A_{3+}\left(ig + \tfrac{1}{2} i\Delta k\right) &= K_3 A_{1+} \\ -A_{3-}\left(ig - \tfrac{1}{2} i\Delta k\right) &= K_3 A_{1-} \end{aligned}} \quad \begin{aligned} &(5) \\ &(6) \end{aligned}$$

Eqs. (3) + (5) are simultaneous eqs for A_{1+} + A_{3-}

$$\begin{pmatrix} i(g + \frac{1}{2}\Delta k) & -K_1 \\ -K_3 & i(g + \frac{1}{2}\Delta k) \end{pmatrix} \begin{pmatrix} A_{1+} \\ A_{3+} \end{pmatrix} = 0$$

A unique solution exists iff the determinant of the matrix vanishes

$$K_3 K_1 = (g - \tfrac{1}{2}\Delta k)(g + \tfrac{1}{2}\Delta k)$$

$$\boxed{g^2 = -K_1 K_3 + \frac{1}{4}\Delta k^2}$$

$$g = +\sqrt{-K_1 K_3 + \tfrac{1}{4}\Delta k^2} \qquad \text{(positive root)}$$

Use initial conditions in order to solve for $A_1(z)$ + $A_3(z)$

$$A_1(0) = A_{1+} + A_{1-} \qquad (7)$$

$$A_3(0) = A_{3+} + A_{3-} \qquad (8)$$

Using (7)(8) + (3)(4) we have for equations for the four unknowns A_{1+} A_{1-} A_{3+} A_{3-} ~~we can then solve~~. Also let us assume there is no initial sum frequency generation

so

$$A_3(0) = 0 \Rightarrow A_{3+} = -A_{3-}$$

Solution :

$$A_1(z) = A_1(0)\left(\cos(gz) + \frac{i\Delta k}{g}\sin(gz)\right)\exp(-i\,\Delta k z/2)$$

$$A_3(z) = A_1(0)\,\frac{K_3}{g}\,\sin(gz)\exp(-i\,\Delta k z/2)$$

For intensities

$$\boxed{\begin{aligned} I_1(z) &= 2n_1\varepsilon_0 c\, I_1(0)\left(\cos^2(gz) + \frac{\Delta k^2}{4g^2}\sin^2(gz)\right) \\ I_3(z) &= 2n_3\varepsilon_0 c\, I_1(0)\,\frac{|K_3|^2}{g^2}\sin^2(gz) \end{aligned}}$$

Characteristic Length g^{-1}

$\Longrightarrow$ g^{-1} decreases as Δk increases

$\Rightarrow$ SHG Intensity as $1/g^2$

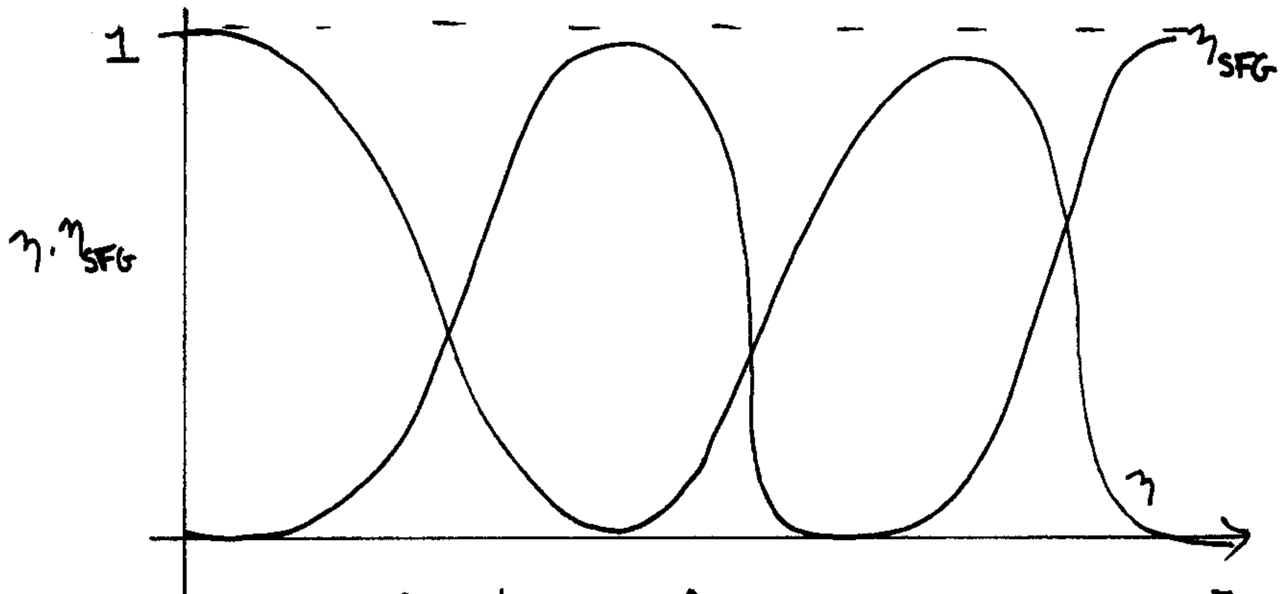
Conversion efficiency $\quad \eta_{SFG} = \dfrac{I_3(z)}{I_1(0)+I_2(0)} \simeq \dfrac{I_3(z)}{I_2(0)}$

Equal to 1 only if $\Delta k L = 0$

$$\eta \equiv \frac{I_1(z)}{I_1(0)}$$

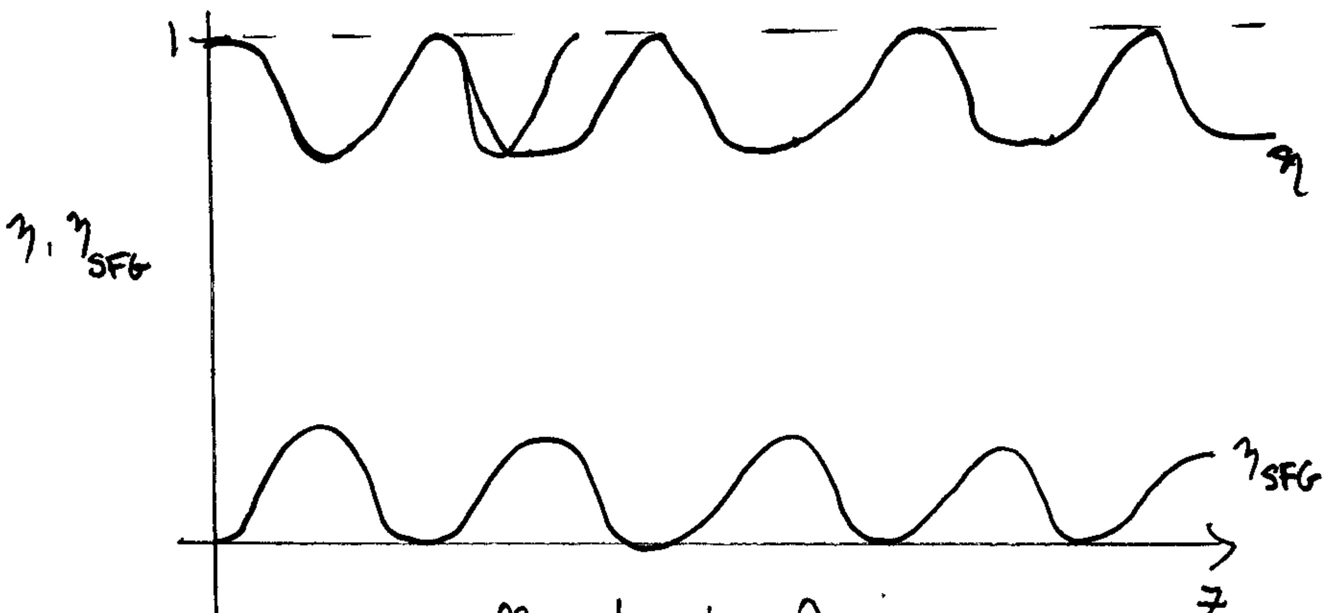
Two Cases

Perfect Phase matching $\Delta k = 0$ (or $\Delta kL = 0$)



Effecient transfer of energy

Non perfect phase matching $\Delta kL = 5$



inefficient transfer of energy.

Case II SHG generation with depleted pumps

$I_1 = I_2$ at $\omega = \omega_1$

I_3 at $2\omega = \omega_3$

$\Delta k = 2k_1 - K_3$

Solve coupled differential equations using a similar but complicated manner as in case II. Assume also $A_3(0) = 0$

~~Solution:~~

$$\boxed{\begin{aligned} \frac{dA_1}{dz} &= \frac{2i\omega_1}{n_1 c} d_{eff} A_1^* A_3 \exp(-i\Delta k z) \\ \frac{dA_3}{dz} &= \frac{2i\omega_3}{n_3 c} d_{eff} A_1^2 \exp(+i\Delta k z) \end{aligned}}$$

Where $\dfrac{I_1(z) + I_3(z)}{I_1(0)} = 1$ and $I_3(0) = 0$

Solution for $\Delta k \neq 0 \Rightarrow$ solutions in terms of elliptic integrals

For $\Delta k = 0$

$$\boxed{I_3(z) = I_1(0) \tanh^2(z/L_{NL})}$$

where

$$\boxed{L_{NL} = \frac{1}{4\pi d_{eff}} \sqrt{\frac{2\epsilon_0 n_1^2 n_3 c \lambda_1^2}{I_1(0)}}}$$

Lecture 8: Phase matching in uniaxial crystals

$\Rightarrow$ Perfect Phasematching implies $\Delta k = 0$!!

However, Δk will be a function of λ + θ where θ is the angle with respect to the optic axis in a uniaxial crystal.

$$\boxed{\Delta k \equiv \text{phase mismatch}}$$

Associated with a uniaxial crystal are the ordinary + extraordinary indices where

$$\boxed{\frac{1}{n_e^2(\theta)} = \frac{\sin^2\theta}{n_e^2} + \frac{\cos^2\theta}{n_o^2}}$$

OR

$$n_e(\theta) = n_o\left[\frac{1+\tan^2\theta}{1+(n_o/n_e)^2\tan^2\theta}\right]^{1/2}$$

To fulfill the $\Delta k = 0$ condition in a second order process we need to have a proper orientation of the input electric fields

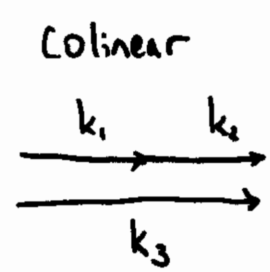
Phase matching

From before $\bar{k}_3 = \bar{k}_2 + \bar{k}_1$ $\qquad$ $\Delta\bar{k} = \bar{k}_3 - \bar{k}_2 - \bar{k}_1$

For a colinear process we can treat $\bar{k}_i$ as scalars

$$k_3 = k_2 + k_1$$

$$\frac{\omega_3 n_3}{c} = \frac{\omega_2 n_2}{c} + \frac{\omega_1 n_1}{c}$$



Phase mismatch $\underset{=0}{}$ $\Rightarrow$ $\boxed{\omega_3 n_3 = \omega_2 n_2 + \omega_1 n_1}$

~~or~~ Specifically:

SHG $\boxed{n(2\omega) - n(\omega)}$ where $\omega_1 = \omega_2 = \omega$

$\omega_3 = 2\omega$

or $\boxed{n_3 = n_1}$

SFG $\boxed{\omega_1 (n_3 - n_1) + \omega_2 (n_3 - n_2) = 0}$ where $\omega_3 = \omega_1 + \omega_2$

For normal isotropic materials the index increases with ω so it is not possible to obtain $n(2\omega) = n(\omega)$

– However, we can use a uniaxial anisotropic material since it has two different indices of refraction. How we use these materials are called the type of phase matching

Type I$^{(-)}$: ooe phase matching (negative uniaxial)

$$\bar{k}_{o1} + \bar{k}_{o2} = \bar{k}_{3e} \quad \begin{cases} E_1 \rightarrow \omega_1 \rightarrow n_o(\omega_1) \\ E_2 \rightarrow \omega_2 \rightarrow n_o(\omega_1) \end{cases}$$

$$E_3 \rightarrow \omega_3 \rightarrow n_e(\omega_3)$$

For SHG $\quad n_e(2\omega) = n_o(\omega)$

Type I $^{(+)}$ eeo phase matching (positive uniaxial)

$$\bar{k}_{1e}(\theta) + \bar{k}_{2e}(\theta) = \bar{k}_{o3}$$

For SHG

$$\boxed{n_e(\omega) = n_o(2\omega)}$$

Type II $^{(-)}$ oee phasematching (negative uniaxial)

$$\bar{k}_{o1} + \bar{k}_{e2}(\theta) = \bar{k}_{e3}(\theta)$$

Type II $^{(-)}$ eoe phase matching (negative uniaxial)

$$\bar{k}_{1e}(\theta) + \bar{k}_{o2} = \bar{k}_{3e}(\theta)$$

Type II $^{(+)}$ oeo phasematching (positive uniaxial)

$$\bar{k}_{o1} + \bar{k}_{e2}(\theta) = \bar{k}_{o3}$$

Type II $^{(+)}$ eoo phasematching (positive uniaxial)

$$\bar{k}_{1e}(\theta) + \bar{k}_{o2} = \bar{k}_{o3}$$

Common uniaxial crystals

Lithium Niobate (negative uniaxial)

beta Barium Borate (BBO) (negative $n_o > n_e$)

Potassium Dihydrogen Phosphate (KDP) (negative)

Potassium Titanyl Phosphate (KDP)

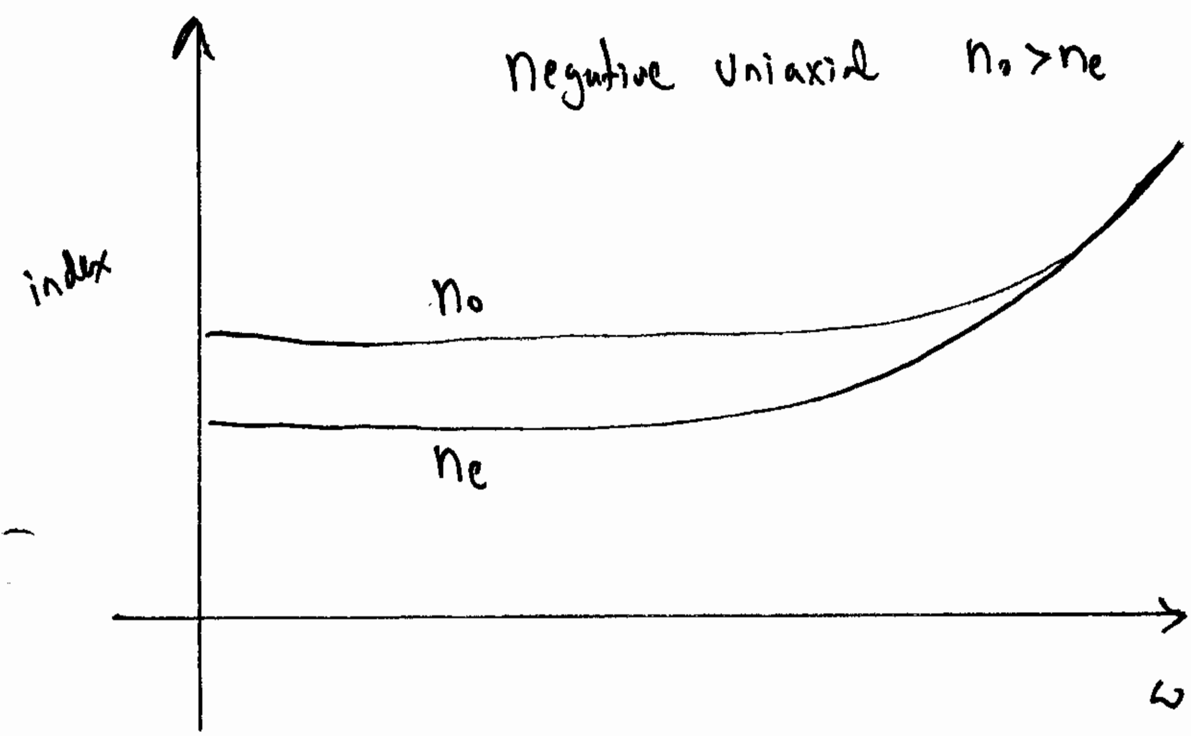
Lithium Iodate (negative)

Proustite (negative)

Lithium Triborate (LBO) (negative)

Common Biaxial

Potassium Iodate



How to compute the phase mismatch? How to compute phase matching angle?

Consider an example of SHG using Type I$^{(-)}$ phasematching (ooe)

$$\Delta k = k_3 - k_1 - k_2$$

For ooe $\begin{cases} E_1 \to \text{ordinary axis} \\ E_2 \to \text{ordinary axis} \\ E_3 \to \text{extraordinary axis} \end{cases}$

But

$$k_3 = \frac{2\pi n_3}{\lambda_3} = \frac{2\pi}{(\lambda/2)} n_e(\theta, \lambda/2)$$

$$k_2 = \frac{2\pi n_2}{\lambda_2} = \frac{2\pi}{\lambda} n_o(\lambda)$$

$$k_1 = \frac{2\pi n_1}{\lambda_1} = \frac{2\pi}{\lambda} n_o(\lambda)$$

So

$$\Delta k(\theta, \lambda) = \frac{2\pi}{(\lambda/2)} n_e(\theta, \lambda/2) - 2\,\frac{2\pi}{\lambda} n_o(\lambda)$$

$$\boxed{\Delta k(\theta, \lambda) = \frac{4\pi}{\lambda}\left[n_e(\theta, \lambda/2) - n_o(\lambda) \right]}$$

$\lambda \equiv$ fundamental wavelength

$\frac{\lambda}{2} = $ SHG

For a given λ, $\Delta k(\theta, \lambda) = 0$ for $\theta = \theta_{pm}$ the phase mismatch angle. How to find θ_{pm}?

1) Solve for θ_{pm} where $n_e(\theta_{pm}, \lambda/2) = n_o(\lambda)$ for the given n_e + n_o for a crystal

2) For ooe process

$$\sin^2\theta_{pm} = \left(\frac{n_e(2\omega)}{n_o(\omega)}\right)^2 \left[\frac{(n_o(2\omega))^2 - (n_o(\omega))^2}{(n_o(2\omega))^2 - (n_e(2\omega))^2} \right]$$

Methods of phase matching

Angle tuning $\Rightarrow$ use n_o + $n_e(\theta, \lambda)$

Temperature tuning $\Rightarrow$ use $n_o(T)$ + $n_e(T, \lambda)$

Problem with Angle tuning : Walkoff!!

Birefringence is temperature dependent. Use temperature tuning.

OR Quasiphasematching

Properties of SHG for non-zero phase matching : Non depletion

Non pump depleted case for cristal of length L

Define conversion efficiency

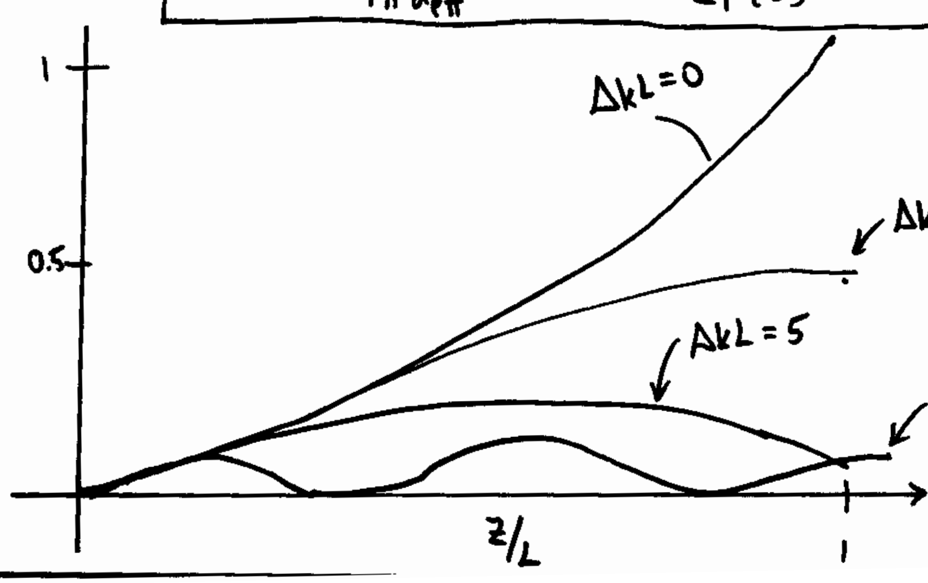
$$\eta \equiv \frac{I_3(z)}{I_1(0)+I_2(0)} \qquad \left(\frac{d_{eff}^2}{n_1 n_2 n_3}\right)$$

Define nonlinear length at $\Delta k = 0$

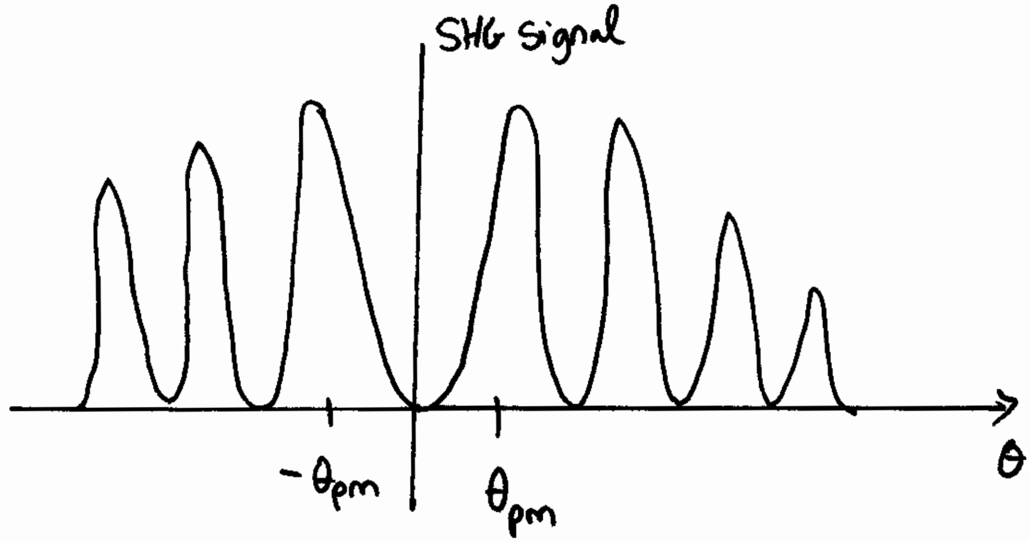
$$\boxed{L_{NL} = \frac{1}{4\pi d_{eff}} \sqrt{\frac{2\varepsilon_0 n_1 n_2 n_3 c \lambda_1}{I_1(0)}}}$$

For SHG
$\lambda_1 = \lambda_2 = \lambda$
$\lambda_3 = \lambda/2$
$n_j \equiv n(\lambda_j)$

Look at conversion efficiency for $\Delta k = 0, 3, 5, 10$

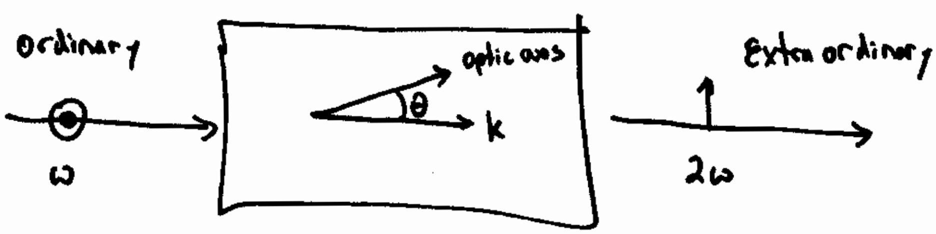


Angle tuning : SHG intensity vs angle.



Orientation of beams into crystal for SHG

Type $I^{(-)}$ ooe phase matching



$$n_e(2\omega, \theta) = n_o(\omega)$$

- SHG is along extraordinary axis
- Fundamental is along ordinary axis

SHG + Fundamental are polarized orthogonal to each other!

Table 10 Angle Phase Matching Formulas for DFG in Uniaxial Crystals

Type I	
ooe	$\sin^2\theta_{pm} = \dfrac{(n_d^e)^2}{(n_d^e)^2 - (n_d^o)^2} \dfrac{[n_{p1}^o - (\lambda_{p1}/\lambda_{p2})n_{p2}^o]^2 - (\lambda_{p1}/\lambda_d)^2(n_d^o)^2}{[n_{p1}^o - (\lambda_{p1}/\lambda_{p2})n_{p2}^o]^2}$
eeo	$\dfrac{n_{p1}^o}{\sqrt{1 + \left[\dfrac{(n_{p1}^o)^2}{(n_{p1}^e)^2} - 1\right]\sin^2\theta_{pm}}} - \dfrac{(\lambda_{p1}/\lambda_{p2})n_{p2}^o}{\sqrt{1 + \left[\dfrac{(n_{p2}^o)^2}{(n_{p2}^e)^2} - 1\right]\sin^2\theta_{pm}}} = (\lambda_{p1}/\lambda_d)$
Type II	
oee	$\dfrac{(\lambda_{p1}/\lambda_d)n_d^o}{\sqrt{1 + \left[\dfrac{(n_d^o)^2}{(n_d^e)^2} - 1\right]\sin^2\theta_{pm}}} + \dfrac{(\lambda_{p1}/\lambda_{p2})n_{p2}^o}{\sqrt{1 + \left[\dfrac{(n_{p2}^o)^2}{(n_{p2}^e)^2} - 1\right]\sin^2\theta_{pm}}} = n_{p1}^o$
eoe	$\dfrac{n_{p1}^o}{\sqrt{1 + \left[\dfrac{(n_{p1}^o)^2}{(n_{p1}^e)^2} - 1\right]\sin^2\theta_{pm}}} - \dfrac{(\lambda_{p1}/\lambda_d)n_d^o}{\sqrt{1 + \left[\dfrac{(n_d^o)^2}{(n_d^e)^2} - 1\right]\sin^2\theta_{pm}}} = (\lambda_{p1}/\lambda_{p2})$
eoo	$\sin^2\theta_{pm} = \dfrac{(n_{p1}^e)^2}{[(\lambda_{p1}/\lambda_d)n_d^o + (\lambda_{p1}/\lambda_{p2})n_{p2}^o]^2} \times \left(\dfrac{(n_{p1}^o)^2 - [(\lambda_{p1}/\lambda_d)n_d^o + (\lambda_{p1}/\lambda_{p2})n_{p2}^o]^2}{(n_{p1}^o)^2 - (n_{p1}^e)^2}\right)$
oeo	$\sin^2\theta_{pm} = \dfrac{(n_{p2}^e)^2}{(n_{p2}^e)^2 - (n_{p2}^o)^2} \dfrac{[n_{p1}^o - (\lambda_{p1}/\lambda_d)n_d^o]^2 - (\lambda_{p1}/\lambda_{p2})^2(n_{p2}^o)^2}{[n_{p1}^o - (\lambda_{p1}/\lambda_d)n_d^o]^2}$

It is noted that for some cases, analytical results for θ_{pm} cannot be obtaine In these situations, the phase matching angle must be calculated numericall This is very straightforward using available software packages.

A simple example is given using the *root* function of Mathcad®.*Type SHG is potassium dihydrogen phosphate (KDP), a negative uniaxial crystal, considered. The fundamental wavelength is 800 nm and the second harmoı wavelength is 400 nm, for which $n_\omega^o = 1.501924$, $n_\omega^e = 1.463708$, $n_{2\omega}^o$ 1.524481, and $n_{2\omega}^e = 1.480244$ [7]. The computation takes only a few secon and the computed angle, 70.204°, is accurate to <0.1%.

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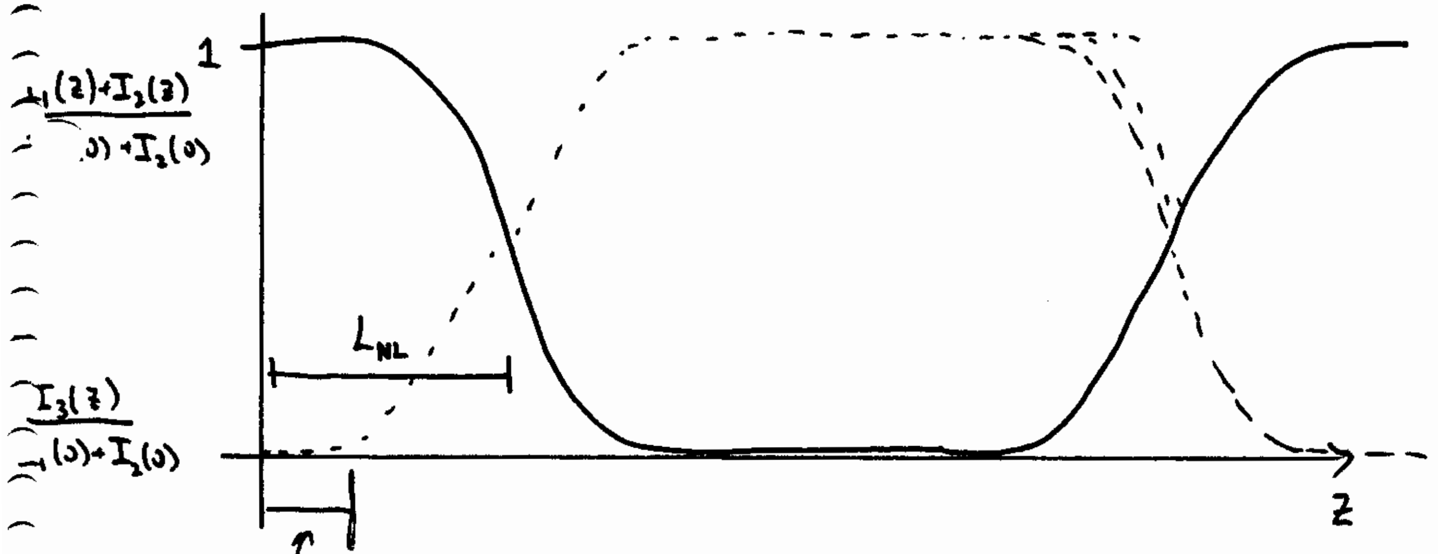
*Mathcad is a registered trademark of MathSoft, Inc., Cambridge, MA.

For SHG (assuming pump depletion)



non pump depletion region $\quad A_1(z) \simeq A_1(0)$

$A_2(z) \simeq A_2(0)$

Analytic expression

$$\frac{I_3(z)}{I_1(0)+I_2(0)} = \tanh\left(L/L_{NL} \right)$$

where

$$L_{NL} = \frac{1}{4\pi d_{eff}} \sqrt{\frac{2\varepsilon_0 n^2(\omega)\, n(2\omega)\, c\, \lambda_1^2}{I_1(0)}}$$

$$\frac{I_3(z=z_{NL})}{I_1(0)+I_2(0)} \simeq 0.58 \quad \text{at} \quad z = z_{NL}$$

Look again at $g(\Delta k)$

$$\boxed{g = \sqrt{-K_1 K_3 + \tfrac{1}{4}\Delta k^2}}$$

g is the smallest when $\Delta k = 0$

also $-K_1 K_3$ is a positive #

$$-K_1 K_3 = -\left(\frac{2i\omega_1 d_{eff}}{n_1 c} A_2^*\right)\left(\frac{2i\omega_3 d_{eff}}{n_3 c} A_2\right)$$

$$= \frac{4 d_{eff}^2 \omega_1 \omega_3}{n_1 n_3 c^2} I_2 \frac{1}{2\varepsilon_0 n_2 c} = \frac{2 d_{eff}^2 \omega_1 \omega_3}{\varepsilon_0 n_1 n_2 n_3 c^3} I_2$$

Which is a positive quantity.

At $\Delta k = 0$

$$g = \sqrt{-K_1 K_3} = \left(\frac{2 d_{eff}^2 \omega_1 \omega_3}{\varepsilon_0 n_1 n_2 n_3 c^3} I_2\right)^{1/2} \quad \left\{ \begin{array}{l} \omega_1 = \dfrac{2\pi c}{\lambda_1} \\ \omega_3 = \dfrac{2\pi c}{\lambda_1} \end{array} \right.$$

$$\frac{1}{g} = \left(\frac{\varepsilon_0 n_1 n_2 n_3 c^3}{2 d_{eff}^2 \omega_1 \omega_3} I_2^{-1}\right)^{1/2} = \left(\frac{\varepsilon_0 n_1 n_2 n_3 c^3}{2 d_{eff}^2 (2\pi c)^2} \lambda_1 \lambda_3 I_2^{-1}\right)^{1/2}$$

$$= \frac{1}{4\pi d_{eff}} \sqrt{\frac{2\varepsilon_0 n_1 n_2 n_3 c \lambda_1 \lambda_3}{I_2(0)}} = L_{NL}$$

General L_{NL}

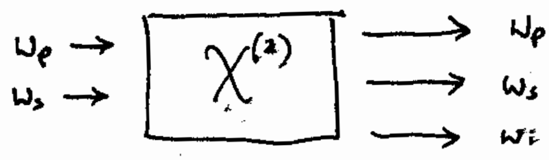
$$L_{NL} = \frac{1}{4\pi d_{eff}} \sqrt{\frac{2\varepsilon_0 n_1 n_2 n_3 \lambda_2 \lambda_3}{I_1(0)}} = \frac{1}{\sqrt{-K_1 K_3}}$$

Has different forms for SHG and SFG

- Can rewrite g for SFG

$$\boxed{g(\Delta k) = \frac{1}{L_{NL}} \sqrt{1 + \frac{\Delta k^2 L_{NL}^2}{4}}}$$

Optical Parametric generation



1) pump + signal beat to give idler field (ω_i)

2) idler + pump beat to give a difference at the signal field (ω_s)

Double beating extra term in polarization which is linear in signal field.

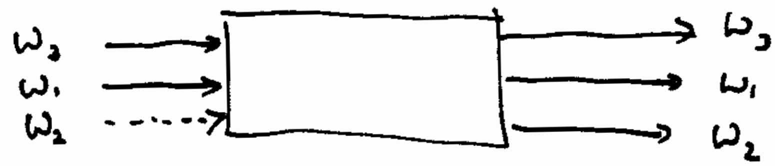
Why parametric?

pump field modulates $\chi^{(1)}$ at pump frequency

↑ parameter

Lecture 10 : Difference frequency generation and OPO

Consider process $\omega_2 = \omega_3 - \omega_1$



Here ω_2 is the generated output.

$$\frac{dA_1}{dz} = \frac{2i\omega_1 d_{eff}}{n_1 c} A_3 A_2^* \exp(i\Delta k z)$$

$$\frac{dA_3}{dz} = \frac{2i\omega_3 d_{eff}}{n_3 c} A_1 A_2 \exp(-i\Delta k z)$$

$$\frac{dA_2}{dz} = \frac{2i\omega_2 d_{eff}}{n_2 c} A_3 A_1^* \exp(i\Delta k z)$$

Solution for $\Delta k \neq 0$ and assuming $A_3(z) \simeq A_3(0)$

$$A_1(z) = \left[A_1(0)\left(\cosh(gz) - \frac{i\Delta k}{2g}\sinh(gz)\right) + \frac{K_1}{g}A_2^*(0)\sinh(gz)\right]e^{i\Delta k/2 z}$$

$$A_2(z) = \left[A_2(0)\left(\cosh gz - \frac{i\Delta k}{2g}\sinh gz\right) + \frac{K_2}{g}A_1^*(0)\sinh gz\right]e^{i\Delta k/2 z}$$

Where $g \equiv \left(K_1 K_2^* - \frac{\Delta k^2}{4}\right)^{1/2}$ $\quad K_j \equiv \frac{2i\omega_j d_{eff}}{n_j c} A_3(0)$

Rember the *

OR for $A_2(0) = 0$

$$A_1(z) = A_1(0)\left(\cosh gz - \frac{i\Delta k}{2g}\sinh gz\right)\exp(i\Delta k/2\, z)$$

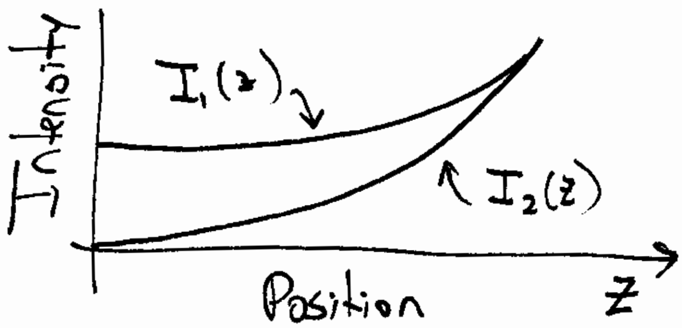
$$A_2(z) = \frac{K_2}{g} A_1(0)\sinh(gz)\exp(i\Delta k z/2)$$

<u>For $\Delta k = 0$</u>

$$\begin{cases} A_1(z) = A_1\cosh(z/L_{NL}) \\ A_2(z) = i\left(\frac{n_1\omega_1}{n_2\omega_2}\right)^{1/2}\frac{A_3}{|A_3|}A_1^*(0)\sinh(z/L_{NL}) \end{cases}$$

<u>Important Points</u>

- Fields + Intensities are <u>not</u> harmonic in z
- Monotonic growth of difference frequency



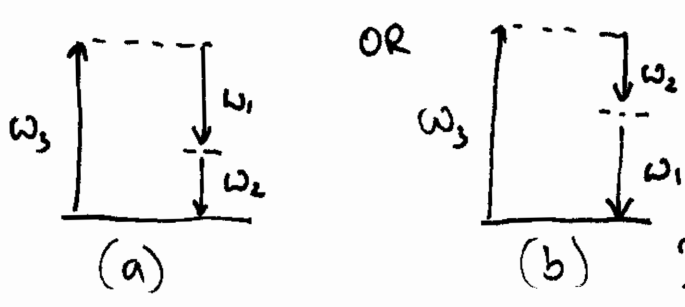
ω_1 is amplified by this process

$\Rightarrow$ This process is known as

<u>Parametric Amplification</u>

- ω_3 (pump)
- ω_1 is amplified by the process (signal wave)
- ω_2 is created by the process (idler wave)

Consider two cases



(a) Presence of ω_1 causes a transition of ω_2

(b) ω_2 stimulates a transition of ω_1.

Both of these process lead to exponential growth.

Optical Parametric Amplifier (OPA)

- Single pass thru crystal
- No feedback
- Modest gain for single pass
- Very useful for tunable wavelength conversion of a CW laser

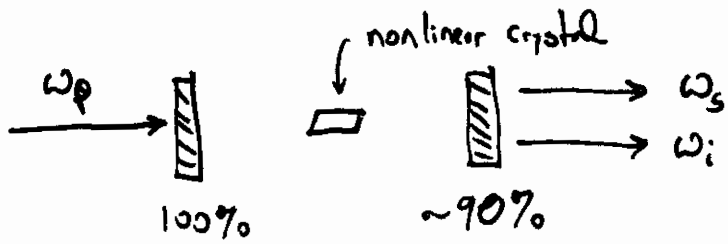
$$\text{gain} \simeq \left(L/L_{NL}\right)^2 \sinh^2\left(\frac{(L/L_{NL})^2 - (\Delta k\, L/2)^2}{(L/L_{NL})^2 - (\Delta k L/2)^2}\right)$$

$$L_{NL} = \frac{1}{4\pi d_{eff}} \sqrt{\frac{2\varepsilon_0\, n_p n_s n_i\, c\, \lambda_s \lambda_i}{I_p(0)}} \quad \left\{ \begin{array}{l} p \equiv \text{pump} \\ s \equiv \text{signal} \\ i \equiv \text{idler} \end{array} \right.$$

For $\Delta k = 0$ $\quad \text{gain} \simeq \left(L/L_{NL}\right)^2$

Optical Parametric Oscillators (OPO)

Put the nonlinear process in a cavity with mirrors that are highly reflecting a ω_1 or ω_2



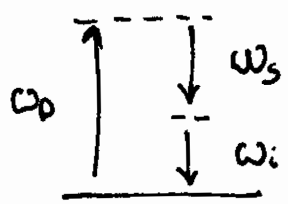
$$\left\{ \begin{array}{l} \omega_p \Rightarrow \omega_3 \\ \omega_s \Rightarrow \omega_1 \\ \omega_i \Rightarrow \omega_2 \end{array} \right.$$

Difference frequency generation leads to the amplification of the lower frequency input field.

$$\omega_i = \omega_p - \omega_s$$

The gain associated with parametric amplification can in the presence of feedback provide an oscillation.

Mirrors can reflect both ω_i and/or ω_s



$\omega_i < \omega_s$

(also called Parametric Down Conversion)

For the case where $\Delta k = 0$ $\quad A_2(0) = 0$ and $A_3(z) \simeq A_3(0)$

$$A_1(z) = A_1(0)\cosh(gz) \quad \text{(an exponential function)}$$

$$A_2(z) = i\left(\frac{n_1\omega_2}{n_2\omega_1}\right)^{1/2} \frac{A_3(0)}{|A_3(0)|} A_1^*(0)\sinh(gz)$$

Both signal + idler experience exponential growth.

** But is an OPO a laser?! **

Is an OPO a laser?

An OPO Has:

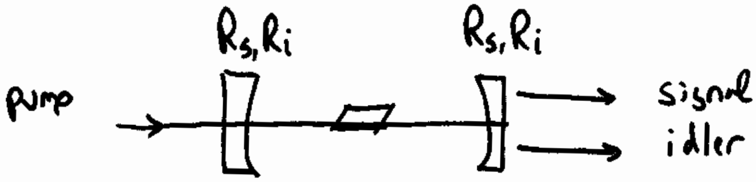
1) A pump source
2) A cavity for feedback
3) Gain at a specific frequency

Answer: No!

A laser has a population inversion caused by the pump. An OPO does not have a population inversion. So it is technically <u>not</u> a laser.

The problem with a laser is saturation when the upper population gets too large. An OPO does not have this problem!

Threshold for Parametric oscillation for a Doubly Resonant OPO



$R_i \equiv$ idler reflectivity
$R_s \equiv$ signal ref.

Threshold of oscillation

Gain per pass $\equiv$ loss per pass

$$\left(\exp(2L/L_{NL}) - 1\right) = (1 - R_s)(1 - R_i)$$

For low loss

$$\left(L/L_{NL}\right)^2 \simeq (1 - R_s)(1 - R_i)$$

For single resonant OPO

$$\left(L/L_{NL}\right)^2 \sim 2(1 - R_s)$$

Tuning and Bandwidth

Wavelength tuning is typically done by temperature tuning.

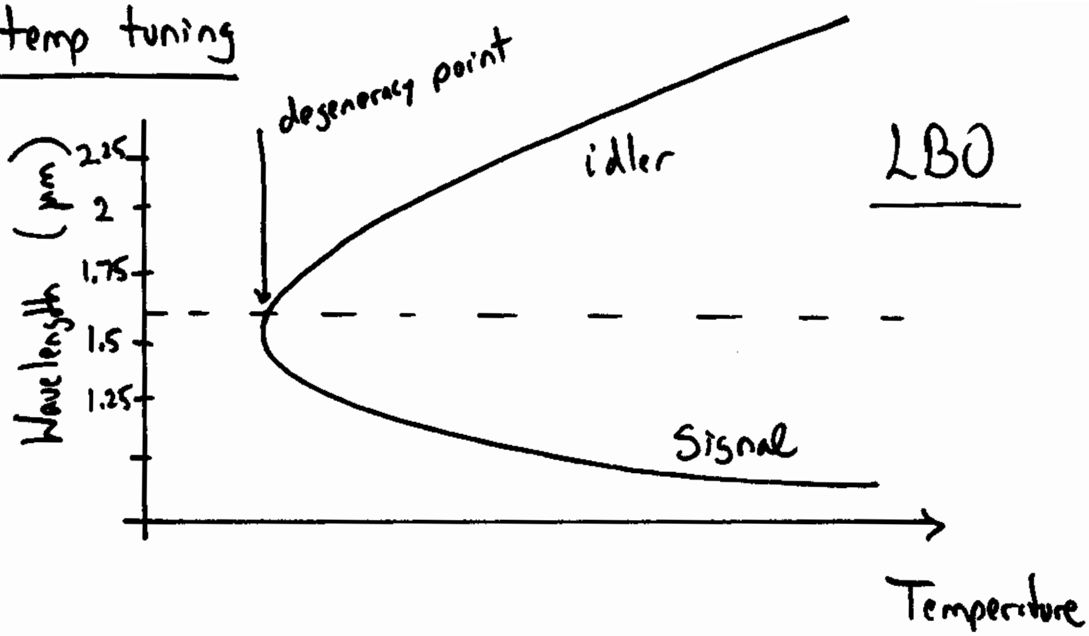
Must satisfy momentum + energy conservation

$$\omega_s + \omega_i = \omega_p \qquad \vec{k}_s + \vec{k}_i = \vec{k}_p$$

Phase Matching $\Rightarrow \omega_p \left[n(\omega_p) - n(\omega_p - \omega_s) \right] = \omega_s \left[n(\omega_s) - n(\omega_p - \omega_s) \right]$

Can also be done using angle tuning

Example of temp tuning



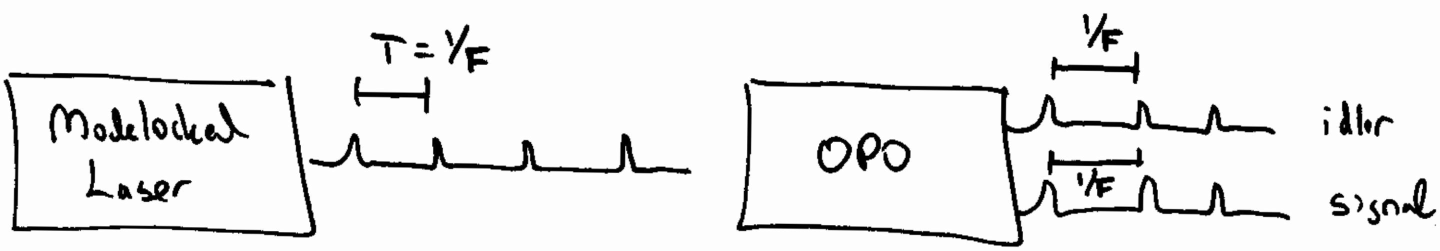
Pump $\lambda_p = 800\,nm$

Huge tuning range $\begin{cases} \text{signal} & 1 - 1.5\ \mu m \\ \text{idler} & 1.5 - 2.25\ \mu m \end{cases}$

Synchronously Pumped OPO

Use a pulsed laser for the pump

Get tunable wavelength pulsed light ⟹ signal + idler



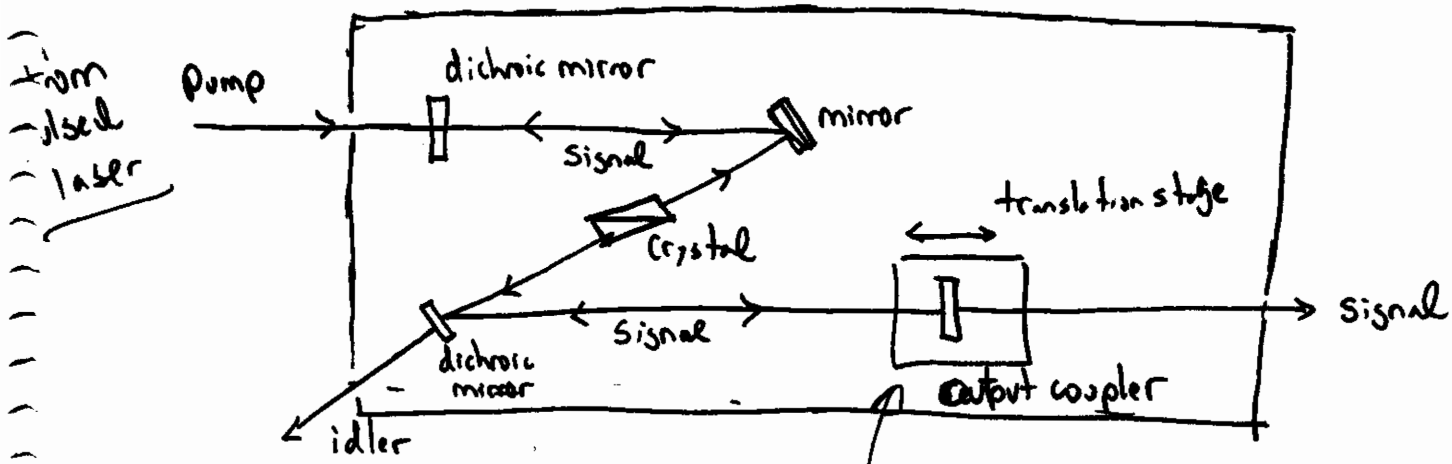
The second order process is modest even for $\Delta k = 0$. For efficiency signal + idler generation, the round trip time of the OPO cavity must be matched to the mode locked laser repetition rate.

$$\tau_{RT} = \frac{2L_c}{c} N_{crystal}(\lambda) + \frac{2L_F}{c} = \frac{1}{F} \qquad \begin{cases} L_c \equiv \text{crystal length} \\ L_F \equiv \text{Cavity length} \end{cases}$$

To match the cavity length & Rep Rate

Use following ~~linear~~ cavity

OPO Cavity



this translation stage with μm resolution is used to ~~get~~ match the cavity length to the repetition rate.

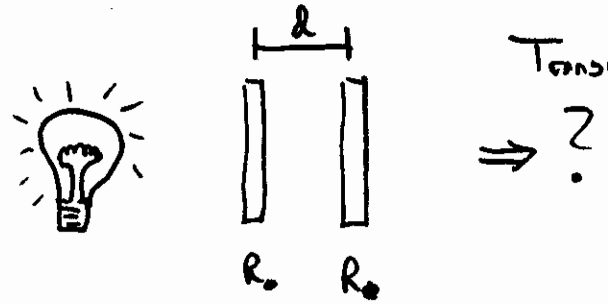
$$\tau_{RT} = \frac{2L_c}{c} N(\lambda) + \frac{2L_F}{c}$$

$$= \frac{1}{F} = T$$

Output coupler (~99% for ω_s) is on translation stage

Review of Cavity modes

Fabry Perot Interferometer : Interference by multiple reflections.



Transmission ?

$r^2 = R$

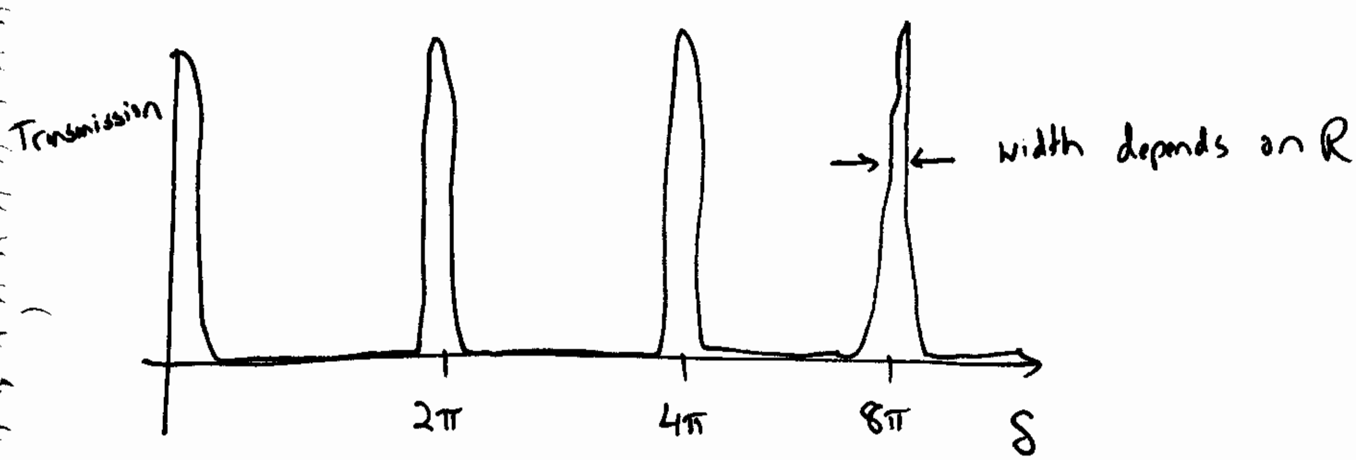
Interfence from reflections within cavity causes a variation in the transmission as a function of frequency.

Transmission

$$\frac{I(\delta)}{I_{max}} = \frac{1}{\left(1 + \frac{4}{\pi^2}\mathcal{F}\sin^2(\delta/2)\right)}$$

$$\mathcal{F} \equiv \text{finesse} \equiv \frac{\text{Free spectral Range}}{\text{Smallest resolvable wavelength}} = \frac{\pi}{2}\left(\frac{2r}{1-r^2}\right)$$

$$\delta \equiv \frac{4\pi d n \cos\theta}{\lambda_0} \quad \text{Phase difference}$$



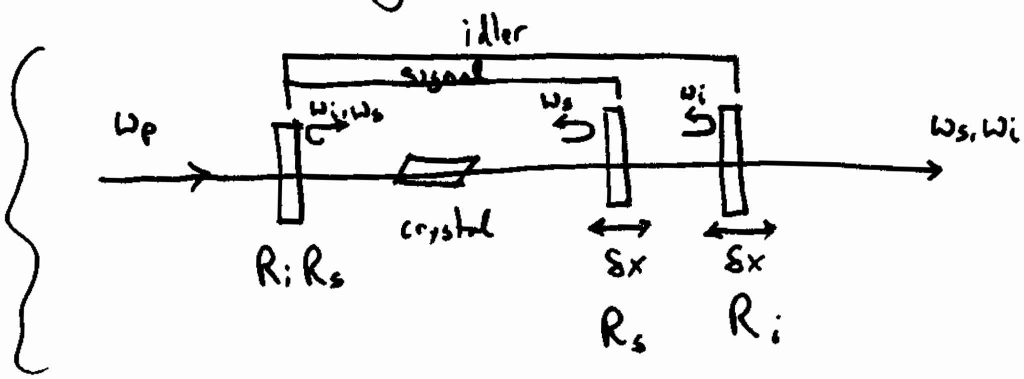
Notice that this is valid for only one wavelength λ_0
If we change λ_0 we need to change the distance d
to maximize the transmission of the Fabry Perot interferometer.

For an OPO

An OPO, the cavity length must correspond to maximum transmission at either the signal or idler wavelength.

In practice, this is difficult to do for both the signal and idler at the same time. Typically this is accomplished using two cavities.

Double Resonant OPO

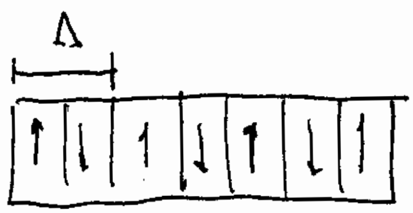


Lecture 8

Quasi-phasematching

Quasi → ?
resembling

Varying Poling   orientate c-axis of material



periodically poled

d_{eff} $-d_{eff}$ ....

Let $$d(z) = d_{eff} \sum_{m=1} \frac{2}{m\pi} \sin\left(\frac{m\pi}{2}\right) \exp(i k_m z)$$

Define $$d_Q \equiv d_{eff} \frac{2}{m\pi} \sin(m\pi/2)$$

$$\Delta k_Q = k_1 + k_2 - k_3 - \frac{2m\pi}{\Lambda}$$

Coupled Eq's $$\frac{dA_1}{dz} = \frac{2 i \omega_1 d_Q}{n_1 c} A_3 A_2^* \exp(\Delta k_Q z)$$

$\vdots$

Optical period when $$\boxed{\Lambda = \frac{2\pi}{k_1 + k_2 - k_3} = \underline{\underline{2 L_c}}}$$

Lecture 11 Quasi Phase matching

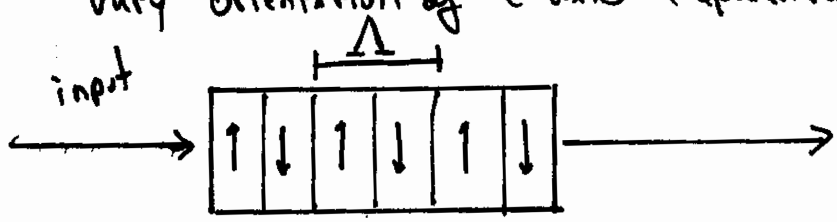
Problems with angle tuning for perfect phase matching

- at some angles propagation is difficult thru the crystal
- walk off
- cannot angle tune cubic crystals, which typically have larger d_{eff}
- cannot use d_{ii} terms, which are typically larger than off diagonal terms (d_{13} etc)
- cannot be used with crystals without birefringence e.g. Gallium Arsenide (GaAs)
- birefringence decreases with increasing ω.

— Use quasi phasematching instead with periodically poled materials

Periodically poled materials

Vary orientation of c axis in crystal



$\Rightarrow$ fabricated structure, not natural

Inversion of c axis changes the sign of d_{eff}

$$+d_{eff} \quad -d_{eff} \quad +d_{eff} \quad -d_{eff} \quad - \cdots$$

The variation of d_{eff} can compensate for $\Delta k \neq 0$!

$$\cancel{d_{Q,eff} = (-1)^{m-1} |d_{eff}| \frac{2}{\pi m}}$$

$m \equiv$ order of phase matching

Write down coupled eqs

$$\boxed{d(z) = d_{eff} \sum G_m \exp(i k_m z)}$$

$(m=1)$

$$\boxed{G_m \equiv 2/m\pi \, \sin(m\pi/2)} \qquad k_m = \frac{2\pi}{\Lambda} m$$

$$\frac{dA_1}{dz} = \frac{2i\omega_1 d_Q}{n_1 c} A_3 A_2^* \exp(\Delta k_Q z)$$

$$\frac{dA_2}{dz} = \frac{2i\omega_2 d_Q}{n_2 c} A_1^* A_3 \exp(\Delta k_Q z)$$

$$\frac{dA_3}{dz} = \frac{2i\omega_3 d_Q}{n_3 c} A_1 A_2 \exp(\Delta k_Q z)$$

Expand $d(z)$ and use only the strongest Fourier component

where $$\boxed{\Delta k_Q = k_1 + k_2 - k_3 - \frac{2\pi m}{\Lambda}, \quad d_Q = d_{eff} G_m}$$

We can determine the optimal period Λ

$$\boxed{\Lambda = \frac{2\pi}{k_1 + k_2 - k_3} \equiv 2L_c}$$

for Lithium niobate $L_c = 3.4 \mu m$ for $\lambda = 1.06 \mu m$

Define $L_c = \frac{\Lambda}{2}$

$$L_c = \frac{\pi}{k_1 + k_2 - k_3}$$

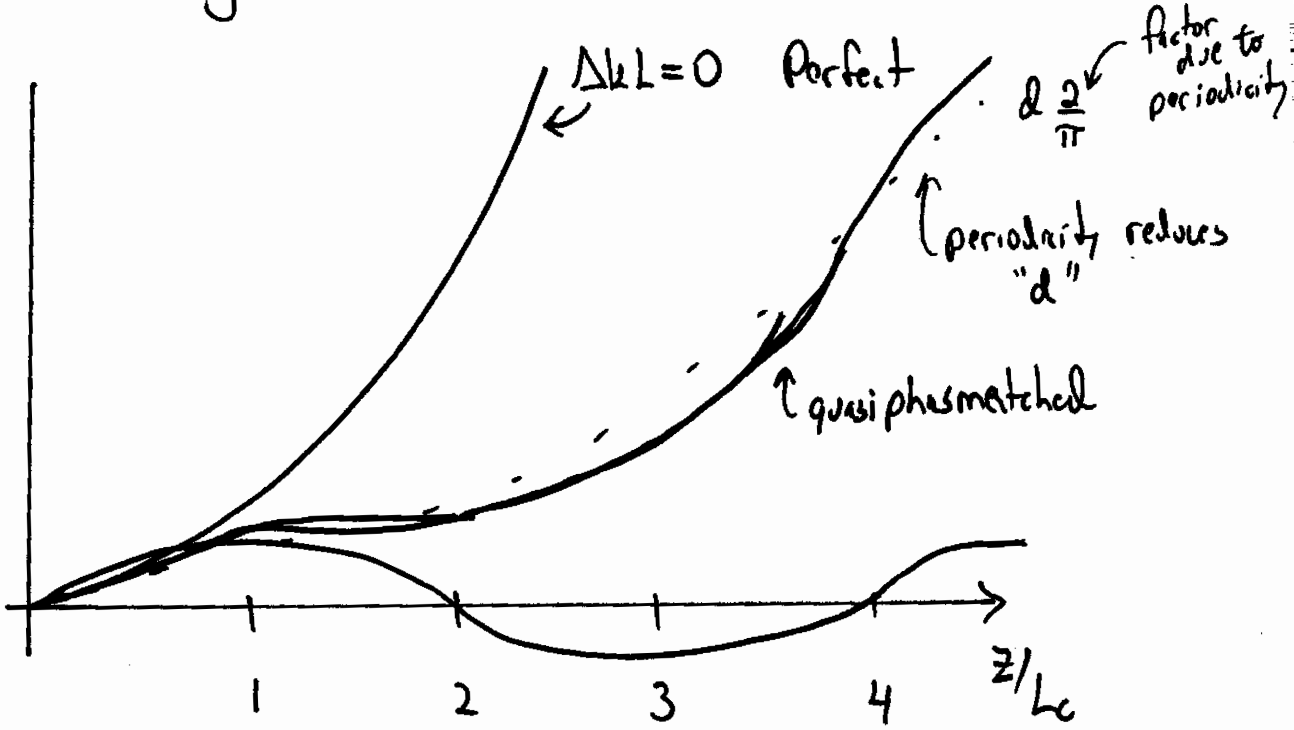
So $\Lambda = 2L_c$

– Quasi-phasematching is less efficient than perfect phase matching by $\left(d \frac{2}{\pi}\right)^2$

– However,, Quasi phase matching allows one to use d_{33} which is typically larger

1st order $\left(\frac{d_{33}}{d_{31}}\right)^2 \left(\frac{2}{\pi}\right)^2 \simeq 15$

Plot Phasematching



From this plot we see ....

QPM is not as efficient as Perfect phase matching but the disadvantage is compensated because with QPM larger die values can be used

$LiNbO_3$: Angle phase matching Type $I^{(-)}$ $\Rightarrow d_{31}$ $\quad$ d_{eff} $(d_{31}\sin\theta) - d_{22}\cos\theta\sin 3\phi$

$d_{22} = 2.4 pm/V$ $\quad$ QPM $\Rightarrow d_{33}$

$d_{33} = 31.5 pm/V$

$d_{31} = -4.52 pm/V$

$$\boxed{\left(\frac{d_{33}}{d_{31}}\right)^2 \left(\frac{2}{\pi}\right)^2 \simeq 15!}$$

1st order QPM

~~Efficien~~ Efficiency of QPM is 15 times larger!

Temperature tuning

One can change the output wavelength by heating the crystal

$$L_c = L_c(T) \qquad T \equiv \text{temperature}$$

Heating the crystal increases Λ thus decreasing the generated wavelength λ_3. It also changes the index. Calculate phase matching λ using SNLO program.

How does one make a periodically poled crystal?

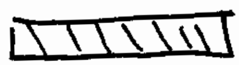
Quasiphase match is an old idea but at the time there was not a method to create the periodic poling. (Circa 1962)

Exposing a crystal to a strong electric field inverts magnetic domains which changes the orientation of d_{eff}. (1993)

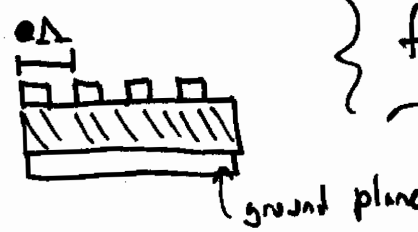
The crystal needs to be ferroelectric

Procedure

1) Lithium Niobate

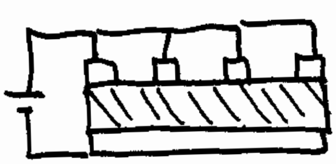


2) Deposit metal mask with desired periodicity



Change domain with $\vec{E}$

3) Apply 21 kV/mm electric field



No iron (ferro)

4) Only material under electrodes get the domain reversal

Advantages for QPM

- Use d_{ii} terms instead of d_{ij} terms. $d_{ii} > d_{ij}$ in general.
- Can be used for crystals that are not birefringent.
- Less walk of than for birefringent phase matching.
- Easier alignment
- Waveguide geometry $\Rightarrow$ Guide the fundamental!!

Disadvantages

- Periodic poling with strong electric fields can only be done in ferroelectric materials.

Review: Quasi phase matching

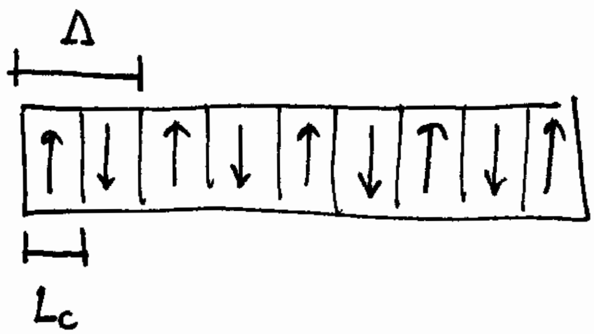
Periodically poled materials (Fabricated structure)
Vary sign of d_{eff} along length of crystal

$$d(z) = d_{eff} \sum_m G_m \exp(i k_m z) \qquad G_m = \frac{2}{\pi m} \sin(m\pi/2)$$

$$\Delta k_a = k_1 + k_2 - k_3 - \frac{2\pi}{\Lambda} m$$

For optimal period $\Delta k = 0$

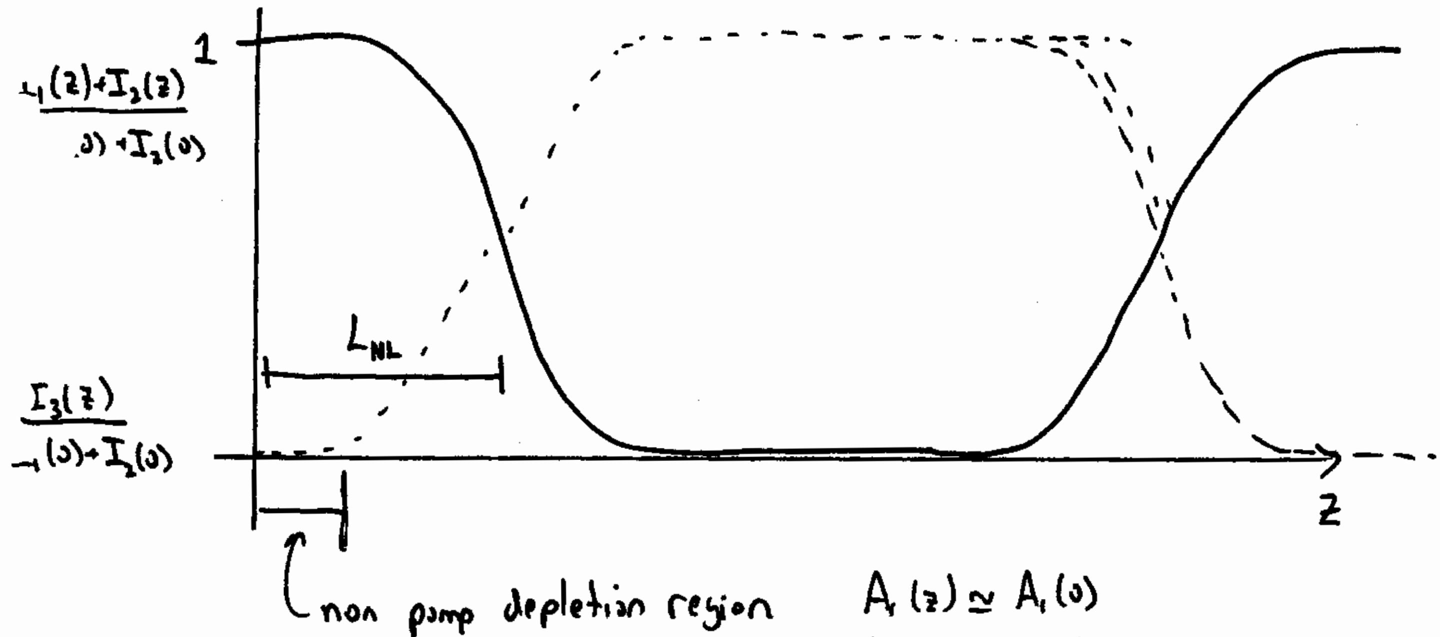
$$\boxed{\Lambda \equiv 2 L_c = \frac{2\pi}{k_1 + k_2 - k_3}}$$



$\Rightarrow$ Solve 3 coupled DE using $d(z)$

Expand $d(z)$ + keep strongest Fourier component.

For SHG (assuming pump depletion)



$A_1(z) \simeq A_1(0)$
$A_2(z) \simeq A_2(0)$

Analytic expression

$$\frac{I_3(z)}{I_1(0)+I_2(0)} = \tanh\left(L/L_{NL} \right)$$

where

$$L_{NL} = \frac{1}{4\pi d_{eff}} \sqrt{\frac{2\varepsilon_0\, n^2(\omega)\, n(2\omega)\, c\, \lambda_1^2}{I_1(0)}}$$

$$\frac{I_3(z = z_{NL})}{I_1(0)+I_2(0)} \simeq 0.58 \qquad \text{at} \quad z = z_{NL}$$

Look again at $g(\Delta k)$

$$\boxed{g = \sqrt{-K_1 K_3 + \tfrac{1}{4}\Delta k^2}}$$

g is the smallest when $\Delta k = 0$

also $-K_1 K_3$ is a positive #

$$-K_1 K_3 = -\left(\frac{2i\omega_1 d_{eff}}{n_1 c} A_2^*\right)\left(\frac{2i\omega_3 d_{eff}}{n_3 c} A_2\right)$$

$$= \frac{4 d_{eff}^2 \omega_1 \omega_3}{n_1 n_3 c^2} I_2 \frac{1}{2\varepsilon_0 n_2 c} = \frac{2 d_{eff}^2 \omega_1 \omega_3}{\varepsilon_0 n_1 n_2 n_3 c^3} I_2$$

Which is a positive quantity.

At $\Delta k = 0$

$$g = \sqrt{-K_1 K_3} = \left(\frac{2 d_{eff}^2 \omega_1 \omega_3}{\varepsilon_0 n_1 n_2 n_3 c^3} I_2\right)^{1/2} \quad \left\{ \begin{array}{l} \omega_1 = \dfrac{2\pi c}{\lambda_1} \\ \omega_3 = \dfrac{2\pi c}{\lambda_1} \end{array} \right.$$

$$\frac{1}{g} = \left(\frac{\varepsilon_0 n_1 n_2 n_3 c^3}{2 d_{eff}^2 \omega_1 \omega_3} I_2^{-1}\right)^{1/2} = \left(\frac{\varepsilon_0 n_1 n_2 n_3 c^3}{2 d_{eff}^2 (2\pi c)^2} \lambda_1 \lambda_3 I_2^{-1}\right)^{1/2}$$

$$= \frac{1}{4\pi d_{eff}} \sqrt{\frac{2\varepsilon_0 n_1 n_2 n_3 c \lambda_1 \lambda_3}{I_2(0)}} = L_{NL}$$

General L_{NL}

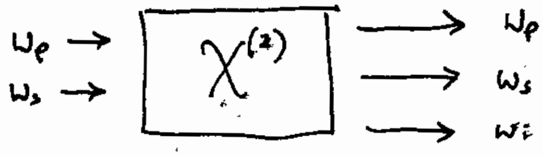
$$L_{NL} = \frac{1}{4\pi d_{eff}} \sqrt{\frac{2\varepsilon_0 n_1 n_2 n_3 \lambda_2 \lambda_3}{I_1(0)}} = \frac{1}{\sqrt{-K_1 K_3}}$$

Has different forms for SHG and SFG

- Can rewrite g for SFG

$$\boxed{g(\Delta k) = \frac{1}{L_{NL}} \sqrt{1 + \frac{\Delta k^2 L_{NL}^2}{4}}}$$

Optical
Parametric generation



1) pump + signal beat to give idler field (ω_i)

2) idler + pump beat to give a difference at the signal field (ω_s)

Double beating: extra term in polarization which is linear in signal field.

Why parametric?

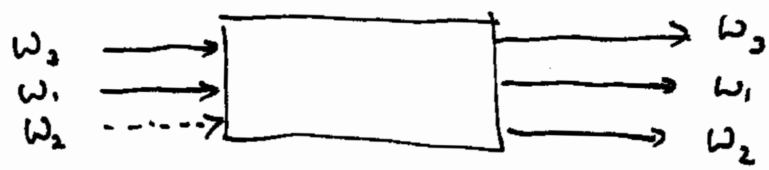
pump field modulates $\chi^{(1)}$ at pump frequency
↑ parameter

OPA : Amplification

OPO : Oscillator

Lecture 10 : Difference frequency generation and OPO

Consider process $\omega_2 = \omega_3 - \omega_1$



Here ω_2 is the generated output.

$$\frac{dA_1}{dz} = \frac{2i\omega_1 d_{eff}}{n_1 c} A_3 A_2^* \exp(i\Delta k z)$$

$$\frac{dA_3}{dz} = \frac{2i\omega_3 d_{eff}}{n_3 c} A_1 A_2 \exp(-i\Delta k z)$$

$$\frac{dA_2}{dz} = \frac{2i\omega_2 d_{eff}}{n_2 c} A_3 A_1^* \exp(i\Delta k z)$$

No pump depletion

Solution for $\Delta k \neq 0$ and assuming $A_3(z) \simeq A_3(0)$

$$A_1(z) = \left[A_1(0)\left(\cosh(gz) - \frac{i\Delta k}{2g}\sinh(gz)\right) + \frac{K_1}{g} A_2^*(0)\sinh(gz)\right] e^{i\Delta k/2\, z}$$

$$A_2(z) = \left[A_2(0)\left(\cosh gz - \frac{i\Delta k}{2g}\sinh gz\right) + \frac{K_2}{g} A_1^*(0)\sinh gz\right] e^{i\Delta k/2\, z}$$

Where $g \equiv \left(K_1 K_2^* - \frac{\Delta k^2}{4}\right)^{1/2}$ $\quad K_j \equiv \frac{2i\omega_j d_{eff}}{n_j c} A_3(0)$

Rember the *

OR for $A_2(0) = 0$

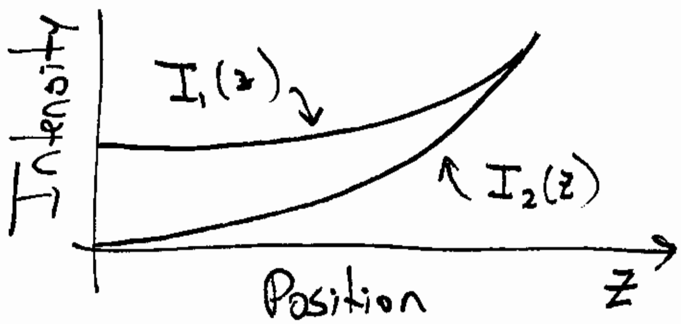
$$A_1(z) = A_1(0)\left(\cosh gz - \frac{i\Delta k}{2g}\sinh gz\right)\exp\left(i\tfrac{\Delta k}{2} z\right)$$

$$A_2(z) = \frac{K_2}{g} A_1(0)\sinh(gz)\ \exp\left(i\Delta k z/2\right)$$

<u>For $\Delta k = 0$</u>
$$\begin{cases} A_1(z) = A_1\cosh(z/L_{NL}) \\ A_2(z) = i\left(\dfrac{n_1\omega_1}{n_2\omega_2}\right)^{1/2}\dfrac{A_3}{|A_3|}A_1^*(0)\sinh(z/L_{NL} \end{cases}$$

<u>Important Points</u>

- Fields + Intensities are <u>not</u> harmonic in z
- Monotonic growth of difference frequency



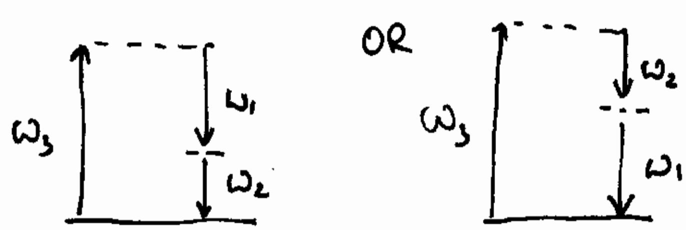
ω_1 is amplified by this process

$\Rightarrow$ This process is known as

<u>Parametric Amplification</u>

- ω_3 (pump)
- ω_1 is amplified by the process (signal wave)
- ω_2 is created by the process (idler wave)

Consider two cases



(a) (b)

(a) Presence of ω_1 causes a transition of ω_2

(b) ω_2 stimulates a transition of ω_1.

Both of these process lead to exponential

Optical Parametric Amplifier (OPA)

Single pass thru crystal

No feedback

Modest gain for single pass

Very useful for tunable wavelength conversion of a CW laser

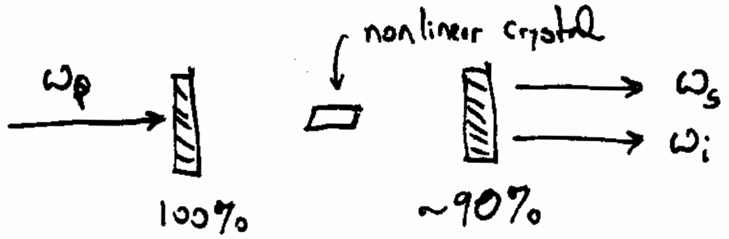
$$\text{gain} \simeq \left(L/L_{NL}\right)^2 \sinh^2\left(\frac{\left(L/L_{NL}\right)^2 - \left(\Delta k\, L/2\right)^2}{\left(L/L_{NL}\right)^2 - \left(\Delta k L/2\right)^2} \right)$$

$$L_{NL} = \frac{1}{4\pi d_{eff}} \sqrt{\frac{2\varepsilon_0 n_p n_s n_i c \lambda_s \lambda_i}{I_p(0)}} \qquad \left\{ \begin{array}{l} p \equiv \text{pump} \\ s \equiv \text{signal} \\ i \equiv \text{idler} \end{array} \right.$$

For $\Delta k = 0$ $\qquad \text{gain} \simeq \left(L/L_{NL}\right)^2$

Optical Parametric Oscillators (OPO)

Put the nonlinear process in a cavity with mirrors that are highly reflecting a ω_1 or ω_2



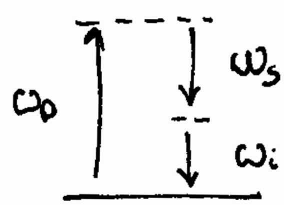
$$\left\{ \begin{array}{l} \omega_p \Rightarrow \omega_3 \\ \omega_s \Rightarrow \omega_1 \\ \omega_i \Rightarrow \omega_2 \end{array} \right.$$

Difference frequency generation leads to the amplification of the lower frequency input field.

$$\omega_i = \omega_p - \omega_s$$

The gain associated with parametric amplification can in the presence of feedback provide an oscillation.

Mirrors can reflect both ω_i and/or ω_s



$\omega_i < \omega_s$ (also called Parametric Down Conversion)

For the case where $\Delta k = 0$ $\quad A_2(0) = 0$ and $A_3(z) \simeq A_3(0)$

$$A_1(z) = A_1(0)\cosh(gz) \quad \text{(an exponential function)}$$

$$A_2(z) = i\left(\frac{n_1\omega_2}{n_2\omega_1}\right)^{1/2} \frac{A_3(0)}{|A_3(0)|} A_1^*(0)\sinh(gz)$$

Both signal + idler experience exponential growth.

** But is an OPO a laser?! **

Is an OPO a laser?

An OPO Has:

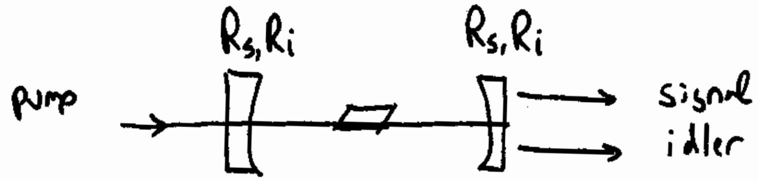
1) A pump source
2) A cavity for feedback
3) Gain at a specific frequency

Answer: No!

A laser has a population inversion caused by the pump. An OPO does not have a population inversion. So it is technically not a laser.

ihe problem with a laser is saturation when the upper population gets too large. An OPO does not have this problem!

Threshold for Parametric oscillation for a Doubly Resonant OPO



$R_i \equiv$ idler reflectivity
$R_s \equiv$ signal ref.

Threshold of oscillation

$$\text{Gain per pass} \equiv \text{loss per pass}$$

$$\left(\exp(2L/L_{NL}) - 1\right) = (1 - R_s)(1 - R_i)$$

For low loss

$$(L/L_{NL})^2 \simeq (1 - R_s)(1 - R_i)$$

For single resonant OPO

$$(L/L_{NL})^2 \sim 2(1 - R_s)$$

Tuning and Bandwidth

Wavelength tuning is typically done by temperature tuning.

Must satisfy momentum + energy conservation

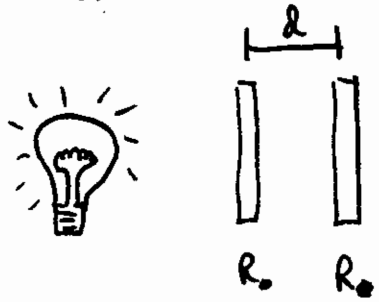
$$\omega_s + \omega_i = \omega_p \qquad \vec{k}_s + \vec{k}_i = \vec{k}_p$$

Phase Matching $\Rightarrow \omega_p\left[n(\omega_p) - n(\omega_p - \omega_s)\right] = \omega_s\left[n(\omega_s) - n(\omega_p - \omega_s)\right]$

Can also be done using angle tuning

Review of Cavity modes

Fabry Perot Interferometer : Interference by multiple reflections.



$\Rightarrow$? Transmission $\Big\{$ $r^2 = R$

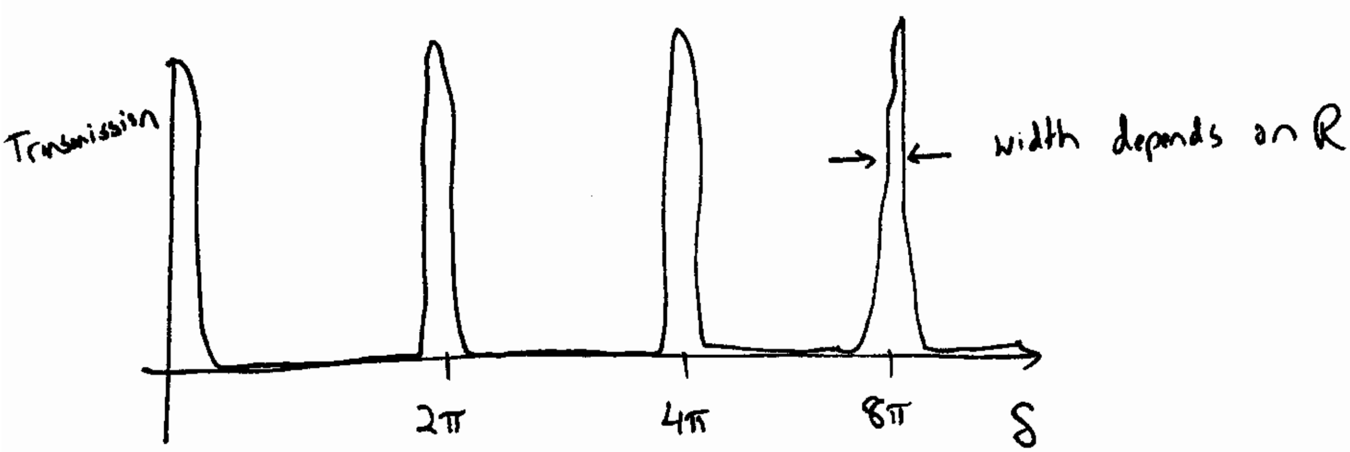
Interference from reflections within cavity causes a variation in the transmission as a function of frequency.

Transmission

$$\frac{I(\delta)}{I_{max}} = \frac{1}{\left(1 + \frac{4}{\pi^2}\mathcal{F}\sin^2(\delta/2)\right)}$$

$$\mathcal{F} \equiv \text{finesse} \equiv \frac{\text{Free spectral Range}}{\text{Smallest resolvable wavelength}} = \frac{\pi}{2}\left(\frac{2r}{1-r^2}\right)$$

$$\delta \equiv \frac{4\pi d n \cos\Theta}{\lambda_0} \quad \text{Phase difference}$$



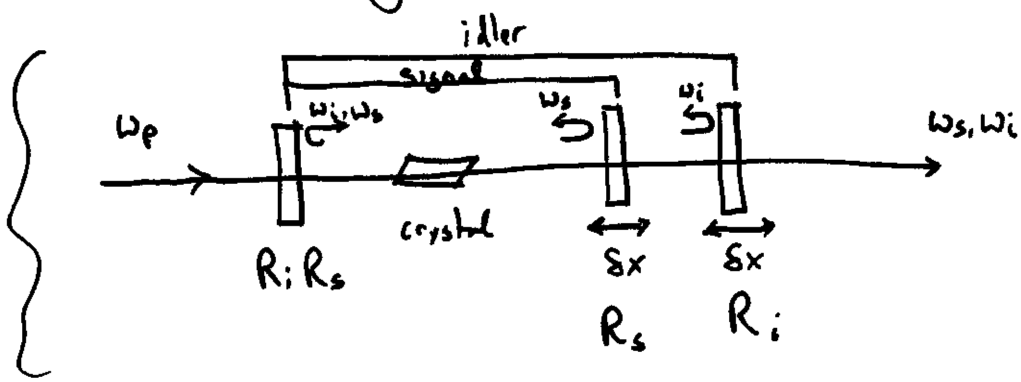
Notice that this is valid for only one wavelength λ_0
If we change λ_0 we need to change the distance d
to maximize the transmission of the Fabry Perot interferometer.

For an OPO

An OPO, the cavity length must correspond to maximum transmission at either the signal or idler wavelength.

In practice, this is difficult to do for both the signal and idler at the same time. Typically this is accomplished using two cavidies.

Double Resonant OPO



Comments on Homework

1. Pulse calculator $\Rightarrow$ www.topica.com

2. Need to write better explanations for solution.
   Add better written discussions to answers.

3. Average vs. Peak Power

$$P_0 = P_{ave} \frac{T}{\Delta t}$$

Better expression

$$P_0 = \frac{P_{ave}}{\frac{1}{T}\int_{-T/2}^{T/2} \operatorname{sech}^2(\eta\, t/\Delta t)\, dt} \qquad \underline{\eta \equiv 2\operatorname{sech}^{-1}(\sqrt{2})}$$

put pulseshape here

$$P_0 \simeq P_{ave} \frac{T}{\Delta t} \frac{1}{1.134} \quad (\operatorname{sech}^2)$$

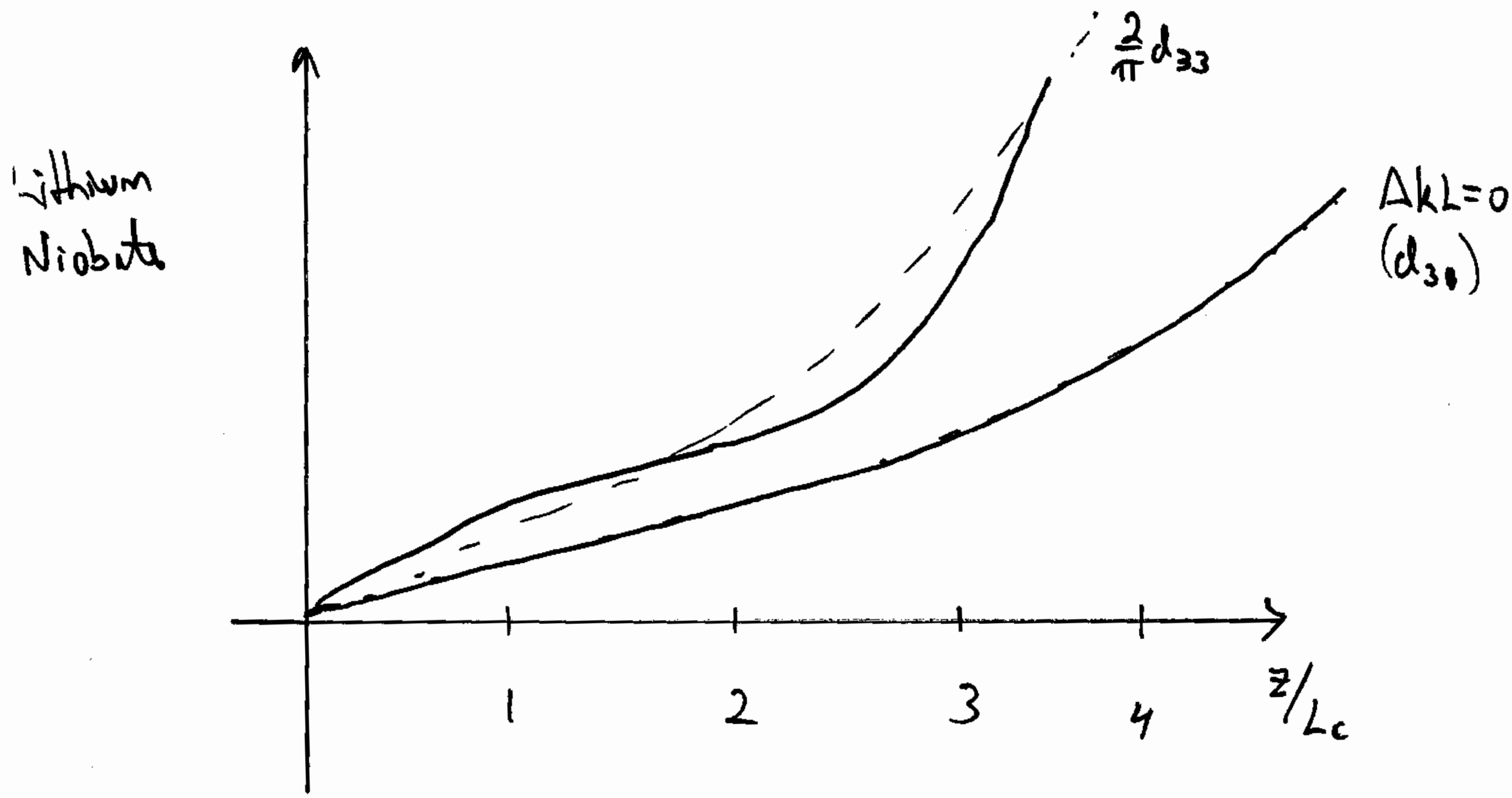
4. Converting $I(\omega)$ to $I(\lambda)$

Need to scale intensities as long as the absuissa!
Energy Conservation requires $|I(\lambda)d\lambda| = |I(\omega)d\omega|$

$$\frac{d\lambda}{d\omega} \Rightarrow \frac{\lambda^2}{2\pi c} \qquad \boxed{I_\omega(\omega) = \frac{\lambda^2}{2\pi c} I_\lambda(\lambda)} \qquad \boxed{\lambda = \frac{2\pi c}{\omega}}$$

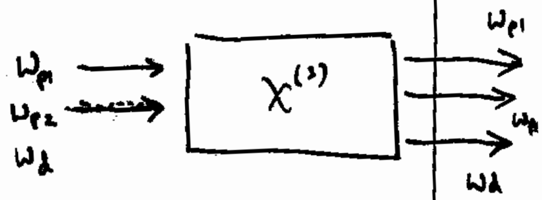
Important for broad band pulses $\Delta\lambda > 30\text{nm}$

This graph is not a proper representation of the situation. QPM allows one to use a larger, on diagonal term of d_{ij} than the off-diagonal term for birefringent phase matching.



Difference Frequency generation

$$\underset{\uparrow \text{idler}}{\omega_d} = \underset{\text{pump}}{\omega_{p1}} - \underset{\text{signal}}{\omega_{p2}}$$



Pump depleted solution

~~Normal~~ solution $(\Delta k = 0)$

$$\eta_d \sim sn^2 [\, i\, z/L_{NL},\ i\gamma \,]$$

Note that $\quad sn^2 [\, i\, z/L_{NL},\ i\gamma \,] \rightarrow \sin(i\, z/L_{NL})$

$$= \sinh(z/L_{NL})$$

as $\gamma \rightarrow 0$

↑ strong pump / No pump depletion

Look at Figures 9 & 10

Figure 9. $\quad \gamma \ll 1$

exponential increase in signal

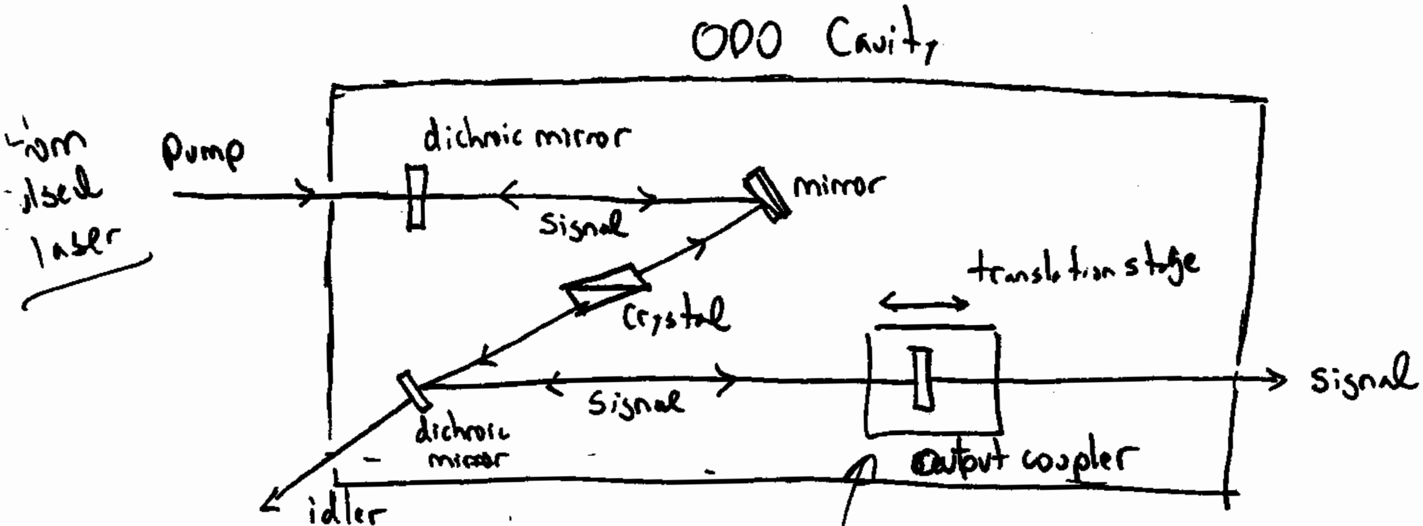
Figure 10 $\quad \gamma \simeq 1$

Equal pump & signal photon Flux

Figure 13 $\quad \gamma \ll 1$ with pump depletion

To match the cavity length & Rep Rate

Use following ~~mirror~~ cavity



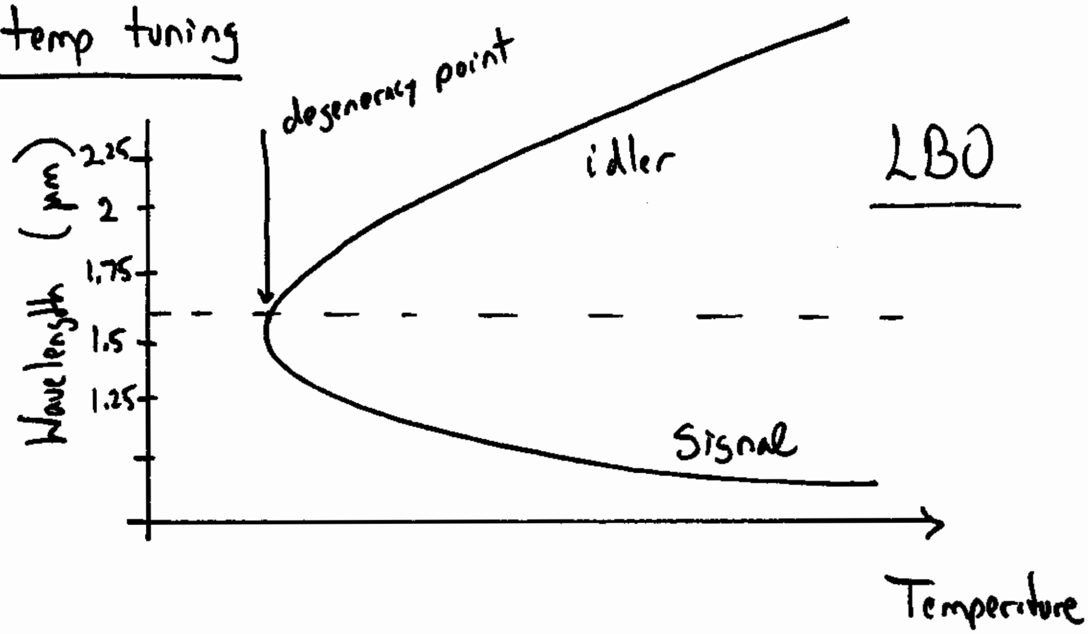
this translation stage with μm resolution is used to ~~get~~ match the cavity length to the repetition rate.

$$\tau_{RT} = \frac{2L_c}{c} N(\lambda) + \frac{2L_F}{c}$$

$$= \frac{1}{F} = T$$

Output coupler (~99% for ω_s) is on translation stage

Example of temp tuning



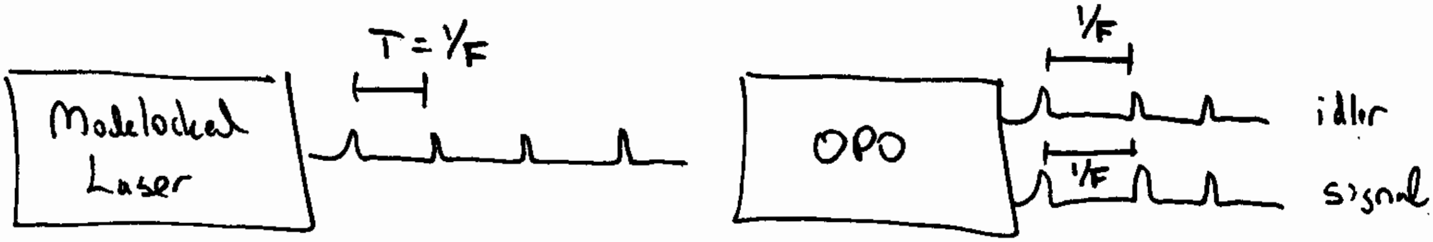
Pump $\lambda_p = 800\,nm$

Huge tuning range: signal $1 - 1.5\,\mu m$; idler $1.5 - 2.25\,\mu m$

Synchronously Pumped OPO

Use a pulsed laser for the pump

Get tunable wavelength pulsed light $\Rightarrow$ signal + idler



The second order process is modest even for $\Delta k = 0$. For efficiency signal + idler generation, the round trip time of the OPO cavity must be matched to the mode locked laser repetition rate.

$$\tau_{RT} = \frac{2L_c}{c} N(\lambda)_{crystal} + \frac{2L_F}{c} = \frac{1}{F}$$

$L_c \equiv$ crystal length

L_F = cavity

3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER
COMET

$$\mathcal{F}\{\exp(i\omega_0 t)\} = \delta(\omega-\omega_0)$$

$$\mathcal{F}\{\cos(\omega_0 t)\} = \delta(\omega-\omega_0) + \delta(\omega+\omega_0)$$

$$\mathcal{F}\{\sin(\omega_0 t)\} = -i\left(\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right)$$

$$\mathcal{F}\{\exp(i\omega_0 t)\} = \mathcal{F}\{\cos\omega_0 t\} + i\,\mathcal{F}\{\sin\omega_0 t\}$$

$$= \delta(\omega-\omega_0)$$

Convolutions & Correlations

Convolution

$$f(t) \circledast g(t) \equiv \int f(\tau)\, g(t-\tau)\, d\tau$$

(flip)

FT Convolution Th$\underline{^m}$

$$\mathcal{F}\{ f(t) \circledast g(t) \} = F(\omega)\, G(\omega)$$

Correlation

$$f(t) \star g(t) \equiv \int f(\tau)\, g^*(\tau - t)\, d\tau$$

(no flip + star)

Autocorrelation Th$\underline{^m}$

$$\mathcal{F}\{ f(t) \circledast f(t) \} = \left| \mathcal{F}\{ f(t) \} \right|^2$$

$$= |F(\omega)|^2$$

Spectrum!

So $\mathcal{F}\{ E(t) \circledast E(t) \} = I(\omega)$

Lecture 12 : SHG with ultrashort pulse

So far we discussed SHG for a monochromatic source (a CW laser). For ultrashort pulses, which a comprised a bandwidth of spectral components, SHG occurs for all components. However, perfect phasematching $\Delta k = 0$ only occurs for <u>one</u> spectral component.

For pulses we discussed the group velocity + group index

$$N(\lambda) = n - \lambda \frac{dn}{d\lambda} \qquad v_g = \frac{c}{N(\lambda)} = \frac{dk}{d\omega}$$

We can define a group velocity for the fundamental ω and SHG 2ω.

$$N_\omega \quad N_{2\omega} \quad \Big\} \quad V_{g\omega} \quad V_{g2\omega}$$

In general the two group velocities will <u>not</u> be the same. This is called the <u>group velocity mismatch</u> (GVM)

$$\boxed{\Delta v_{GVM} = -V_{g,2\omega} + V_{g,\omega}} = -\frac{c}{N(\lambda/2)} + \frac{c}{N(\lambda)}$$

It is a measure of the delay between the fundamental and SHG pulse. This mismatch leads to a finite phase matching bandwith between the fundamental + SHG.

But for Type I ooe phase matching

$$\boxed{\Delta k(2\omega, \theta) = \frac{2\omega}{c}\left[n_e(2\omega, \theta) - n_o(\omega) \right]}$$

We can also write the GVM as

$$\boxed{\Delta v_g = \frac{c}{N(2\omega)} - \frac{c}{N(\omega)}}$$

SHG Spectral filtering

The group velocity mismatch leads to a finite spectral bandwith for SHG. Find this bandwith.

$$E_1(z,t) = A_1\left(t - \frac{1}{v_g(\omega_0)} z\right) \exp(-i(\omega_0 t - k(\omega_0) z))$$

$$E_2(z,t) = E_1(z, t-\tau) \Leftarrow \text{Delayed version of } E_1$$

$$E_{SHG}(z,t) = A_{SHG}\left(z, t - \frac{1}{v_g(2\omega_0)} z\right) \exp(-i(2\omega_0 t - k(2\omega_0) z))$$

$$P_{NL}(z,t) = \epsilon_0 \chi^{(2)} E_1 E_2 \Rightarrow \text{assume } \chi \text{ is fast.}$$

should be

$$P_{NL}(z,t) = \epsilon_0 \iint \chi^{(2)}(t-t', t-t'') E_1(z, t-t') E_2(z, t-t'-\tau)\, dt' dt''$$

Example of GVM

0.3 mm KDP crystal

$\lambda_0 = 620\,nm \qquad \lambda_0/2 = 310\,nm$

$$\text{group delay mismatch} = \frac{1}{\Delta v_g} L = 56\,fs$$

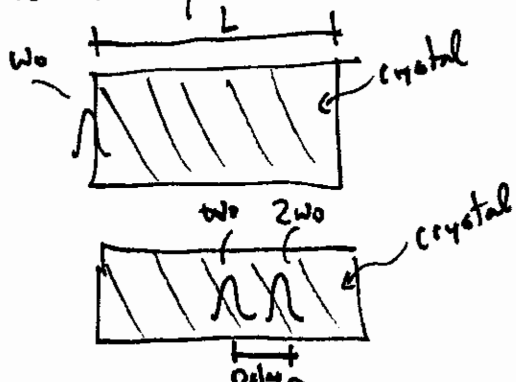
We wish to see how the effect of GVM changes the generated second harmonic spectrum for SHG.

$$I_{SHG}(\omega) = ?$$

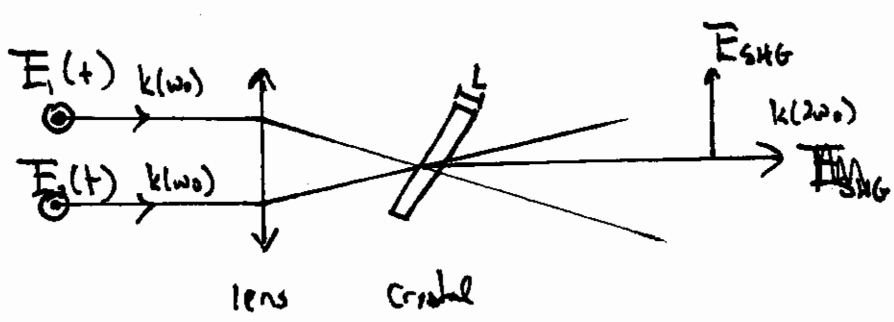
However, due to GVM the fundamental & SHG pulse will not maintain temporal synchronicity for the entire length of the crystal.

$t = 0$

$t =$ later



We wish to derive an expression for $I_{SHG}(\omega)$ for a noncolinear coupling into the crystal.



$E_1 + E_1$ along ordinary axis. E_{SHG} along extraordinary. Note $E_2(t) = E_1(t-\tau)$

The result for $I_{SHG}(\omega)$ is given by:

$$\boxed{I_{SHG}(\omega) = \frac{\sin^2(\Delta k L/2)}{(\Delta k L/2)} I(\omega) \otimes I(\omega)}$$

$\Delta k \equiv$ total phase mismatch. Write the total mismatch for SHG with a spectral bandwidth.

<u>For SHG</u> with pulse. We need to Taylor expand $\Delta k(\omega)$

$$\Delta k(\omega) = \Big[\underbrace{k(\omega_0) + k(\omega_0) - k(2\omega_0)}_{\text{phase matching for } \omega_0 \text{ \& } 2\omega_0 \text{ only}} \Big]$$

phase velocity mismatch

$$+ \underbrace{\left[\frac{\partial k}{\partial \omega}\bigg|_{\omega_0} - \frac{\partial k}{\partial \omega}\bigg|_{2\omega_0} \right]}_{\text{Group velocity mismatch}} \omega$$

$$+ \underbrace{\frac{1}{2}\omega^2 \left[\frac{\partial^2 k}{\partial \omega^2}\bigg|_{\omega_0} - \frac{\partial^2 k}{\partial \omega^2}\bigg|_{2\omega_0} \right]}_{\text{group delay dispersion}}$$

This expansion gives terms that will be nonzero at the phase matching angle. This implies that perfect phasematching cannot be applied for all spectral components of the pulse.

Consider only phase + group velocity mismatch

$$\boxed{\Delta k \cong \Delta k_0 + \frac{1}{\Delta V_g}\omega}$$

GVM

Write each in terms of its spectrum

$$E_1(z,t) = \mathcal{F}^{-1}\{A_1(\omega)\}\exp(-i\,\omega_0 t - k(\omega_0)z)$$

$$E_2(z,t) =$$

$$P_{NL}(z,t) = \epsilon_0 \chi^{(2)} \exp(-i(2\omega_0 t - 2k(\omega_0)z))\, e^{-i\omega_0\tau}$$

$$* \;\mathcal{F}\left\{ \int A_1(\omega-\omega')\, A_1(\omega')\exp(-i\omega'\tau) \right\}$$

Differential Eq using the SVEA

$$\boxed{\partial_z A_{SHG}(z,t) = -\frac{i2\omega_0\mu_0 c}{2n} P_{NL}(z,t)\exp(i\Delta k_0 z)}$$

where $\Delta k_0 = 2k(\omega_0) - k(2\omega_0)$

Write DE in frequency domain

$$\boxed{\partial_z A_{SHG}(z,\omega) = \frac{-i\omega_0\chi^{(2)}}{nc\,2\pi}\exp(-i\omega_0\tau)\exp(i\Delta k z) * \int A_1(\omega-\omega')\,A(\omega')\, e^{-i\omega'\tau}\, d\omega'}$$

where $\Delta k = \Delta k_0 + \frac{1}{\Delta v_g}\omega$

Integrate from 0 to L

$$\boxed{A_{SHG}(\omega, L) \simeq e^{-i\omega_0\tau}\left[\exp(i\Delta k L/2)\right]\left[\frac{\sin(\Delta k L/2)}{\Delta k L/2}\right] \times \int A_1(\omega-\omega')A(\omega')e^{-i\omega'\tau}d\omega'}$$

We want the intensity $I_{SHG}(\omega, L) \simeq |A_{SHG}(\omega, L)|^2$ over all τ.

$$I_{SHG}(\omega) = \frac{\sin^2(\Delta k L/2)}{(\Delta k L/2)^2}\left[\int\left[\int A(\omega-\xi)A(\xi)e^{-i\xi\tau}d\xi\right] \times \left[\int A^*(\omega-\eta)A(\eta)e^{-i\eta\tau}d\eta\right]d\tau\right]$$

But $\delta(\eta-\xi) = \int \exp(i(\eta-\xi)\tau)\,d\tau$ (FT of a delta function). Which allows us to do the integral over τ to get rid of ξ.

$$I_{SHG}(\omega) = \text{sinc}(\Delta k L/2)\int A^*(\omega-\eta)A(\omega-\eta)A^*(\eta)A(\eta)\,d\eta$$

But $A^*(\ )A(\ ) = I(\ )$

So

$$\boxed{\begin{aligned} I_{SHG}(\omega) &= \mathrm{sinc}^2(\Delta k L/2) \int I_1(\omega - \eta)\, I(\eta)\, d\eta \\ &\text{OR} \\ I_{SHG}(\omega) &= \mathrm{Sinc}^2(\Delta k L/2)\; I_1(\omega) \otimes I_1(\omega) \\ &\text{OR} \\ I_{SHG}(\omega) &= H(2\omega)\; I_1(\omega) \otimes I_1(\omega) \end{aligned}}$$

where $H(2\omega) = \dfrac{\sin^2\left(\Delta k(2\omega, \theta) L/2\right)}{\left(\Delta k(\omega) L/2\right)^2}$

The filter function $H(2\omega)$ is evaluated for the case of perfect phase matching

$$\theta = \theta_{pm} \quad \text{where} \quad \Delta k_0 = 0$$

Major Points

1) The width of the SHG spectrum is related to the autoconvolution of the fundamental spectrum.

2) Width of $H(\omega)$ depends on GVM $\Rightarrow \Delta v_g^{-1} L$ and the crystal length.

3) Center of $H(\omega)$ depends on Δk_0

How to calculate the spectral filtering?

1) Find $I_1(\omega)$ and $\Delta\lambda$ of the fundamental pulse

2) Determine $\Delta k(2\omega, \theta)$ given $n_e(\lambda)$ + $n_o(\lambda)$ of your given crystal. Set $\theta = \theta_{pm}$ (perfect phase matching).

3) Find
$$H(2\omega) = \frac{\sin^2\left(\Delta k(2\omega, \theta_{pm}) L/2\right)}{\left(\Delta k(2\omega, \theta_{pm}) L/2\right)}$$

4) Determine the SHG spectrum from $I_1(\omega) \otimes I_1(\omega)$. Find its spectral width $\Delta\lambda_{SHG}$

5) Determine the filtered SHG spectrum using
$$H(2\omega)\left(I(\omega) \otimes I(\omega)\right)$$

6) Make sure the filtered spectrum $\Delta\lambda_{SHG,\,filtered}$ is not significantly different than $\Delta\lambda_{SHG}$

Lecture 15 Applications for SHG

Back to $I_{SHG}(\omega) \Rightarrow$ Problem with Weiner result for chirped pulses

$$\boxed{I_{SHG}(\omega) \simeq \text{sinc}^2(\Delta k L/2) \left| \int E_1(q) E_1(\omega - q)\, dq \right|^2}$$

for transform-limited pulses (or nearly transform-limited pulses)

$$I_{SHG}(\omega) \simeq \text{Sinc}^2(\Delta k L/2) \left[I_1(\omega) \otimes I_1(\omega) \right]$$

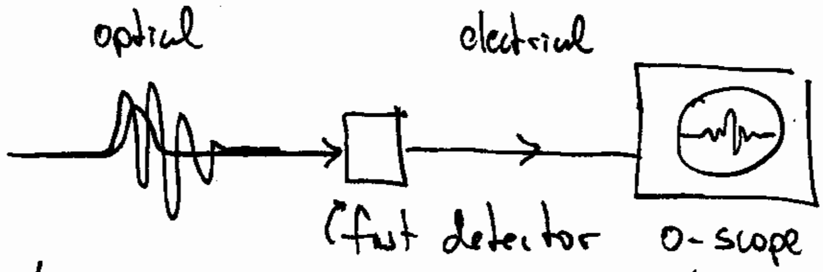
The presence of a phase distortion does not modify the spectral width but does modify the spectral shape

- What about $I_{SHG}(t)$?

The SHG intensity will also be a function of chirp.

Applications Pulse measurement

We really want

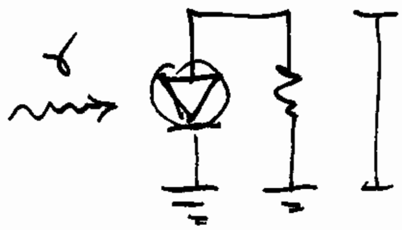


Why can't we do this! Motion of electrons in the detection cannot follow the osillations of the optical electric field.

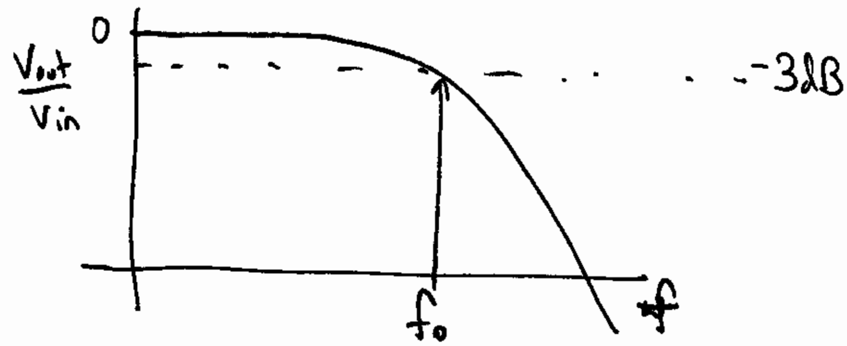
$$V(t) = \mathcal{R}\, h(t) \otimes I(t)$$

Really Fast optical detector

Converts optical power to a current, or a voltage



"Bandwidth" $f_0 \sim$ 100 GHz



$\frac{1}{f_0} \sim 0.1$ ps $\Leftarrow$ not fast enough to resolve the oscillations of the pulse's electric field oscillations

(really more like 1ps)

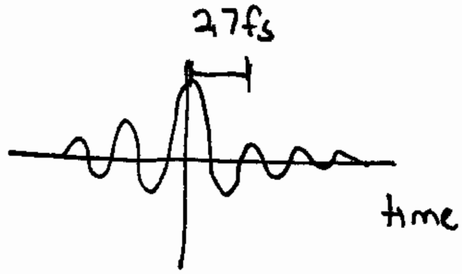
Example

$\lambda_0 = 800$ nm

$\omega_0 = \frac{2\pi c}{\lambda_0} = 2.35$ 1/fs

$f_0 = 0.375$ 1/fs

$1/f_0 = 2.7$ fs (single cycle)



We need a better method

What do we want to measure?

$$\boxed{E(t) \cong \sqrt{I(t)}\ \exp(\, i\phi(t)\,)}$$

$I(t) \equiv$ Temporal Intensity $\quad \phi(t) \equiv$ Temporal Phase

OR we can write

$$\boxed{E(\omega) \cong \sqrt{I(\omega)}\ \exp(\, i\,\phi(\omega)\,)}$$

$I(\omega) \equiv$ Spectral intensity $\quad \phi(\omega) \equiv$ Spectral phase

Can we measure $E(\omega)$ + $E(t)$ directly $\Rightarrow$ Difficult.

earlier measurements provided limited information about $E(t)$

~~Intensity Autocorrelation~~

~~Gives an estimate of the pulse duration and shape~~

~~Really a "guess-estimate"~~

~~Measure $\underbrace{I_{ac}(\tau)}_{\text{autocorrelation}} = \int I(t)\, I(t-\tau)\, dt$~~

~~How to do this $\Rightarrow$ use SHG generation~~

Prove

$$\mathcal{F}\{\Gamma^{(1)}(t)\} = \mathcal{F}\{\int E(t)\, E^*(t-\tau)\,dt\} = I(\omega)$$

$$\mathcal{F}\{\int E(t)\, E^*(t-\tau)\, dt\} = \mathcal{F}\{E(t) \underset{\text{convolution}}{\otimes} E(t)\}$$

From the convolution thm $\mathcal{F}\{g \otimes f\} = \mathcal{F}\{g\}\mathcal{F}\{f\}$

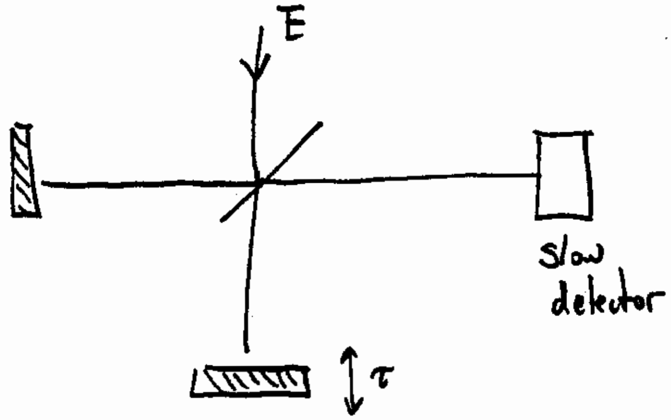
So

$$= \mathcal{F}\{E(t)\}\ \mathcal{F}\{E^*(t)\}$$

$$= E(\omega)\ E^*(\omega)$$

$$= I(\omega)$$

Can we use a Michelson Interferometer?!



Measure interferogram

$$I_M(\tau) \sim \int \left| E(t) - E(t-\tau) \right|^2 dt$$

$$I_m(\tau) \sim \underbrace{2\int |E(t)|^2 dt}_{\text{pulse intensity}} - \underbrace{2\,\mathrm{Re}\left\{ \int E(t)\, E^*(t+\tau)\, dt \right\}}_{\substack{\text{interferogram} \\ \text{Field autocorrelation} \\ (\Gamma^{(2)}(t))}}$$

$$\mathcal{F}\{E(t) \otimes E(t)\} = E(\omega) E(\omega) = I(\omega)$$

$$\text{But} \quad \mathcal{F}\{\Gamma^{(2)}(t)\} = I(\omega)$$

(the spectrum!)

Measuring interferometer is same as measuring the spectrum.

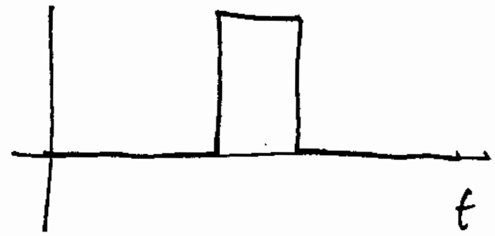
- If you do not have a detector or modulator that is fast compared to the pulse you cannot measure the pulse intensity + phase

Need something that has a faster response $\Rightarrow$ $\chi^{(2)}$ effects!!

Mathematical Picture of an autocorrelation.

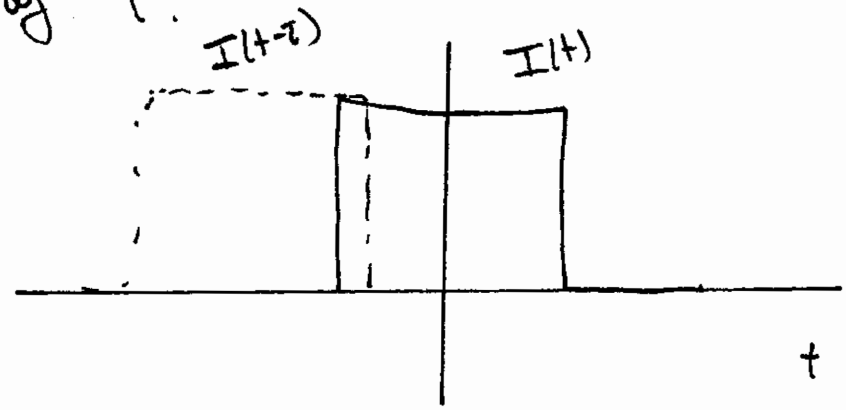
Pulse

$$I(t) = \begin{cases} 1 & |t| \le \Delta t/2 \\ 0 & \text{elsewhere} \end{cases}$$



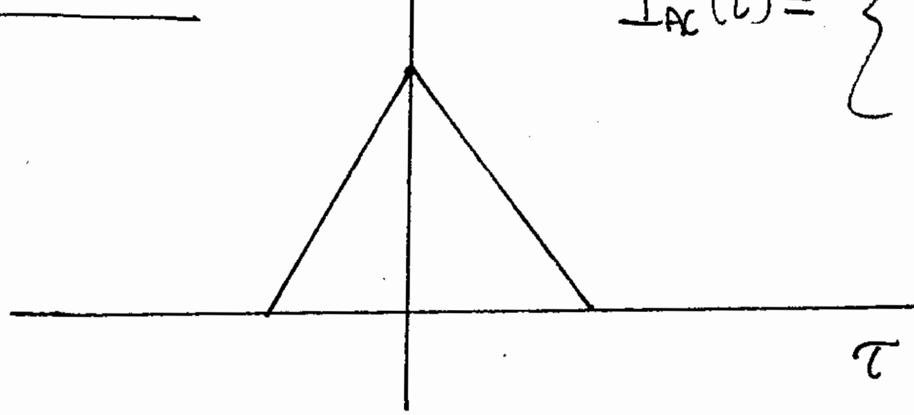
Autocorrelation

Scan a replica of the pulse $I(t-\tau)$ for all values of t.



Autocorrelation

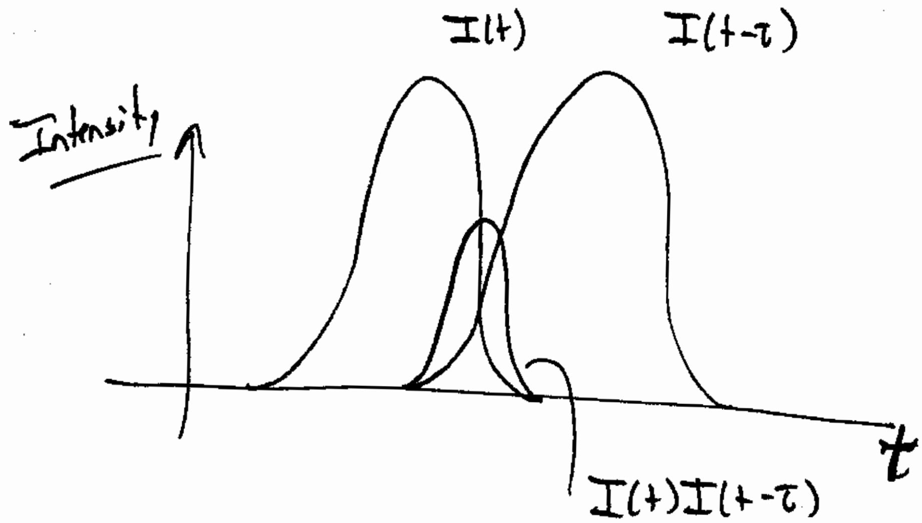
$$I_{AC}(\tau) = \begin{cases} 1 - \left|\dfrac{\tau}{\Delta t_{AC}}\right| & \tau < \Delta\tau \\ 0 & \end{cases}$$



Where $\Delta\tau_{AC} = \Delta t$

Pulse measurement in time domain: Intensity AC

Wish to overlap two pulses in a crystal as a function of delay τ



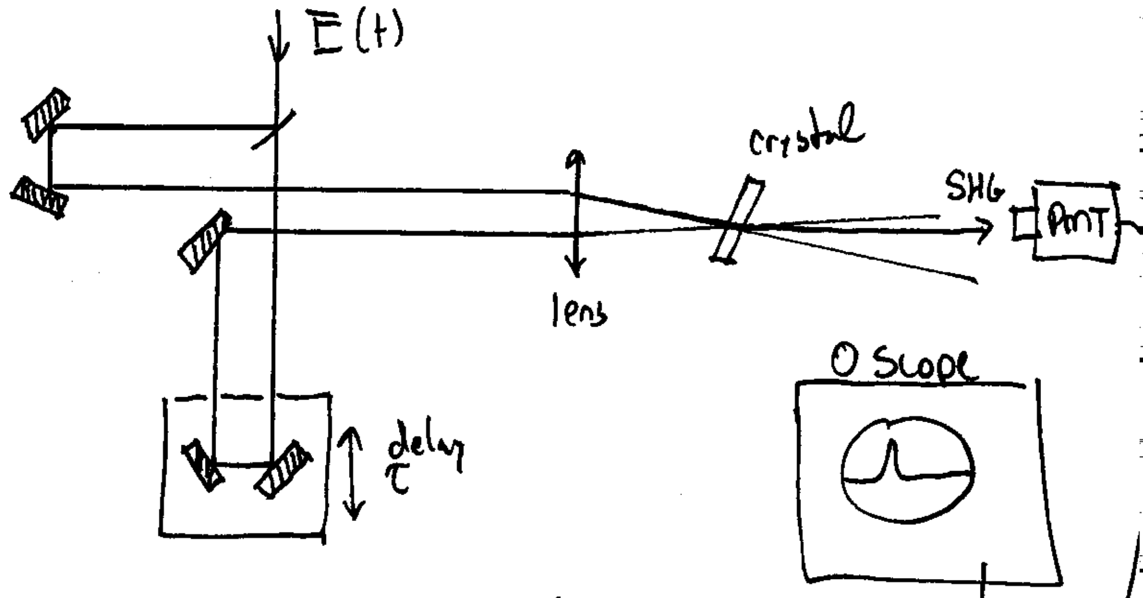
$$I_{AC}(\tau) \equiv \int I(t)\, I(t-\tau)\, dt$$

The second harmonic intensity is a function of τ and the temporal overlap in the nonlinear crystal

The response of the medium provides the fast temporal resolution to measure the pulse duration.

Note that $\quad I_{AC}(\tau) = I_{AC}(-\tau)$

Setup



Scan delay τ and measure intensity $I(\tau)$

Due to the non colinear Focusing in the crystal, SHG ~~MM~~ into the PMT (photomultipler tube) will be detected for an ~~over~~ temporal overlap. The SHG intensity as a function of delay will be the autocorrelation of the fundamental intensities.

$$I_{AC}(\tau) \sim \int I(t)\, I(t-\tau)\, d\tau$$

Advantages

- Can scan τ at any rate
- gives a "pretty good" estimate of maximum Δt.

Disadvantages

- Need to guess form of $E(t)$ to get estimate of Δt
- Does not measure $E(t)$ or even $I(t)$
- Autocorrelations are not unique $\Rightarrow$ multiple $E(t)$ will produce the same autocorrelation.
- Cannot get $\phi(t)$ or $\phi(\omega)$

Comparison of Ultrashort Pulse Functional Forms for the electric field: Gaussian and Sech

Brian Washburn version 1 9/21/07

```
Off[General::spell];
<< Graphics`Graphics`
```

I wish to plot a sech^2pulse and a Gaussian pulse with the same intensity full width at half maximum and peak power.

```
Δt = 100;  Po = 1;
```

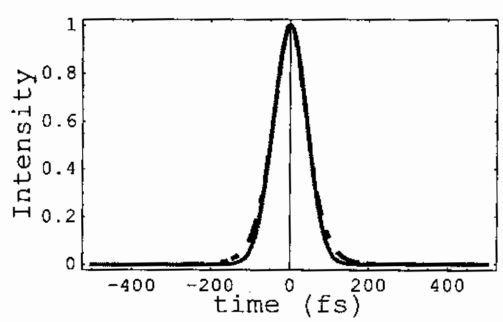
$$eg[t\_] = \sqrt{P_o}\ Exp\left[-2\ Log[2]\ \left(\frac{t}{\Delta t}\right)^2\right];\quad es[t\_] = \sqrt{P_o}\ Sech\left[2\ ArcSech\left[\sqrt{0.5}\right]\ \frac{t}{\Delta t}\right];$$

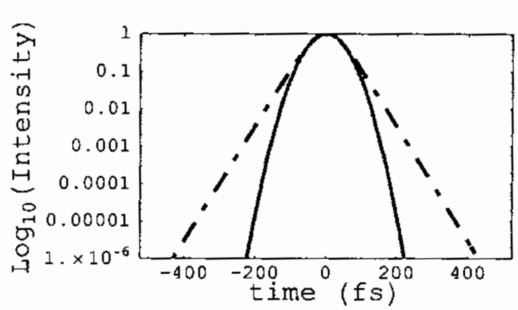
```
Ig[t_] = eg[t] * Conjugate[eg[t]];   Is[t_] = es[t] * Conjugate[es[t]];
```

Here I plot both pulse shapes. The Gaussian is the solid line and the sech^2 is the dotted line. The hyperbolic secant pulse has wider wings, which is quite pronounced on the Log plot.

```
p1 = Plot[{Ig[t], Is[t]}, {t, -500, 500},
   Frame -> True, PlotRange -> {All, All}, FrameLabel ->
    {StyleForm["time (fs)", FontSize → 14], StyleForm["Intensity", FontSize → 14] },
   PlotStyle → {{RGBColor[1, 0, 0], Thickness[0.01]}, {RGBColor[0, 0, 1],
      Dashing[{0.01, 0.05, 0.05, 0.05}], Thickness[0.01]}}, DisplayFunction → Identity];
p2 = LogPlot[{Ig[t], Is[t]}, {t, -500, 500}, Frame -> True, PlotRange -> {All, {10^-6, 1}},
   FrameLabel -> {StyleForm["time (fs)", FontSize → 14],
     StyleForm["Log10(Intensity)", FontSize → 14] },
   PlotStyle → {{RGBColor[1, 0, 0], Thickness[0.01]}, {RGBColor[0, 0, 1],
      Dashing[{0.01, 0.05, 0.05, 0.05}], Thickness[0.01]}}, DisplayFunction → Identity];

Show[GraphicsArray[{p1, p2}]];
```





Lecture 10

AC width

$$\Delta t_{AC} = 100\,fs$$

Assume sech^2 0.6482

$$(100\,fs)\,0.6482 = \frac{100\,fs}{1.543} = 64.8\,fs$$

Assume Gaussian 0.7071

$$100\,fs\,(0.7071) = \frac{100\,fs}{\sqrt{2}} = 70.71\,fs$$

If actually gaussian

$$\frac{|64.8 - 70.71|}{70.1} \times 100\% = \boxed{8.4\%}$$

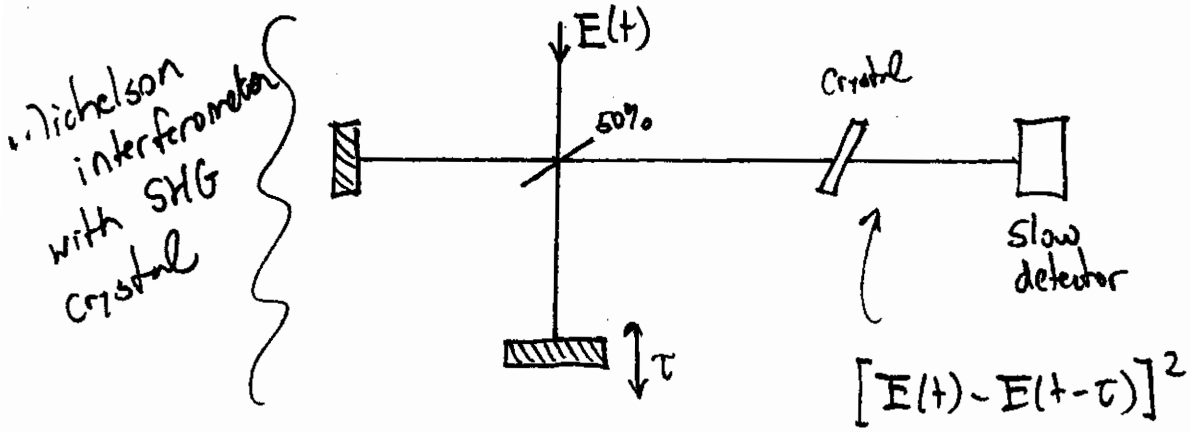
~~AC(Gaussian) → Gaussian~~

$$AC\left(\text{sech}^2\left(\tfrac{t}{T}\right)\right) \rightarrow \frac{3}{\sinh^2(t/T)}\left(\frac{t}{T}\coth t/T - 1\right)$$

$$AC\left(e^{-t^2/T^2}\right) = e^{-t^2/2T^2}$$

$$\boxed{\text{Look at Mathematica!!}}$$

Another method: Interferometric Autocorrelator



$$I_{IAC}(\tau) = \int_{-\infty}^{\infty} \left| [E(t) - E(t-\tau)]^2 \right|^2 dt$$

Notice the difference from the Michelson with + without the SHG crystal

without $\quad I_M(\tau) \sim \int_{-\infty}^{\infty} |E(t) - E(t-\tau)|^2 \, dt$

with $\quad I_{IAC}(\tau) \sim \int_{-\infty}^{\infty} \left| [E(t) - E(t-\tau)]^2 \right|^2 dt$

$$I_{IAC}(\tau) \equiv \int_{-\infty}^{\infty} |E^2(t) + E^2(t-\tau) - 2E(t)E(t-\tau)|^2 \, dt$$

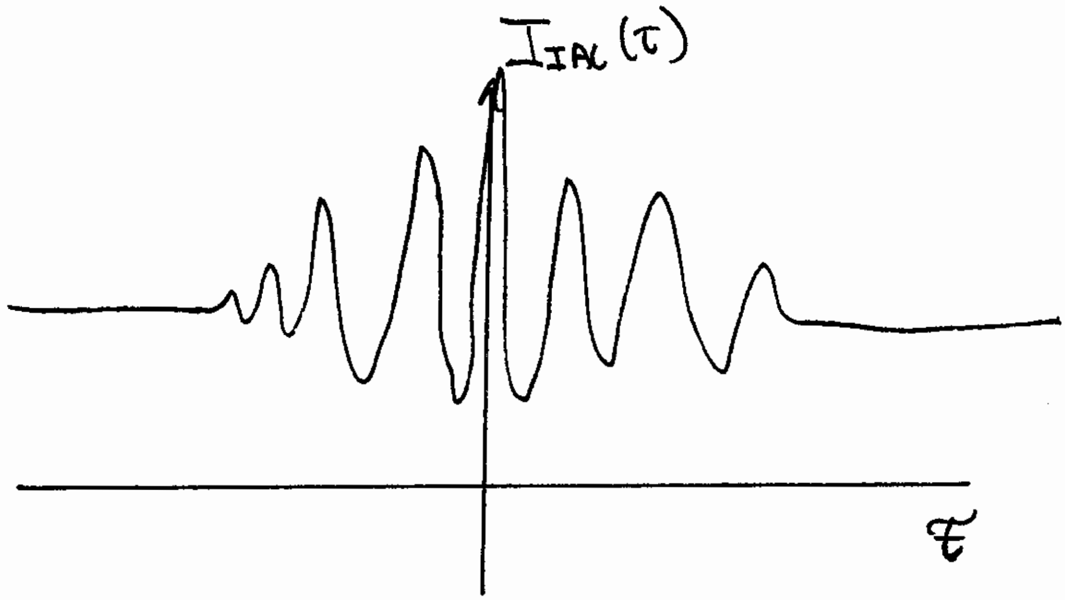
Expand this

$$I_{IAC}(\tau) = \int_{-\infty}^{\infty} \left(I^2(t) + I^2(t-\tau) \right) dt \qquad \text{(constant)}$$

$$+ 4 \int_{-\infty}^{\infty} I(t)\, I(t-\tau)\, dt \qquad \text{(Intensity AC)}$$

$$+ 2 \int_{-\infty}^{\infty} \left[I(t) + I(t-\tau) \right] E(t) E^*(t-\tau)\, dt + c.c. \qquad \left(\begin{array}{c} \text{Sum of intensities} \\ \text{weighted interferogram} \\ (\omega_0) \end{array} \right)$$

$$+ \int_{-\infty}^{\infty} E^2(t) E^{2*}(t-\tau)\, dt + c.c. \qquad \left(\begin{array}{c} \text{Interferogram of second} \\ \text{harmonic} \\ 2\omega_0 \end{array} \right)$$



Advantages:

- Measure pulse duration.
- Qualitative test of phase modulation, Quantitative for linear chirp
- Method to get complete $E(t)$ using pulse spectrum (PICASSO)

Lecture 16 More Applications of SHG : FROG

Frequency Resolved Optical Gating

Pulse characterization — What do we wish to measure?

<u>Intensity</u> + <u>Phase</u>

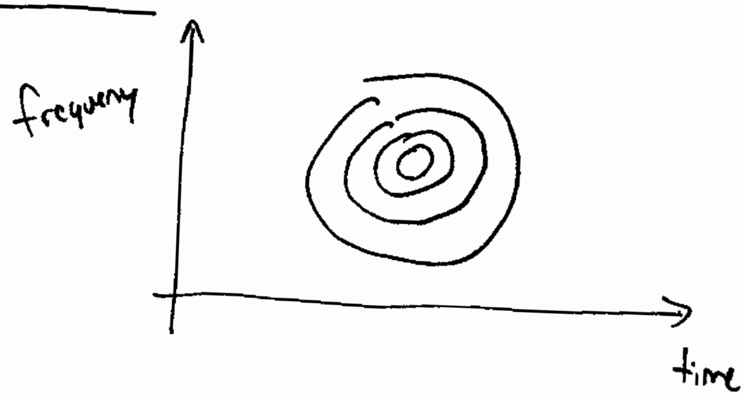
$$I(t) \qquad \phi(t)$$
$$+$$
$$I(\omega) \qquad \phi(\omega)$$

Intensity AC does not provide this information

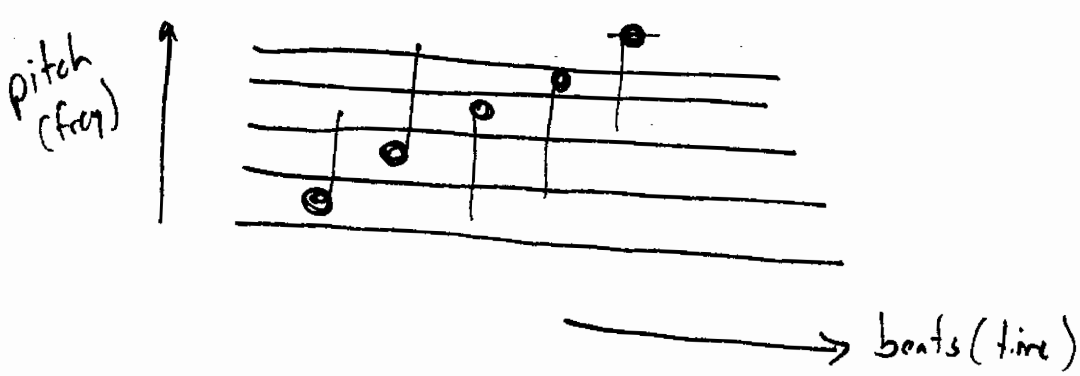
$$I_{AC}(\tau) \sim \int I(t)\, I(t-\tau)\, dt$$

Not enough data here to provide intensity + phase. Can we somehow get more data?

<u>Time frequency Domain</u>



Like a musical score



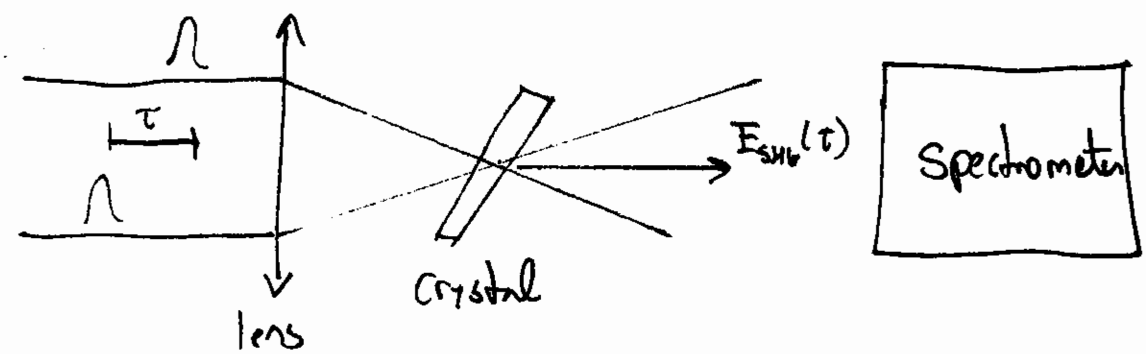
We can set up an experiment to measure the spectrogram or time-frequency representation of the pulse

$$I^{SHG}_{FROG}(\omega, \tau) = \left| \int_{-\infty}^{\infty} E(t)\, E(t-\tau)\, e^{-i\omega\tau}\, dt \right|^2$$

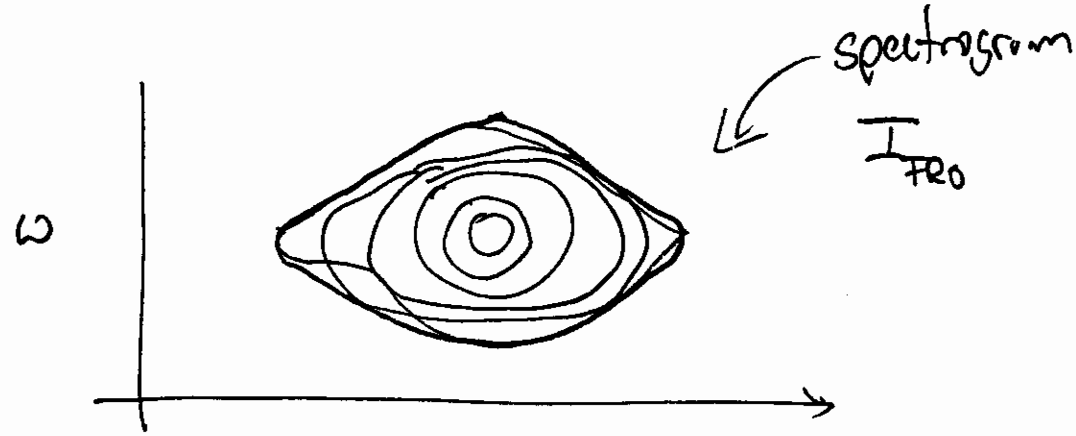
Form of nonlinearity

SHG FROG TRACE

How to do this? Use our intensity autocorrelator . . . .



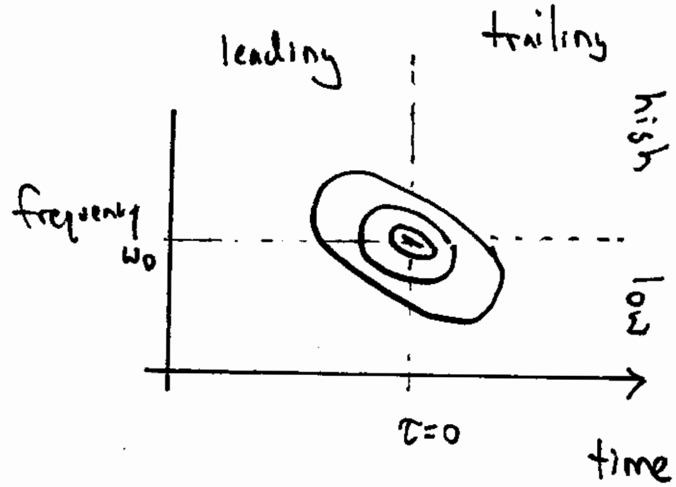
Here we spectrally resolve the SHG as a function of τ



Spectrally resolved intensity autocorrelation.

FROG traces are very intuitive

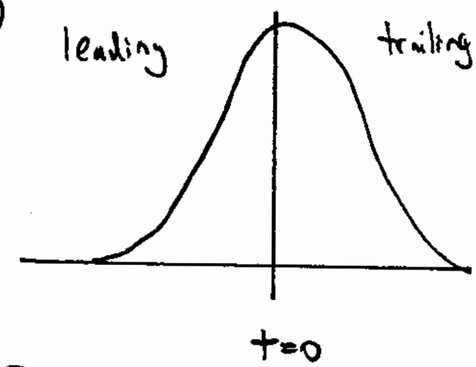
FROG

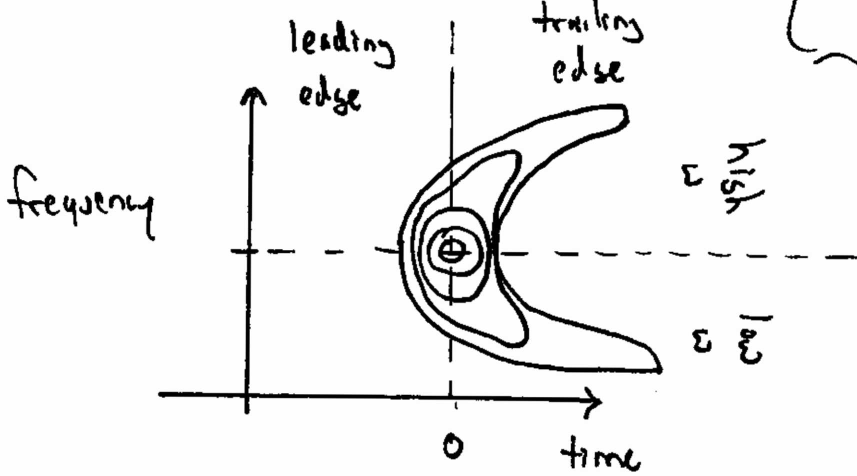


This spectrogram tells me that the higher frequency components of a pulse arrive before the lower frequency components

$\Rightarrow$ negative chirp

Aside: Pulse in time





This shape tells me that the center frequency arrives 1st the the ~~[illegible]~~ high + low

<u>Except!</u> SHG spectrograms are always symmetric about τ + ω!!

Not intuitive!!

FROG Algorithm : 2D Phase retrieval

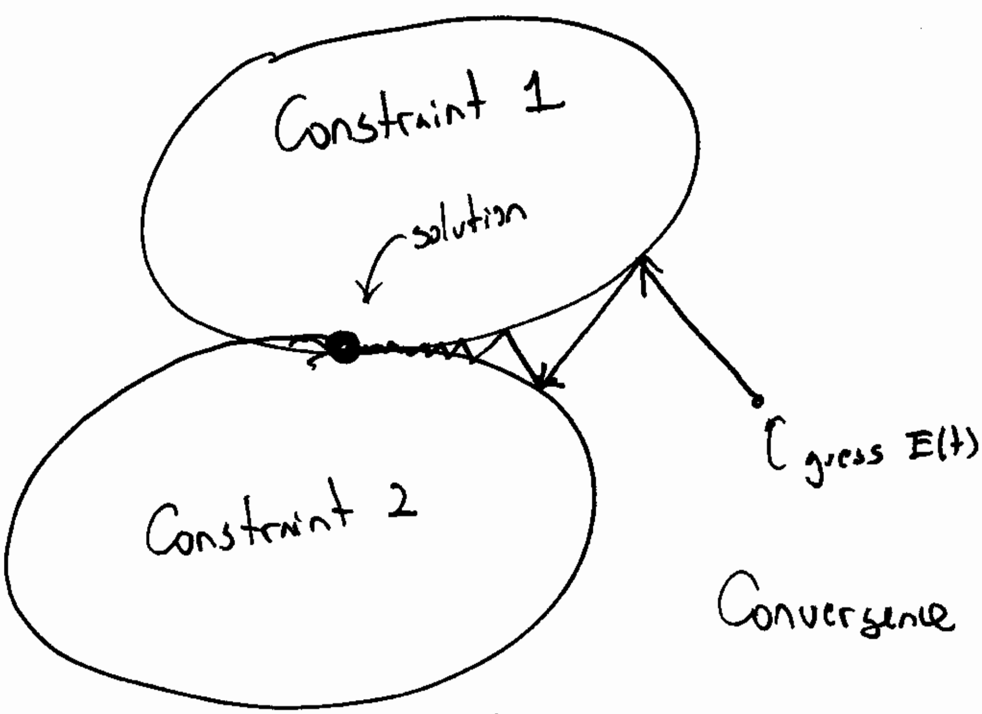
An iterative method to find Intensity + phase.
Solution must satisfy two constraints:

Constraint 1 : Set of $E(t)$ that satisfy

$$E_{sig}(t,\tau) \sim E(t)\,E(t-\tau)$$

Constraint 2 : Set of $E(t)$ that satisfy

$$I_{FROG}(\tau,\omega) = \left| E_{sig}(t,\tau)\exp(-i\omega t)\,dt \right|^2$$



Convergence is <u>not</u> guarenteed.

SHG FROG Experimentally is a spectrally resolved intensity autocorrelation.

In general FROG can be used with other nonlinearities.

$$I_{FROG}(\tau,\omega) = \left| \int E_{sig}(t,\tau)\, \exp(-i\omega t)\, dt \right|^2$$

$$E_{sig}(t,\tau) \sim \begin{cases} E(t)\,|E(t-\tau)|^2 & \text{polarization gate} \\ E^2(t)\, E^*(t-\tau) & \text{self diffraction} \\ E(t)\, E(t-\tau) & \text{SHG} \\ E^2(t)\, E(t-\tau) & \text{Third harmonic generation} \end{cases}$$

FROG consists of two parts

1) Measurement apparatus

2) 2D Phase retrieval algorithm
(FROG Algorithm)

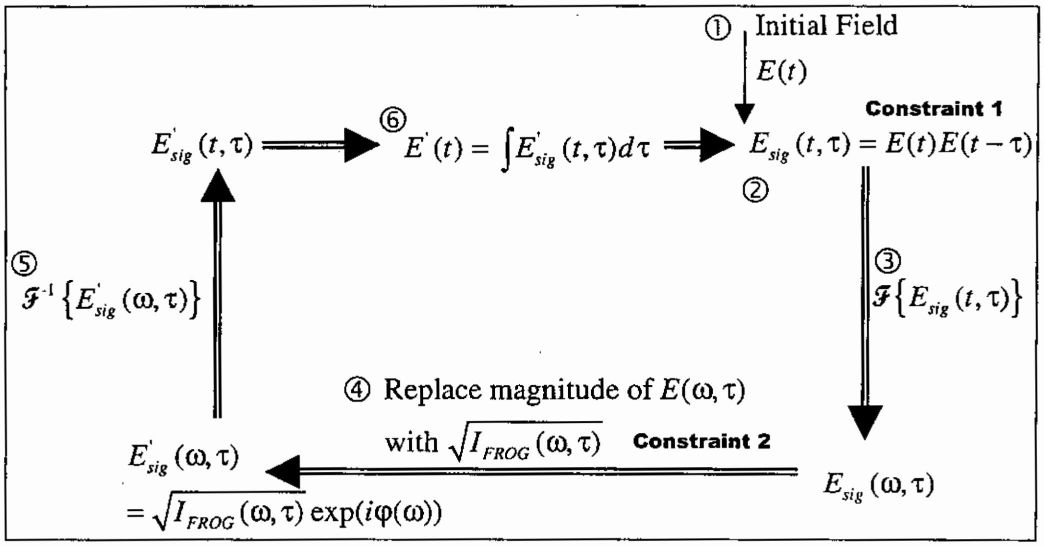


Figure 2.15 The FROG algorithm with generalized projections.

The steps of the FROG algorithm are:

① First, an initial guess electric field, $E(t)$, is generated, typically intensity noise or Gaussian profile.

② The quantity $E_{sig}(t,\tau)$ is calculated by Eq. (2.15), applying Constraint #1.

③ The quantity $E_{sig}(\omega,\tau)$ is determined using the 1D Fourier transform with respect to t.

④ In the frequency domain, the magnitude of $E_{sig}(\omega,\tau)$ is replaced by the experimental spectrogram $\sqrt{I_{FROG}(\omega,\tau)}$ while the phase is kept the same, applying Constraint #2.

⑤ The 1D inverse Fourier transform is performed to obtain $E'_{sig}(t,\tau)$.

⑥ Finally, a new $E'(t)$ is calculated from $E'_{sig}(t,\tau)$.

The new field $E'(t)$ is used as the new input to step ② and the process repeats. At the k^{th} iteration the FROG error G is calculated by

$$G = \sqrt{\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left[I_{FROG}^{(k)}(\omega_i,\tau_j) - I_{FROG}(\omega_i,\tau_j)\right]^2} \ . \tag{2.37}$$

FROG Advantages

- Provides information on intensity & phase (if you trust the algorithm)
- Has "self-checks" built in

  temporal & frequency marginals

FROG Disadvantages

- A bit of a complicated experiment
- Algorithm a "black box"
- Susceptible to systematic errors

  crystal thickness, misalignment, etc.

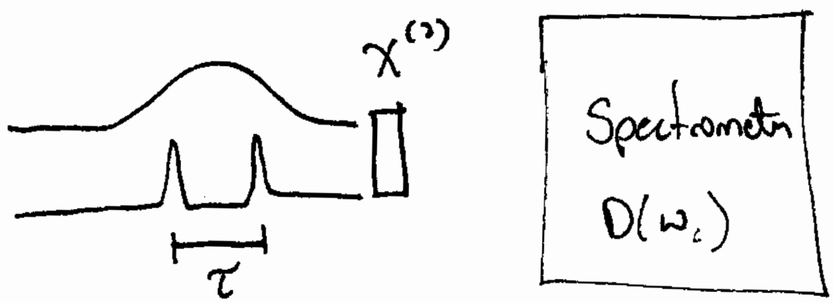
- SHG FROG does not give the sign of phase distortion.

Other Pulse measurement techniques

Spectral phase interferometry for direct electric field reconstruction (SPIDER)

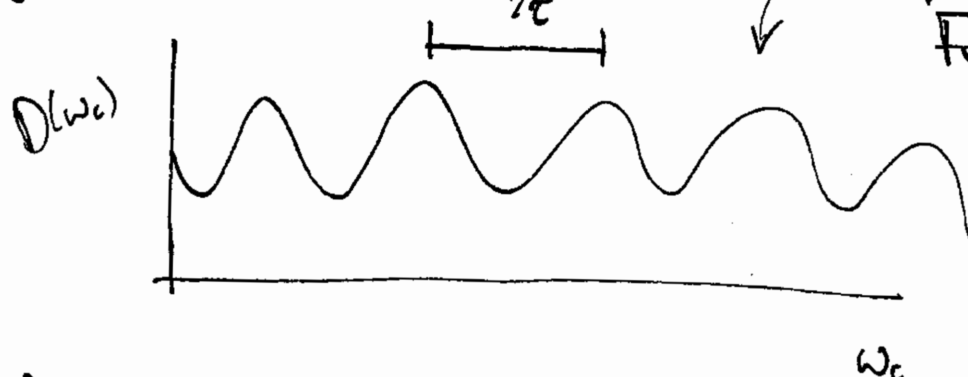
Idea:

Two replicas of the pulse are mixed with a highly chirped pulse in a nonlinear crystal



$$D(\omega_c) = \left|E_{1SHG}(\omega_c) + E_{2SHG}(\omega_c)\right|^2$$

Get interferogram



Get phase from Fringes

Spectral interferometry + mixing in a nonlinear crystal

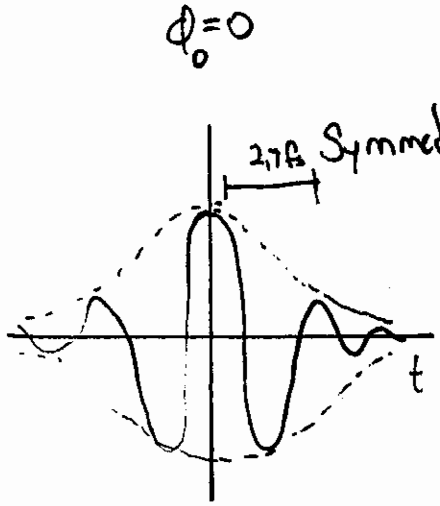
Lecture 17 Carrier Envelope Phase

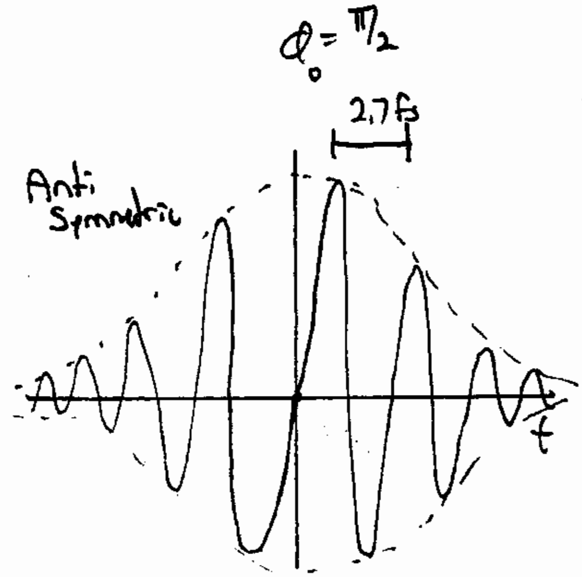
⇒ The Frequency comb

Consider the "real" electric field of a pulse

absolute phase

$$\mathcal{E}(t) = \mathcal{E}_0 \operatorname{sech}(t/\Delta t) \cos(\omega_0 t + \phi_0)$$



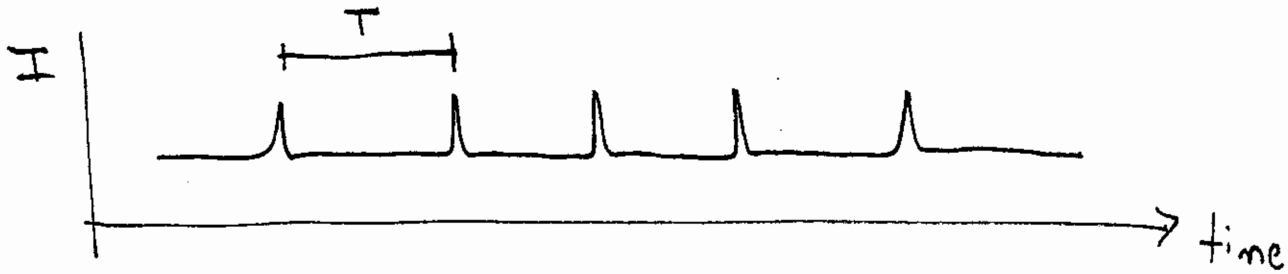


For a 800nm the period of a single cycle is 2.7 fs

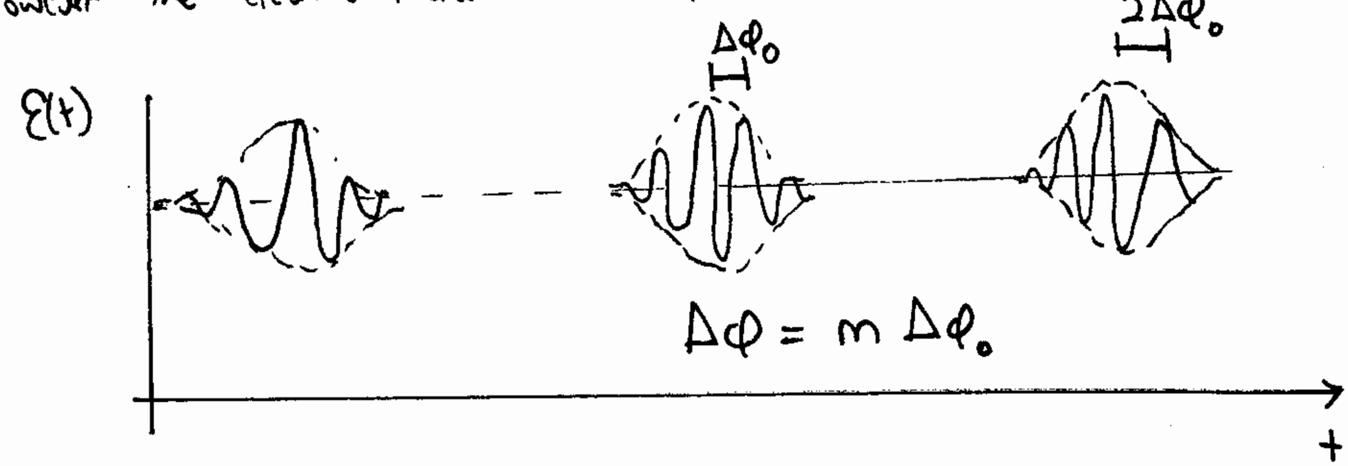
Perhaps we wish to control the absolute phase ϕ_0 of a pulse.

But we never have just <u>one</u> pulse. A mode locked laser produces many pulses, millions per second!!

We have a pulse train



However the electric field looks quite different



In the laser cavity the group + phase velocities are not the same and <u>not</u> constants over time.

$$\boxed{\Delta\phi_0 = \left(\frac{1}{v_g} - \frac{1}{v_p}\right) L\,\omega_0}$$

"ational Ignition Facility
1.8 MJ
3 times a day

$\Longrightarrow$ But $v_g(t)$ + $v_p(t)$ due to cavity flucuations

Each sucessive pulse will pick up an extra $\Delta\phi$ <u>even if</u> there are no cavity flucuations.

3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER
COMET

Simple derivation of Comb

pulse train

$$E(t) = \mathcal{E}_0 \operatorname{sech}(t/\Delta t)\, e^{i(\omega_0 t + \phi)} \circledast Ш(t-T)$$

↑ Shah function

$$\boxed{E(t) = \mathcal{E}_0 \operatorname{sech}(t/\Delta t)\, e^{i(\omega_0 t + \phi)} \circledast \sum_{m=-\infty}^{\infty} \delta(t - m/F)}$$

Shah $\quad Ш(t-T) = \sum_{m=-\infty}^{\infty} \delta\left(t - \frac{m}{F}\right) \qquad F = \frac{1}{T}$

$$\mathcal{F}\{E(t)\} = \mathcal{F}\left\{ \mathcal{E}_0 \operatorname{sech}(t/\Delta t)\, e^{i(\omega_0 t + \phi)} \circledast Ш\left(t - \frac{m}{F}\right)\right\}$$

$$= \mathcal{F}\{ \mathcal{E}_0 \operatorname{sech}(t/\Delta t)\, e^{i(\omega_0 t + \phi)}\}\ Ш(f - mF)$$

$$= e^{i\phi_0}\ \mathcal{E}_0 \operatorname{Sech}\left(\frac{\omega-\omega_0}{\Delta\omega}\right) Ш(f-mF)$$

Spectrum * comb

$$\boxed{E(\omega) = e^{i\phi_0}\, \mathcal{E}_0 \operatorname{sech}\left(\frac{\omega-\omega_0}{\Delta\omega}\right) \sum_{m=-\infty}^{\infty} \delta(f-mF)}$$

For more complete need to use

$$\sum_{m=-\infty}^{\infty} \delta(t - m/F)\exp(i m \Delta\phi_0)$$

$$\boxed{E(t) = \mathcal{E}_0 \operatorname{sech}(t/\Delta t)\, \underset{\wedge}{e^{-i(\omega_0 t + \phi)}} \circledast \sum_{m=-\infty}^{\infty} \delta(t - m/F)\exp(i m \Delta\phi_0)}$$

$e^{-i\phi(t)}$

Derivation of Frequency Comb From a pulse Train

Consider a pulsed electric field in the time domain, (sech^2)

$$\boxed{\begin{aligned} E(t) &= E_0 \,\text{sech}(\eta t/\Delta t)\, \exp(-i\varphi(t)) \exp(-i(\omega_0 t + \varphi_0)) \\ &\circledast \sum_{m=-\infty}^{\infty} \delta(t - m/F)\, \exp(-im\Delta\varphi_0) \end{aligned}} \qquad (1)$$

Where $\Delta t \equiv$ FWHM, $\Delta\varphi_0 \equiv$ CEO phase, $\eta \simeq 1.763$

$\omega_0 \equiv$ carrier frequency, $F \equiv$ Rep. Rate

$\varphi_0 \equiv$ Absolute phase of 1st pulse, $\varphi(t) =$ Temporal phase

$\circledast \equiv$ convolution, $\delta(\,) \equiv$ Delta function

Fourier Transform this result

$$\mathcal{F}\{E(t)\} =$$

$$\mathcal{F}\Big\{ E_0 \,\text{sech}(\eta t/\Delta t)\, \exp(-i\varphi(t))\, \exp(-i(\omega_0 t + \varphi_0)) \circledast \sum \delta(t - m/F)\, \exp(-im\Delta\varphi_0) \Big\}$$

Use convolution Th^m

$$\boxed{\begin{aligned} \mathcal{F}\{E(t)\} &= \mathcal{F}\{E_0 \,\text{sech}(\eta t/\Delta t)\, \exp(-i\varphi(t))\, \exp(-i(\omega_0 t + \varphi_0))\} \\ &\times \mathcal{F}\Big\{\sum \delta(t - m/F)\, \exp(-im\Delta\varphi_0)\Big\} \end{aligned}} \qquad (2)$$

To compute both FT, the shift Th^m will be used multiple times

$$\mathcal{F}\{ f(t-a) \} = \exp(i\omega a)\, f(\omega) \qquad \text{shift}$$

OR $$\mathcal{F}\{ f(t)\exp(+iat) \} = f(\omega - a) \qquad \text{Modulation}$$

(the converse Th^m is also called the modulation Th^m)

Derive the modulation thm

A phase shift in the time domain is a shift in absolute frequency in the frequency domain.

$$\mathcal{F}\{ f(t)\exp(+iat) \}$$

$$= \int f(t)\exp(+iat)\exp(-i\omega t)\,dt$$

$$= \int f(t)\exp(-i(\omega-a)t)\,dt$$

$$= f(\omega-a)$$

- the modulation thm on the 1st term of Eq 2.

$$\mathcal{F}\{ E_0 \operatorname{sech}(\eta t/\Delta t)\exp(-i\phi(t))\exp(-i\omega_0 t - i\phi_0) \}$$

$$= \exp(-i\phi_0)\,\mathcal{F}\{ \overbrace{E_0 \operatorname{sech}(\eta t/\Delta t)\exp(i\phi(t))}^{\text{complex envelope}}\ \overbrace{\exp(-i\omega_0 t)}^{\text{carrier}} \}$$

$$= \boxed{\exp(-i\phi_0)\ E(\omega-\omega_0)} \quad \text{or} \quad \boxed{\exp(i\phi_0)\operatorname{sech}((\omega-\omega_0/\Delta\omega))\exp(i\phi(\omega))} \quad (3)$$

where $E(\omega-\omega_0)$ is the spectral representation of the electric field centered at ω_0

i.e. $E(\omega-\omega_0) = \sqrt{I(\omega-\omega_0)}\ \exp(i\phi(\omega))$

$\uparrow$ spectrum $\uparrow$ spectral phase

- Modulation thm on the 2nd term of Eq 2.

$$\mathcal{F}\{ \sum \delta(t-m/F)\exp(-im\Delta\phi_0) \}$$

The FT of a comb function is another comb function, with different separation

$$\mathcal{F}\{ \sum_m \delta(t-m/F) \} = \sum_j \delta(\nu - jF) = \sum_j \delta(\omega - j2\pi F)$$

Using the modulation thm, the exponential term will shift the comb by $\Delta\phi_0/2\pi$, thus

$$\cancel{\mathcal{F}\left\{\sum \delta(t - m/F)\right\} =}$$

$$\mathcal{F}\left\{ \sum_m \delta(t - m/F) \exp(-im\Delta\phi_0) \right\}$$

$$= \sum_j \delta\left(\omega - j2\pi F + \Delta\phi_0/2\pi\right)$$

$$= \boxed{\sum_j \delta(\omega - j2\pi F + \underset{\uparrow \text{ CEO frequency}}{\delta_0})} \qquad (4)$$

Combining (3) & (4)

$$\cancel{A(\omega) = \exp(-i\phi_0)\mathcal{F}\{E_0 \operatorname{sech}}$$

$$\cancel{E(\omega) = \tilde{E}_0 \operatorname{sech}(\eta (\omega-\omega_0)/\Delta\omega) \exp(-i\phi(\omega))}$$

$$\boxed{E(\omega) = \tilde{E}_0 \operatorname{sech}\left((\omega-\omega_0)/\Delta\omega\right) \exp(i\phi(\omega)) \left[\sum_j \delta(\omega - j2\pi F + \delta_0)\right] \exp(i\phi_0)}$$

$\Delta\omega \equiv$ Spectral FWHM, $\delta_0 \equiv$ CEO frequency

$\phi(\omega) \equiv$ Spectral phase, $\tilde{E}_0 \equiv$ spectral magnitude.

The goal is to make every pulse in the pulse train to look identical $\Rightarrow$ Same form of $\mathcal{E}(t)$.

Why? $\Rightarrow$ Some experiments with (< 20 fs pulses @ 800nm)
Short pulses depend on the ~~peak~~ time of the peak of $E(t)$.

How to write the electric field of a pulse train?

$$\boxed{E_{i}(t) = \left[E_0 \operatorname{sech}(t/\Delta t\,\eta)\exp(-i\phi(t))\exp(-i(\omega_0 t + \phi_0)) \circledast \sum_{m=-\infty}^{\infty} \delta(t - m/F)\exp(-i m \Delta\phi_0)\right] sq(T_w)}$$

where

$F \equiv$ frep rate $= 1/T$

$\circledast \equiv$ Convolution

$sq() =$ unit Square function

$\eta \simeq 2\operatorname{sech}^{-1}\sqrt{1/2} \simeq 1.763$

$\omega_0 \equiv$ Carrier frequency

$\delta \equiv$ delta function

$T_w \equiv$ time the laser is on

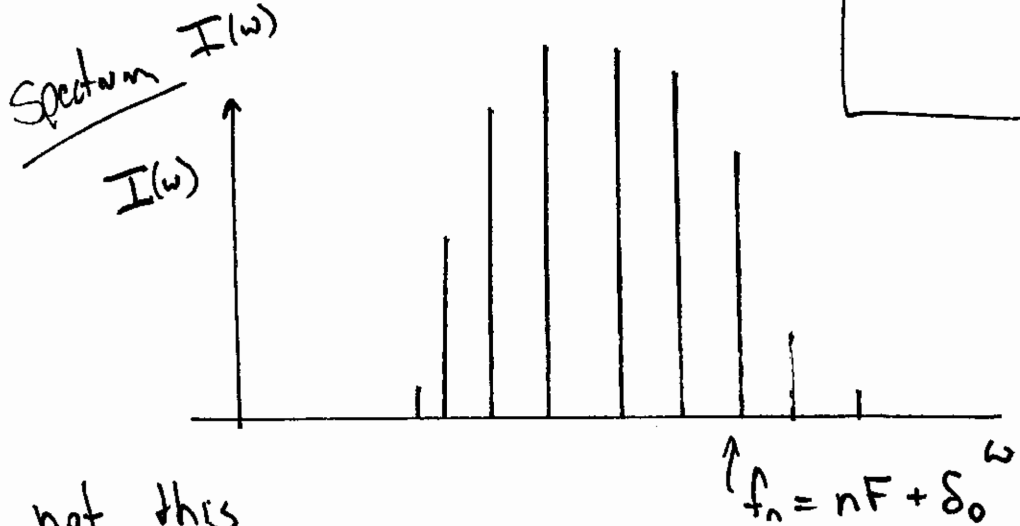
The term

$$\boxed{\sum_{m=-\infty}^{\infty} \delta(t - m/F)\exp(-i m \Delta\phi_0)}$$

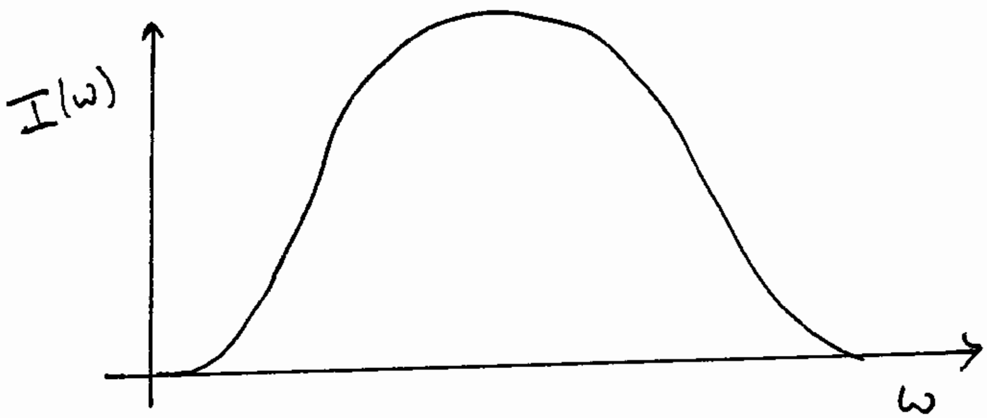
is an array of delta functions each with phase $m\Delta\phi_0$

The end result is the Fourier transform is the frequency comb.

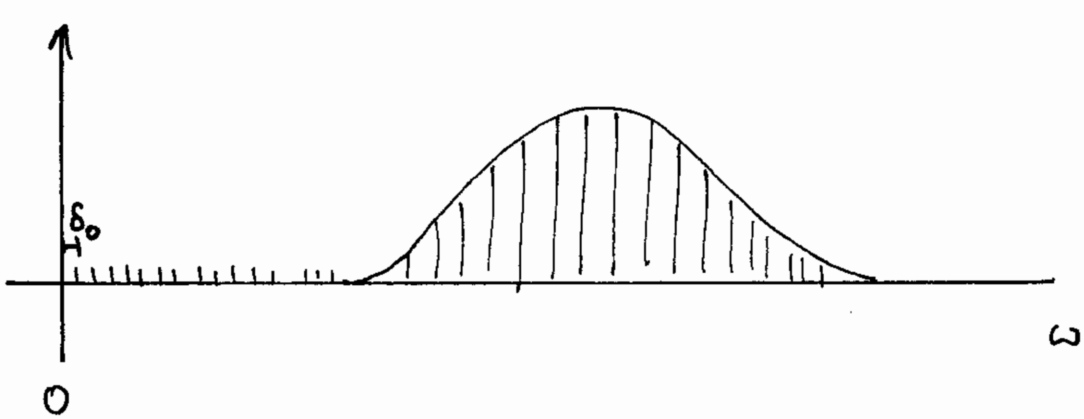
$$\boxed{E(\omega) \simeq A(\omega)\, e^{-i\varphi_0} \sum_j \delta(\omega - j 2\pi F + \delta_0)}$$



not this



the non zero $\Delta\varphi$ lets the extended comb not to fall at $\omega = 0$



The rate of change of $\Delta\varphi$ depends on δ_0.

If $\delta = 0$ then $\partial_t \Delta\varphi = 0$

To get every pulse in the train to have the same electric field we need to detect + fix δ_0

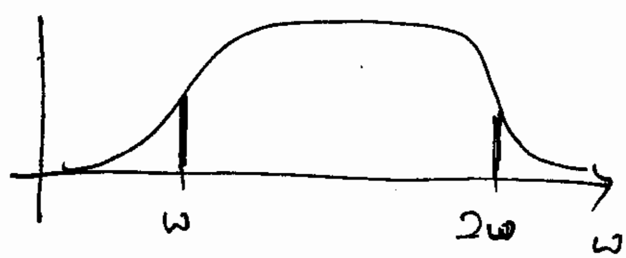
$$\boxed{\delta_0 = \frac{1}{2\pi} F \Delta\phi} \qquad (\delta_0, f_0, f_{ceo})$$

To do thing correct we need to detect + fix F as well.

How to detect F? $\Rightarrow$ Easy, fast photodiode

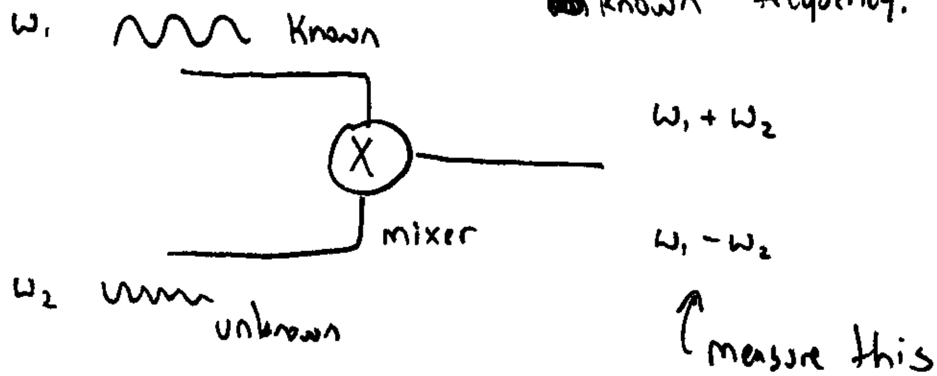
How to detect δ_0 $\Rightarrow$ Hard, use SHG with an octave spanning bandwidth.

Octave spanning $\Rightarrow$ spectrum covers ω_0 + $2\omega_0$

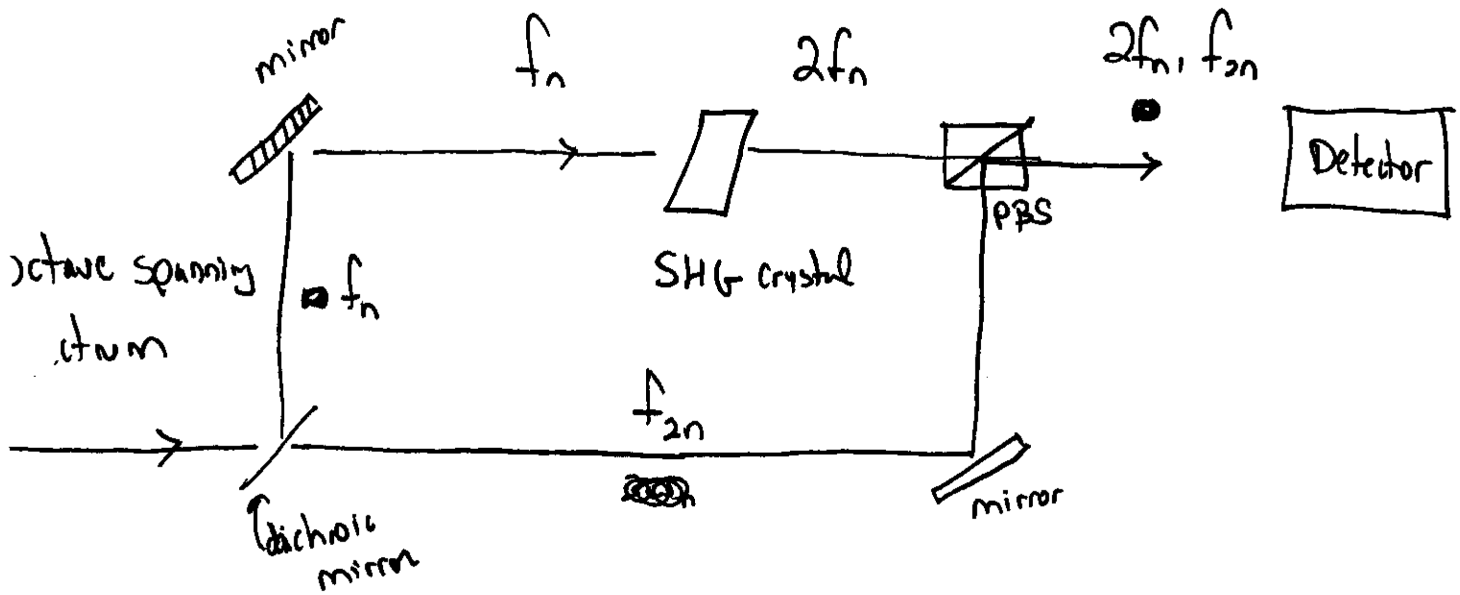


Use SHG generation of ω and beat against 2ω portion of the spectrum.

Hetrodyne detection : A method to measure a frequeny with an known frequenuy.



Beat notes



$$\boxed{f_n = nF + \delta_0}$$

Thru SHG we get $\Rightarrow$ $\boxed{2f_n = 2nF + 2\delta_0}$

But $\Rightarrow$ $\boxed{f_{2n} = 2nF + \delta_0}$

We hetrodyne $2f_n + f_{2n}$

$$2f_n - f_{2n} = 2nF + 2\delta_0 - 2nF - \delta_0 = \boxed{\delta_0}\ !!$$

So we can detect both F + δ_o. Can we control them?

Control $F \Rightarrow$ Easy, just change the cavity length of the mode locked laser.

Control $\delta_o \Rightarrow$ Difficult, we need to change the group velocity independent of the phase velocity

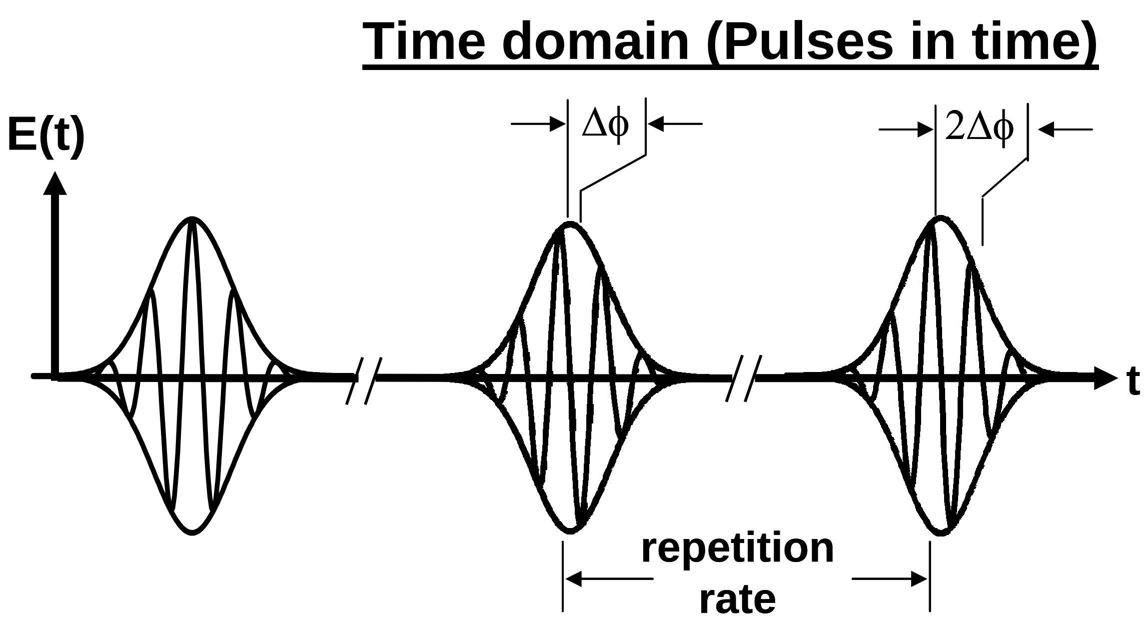
$$\boxed{\Delta\varphi_o = \left(\frac{1}{V_g} - \frac{1}{V_p}\right) L\, \omega_c}$$

Many ways to do this

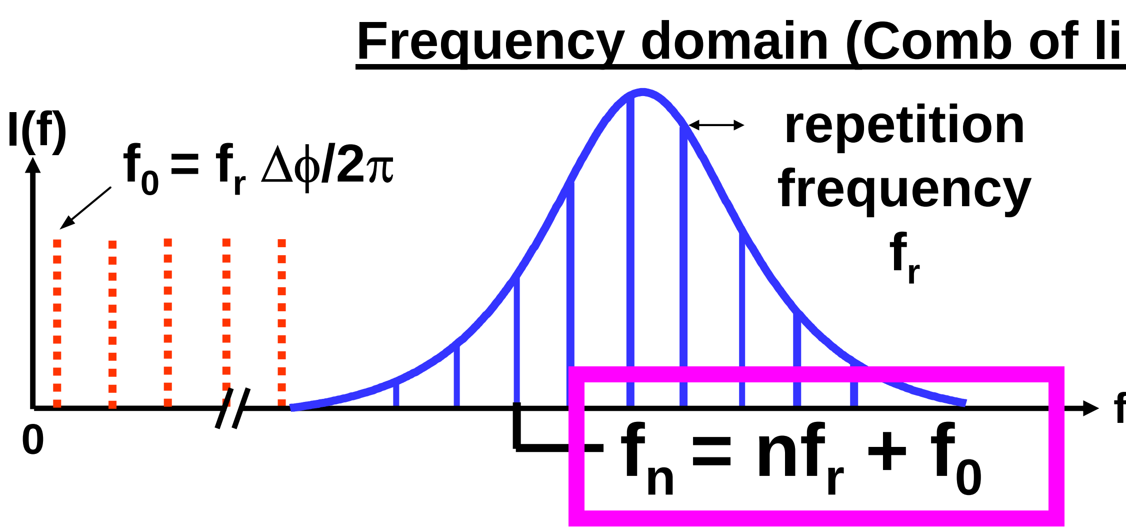
1) Pump power modulation
2) Mirror tilt in the Fourier plane of a prism pair

Note that the detection of δ_o depends on this octave spanning spectrum. Typically lasers do not produce an octave spanning spectrum, we need to do something to the pulse to broaden its spectrum. Thus we need another nonlinear effect to produce the octave spanning spectrum. That nonlinear effect will be a <u>third</u> <u>order</u> <u>nonlinearity</u>.

The Frequency Comb

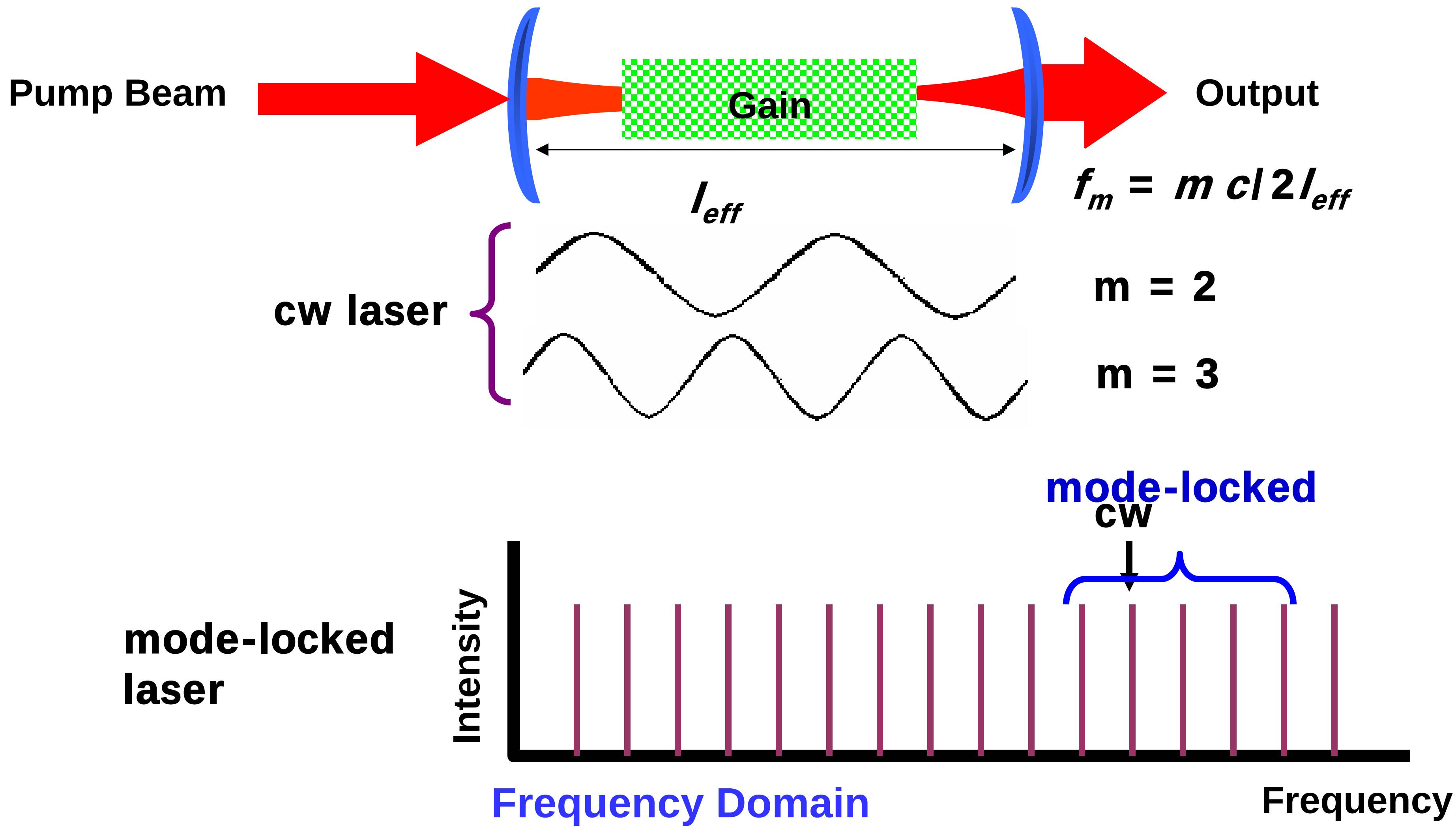


Carrier-envelope phase slip from pulse to pulse because group and phase velocities differ



Stable frequency comb if
1) Repetition rate (f_r) locked
2) Offset frequency (f_0) (phase slip) locked

Pulsed lasers must give a frequency comb



Comb-like nature of ultrafast lasers

1979- mode-locked (pulsed) dye laser used as comb

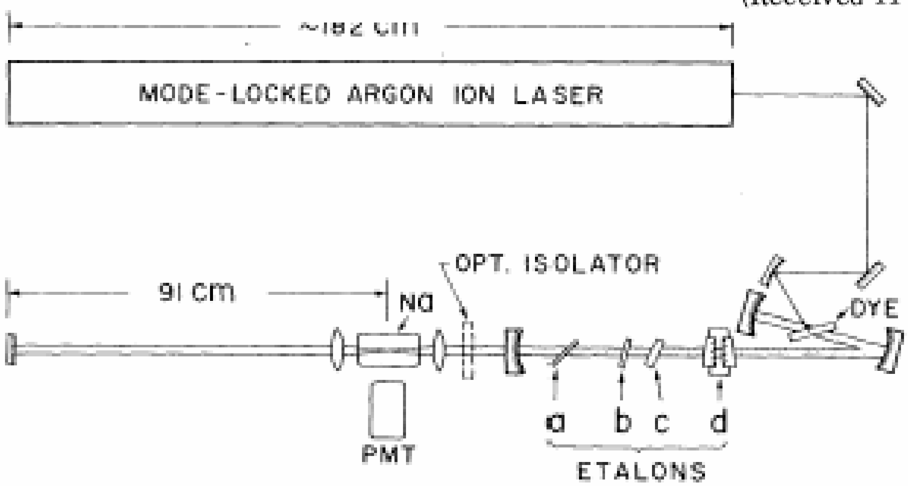
- **500 ps pulses ~0.003 nm, ~1 GHz wide**

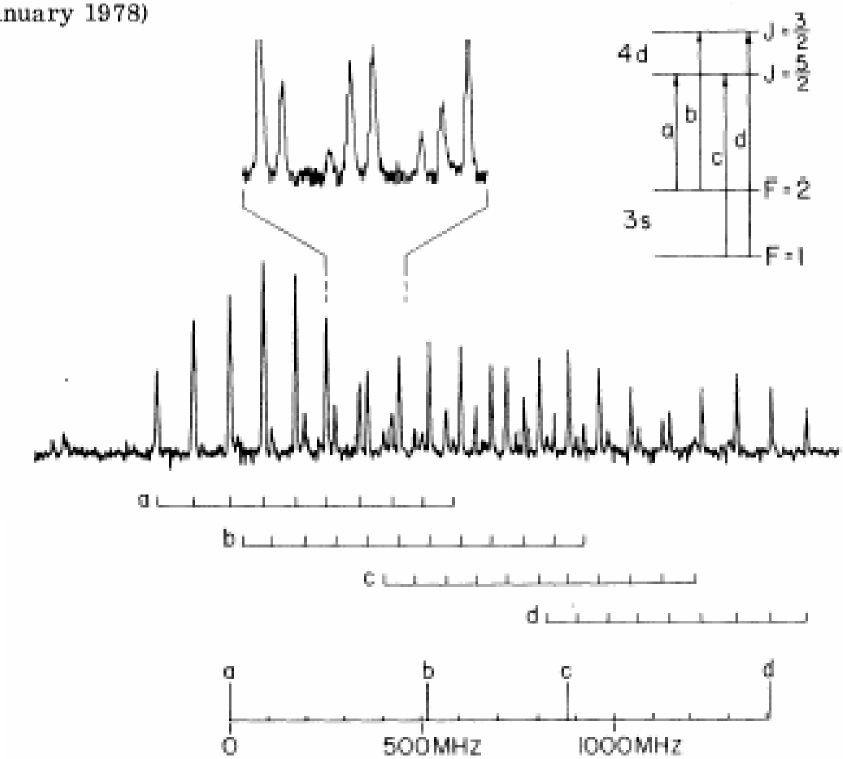
VOLUME 40, NUMBER 13 PHYSICAL REVIEW LETTERS 27 MARCH 1978

High-Resolution Two-Photon Spectroscopy with Picosecond Light Pulses **Na**

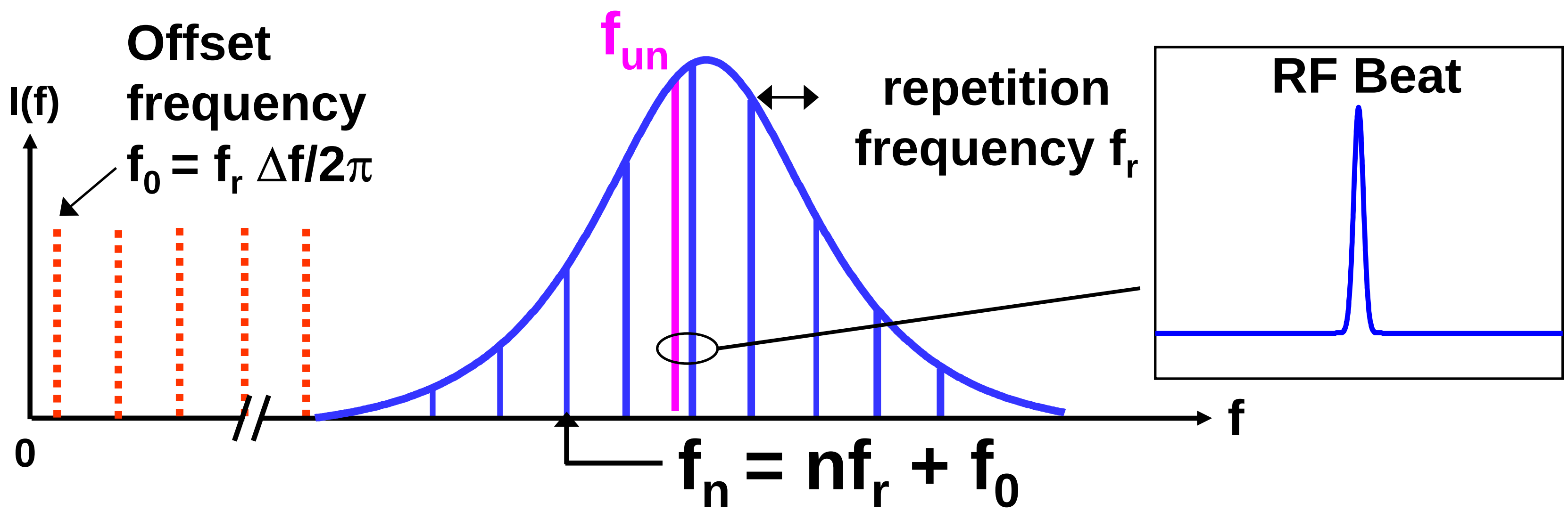
J. N. Eckstein, A. I. Ferguson, and T. W. Hänsch
Department of Physics, Stanford University, Stanford, California 94305

(Received 11 January 1978)





Optical Frequency Metrology



Frequency comb as a spectral ruler

References:

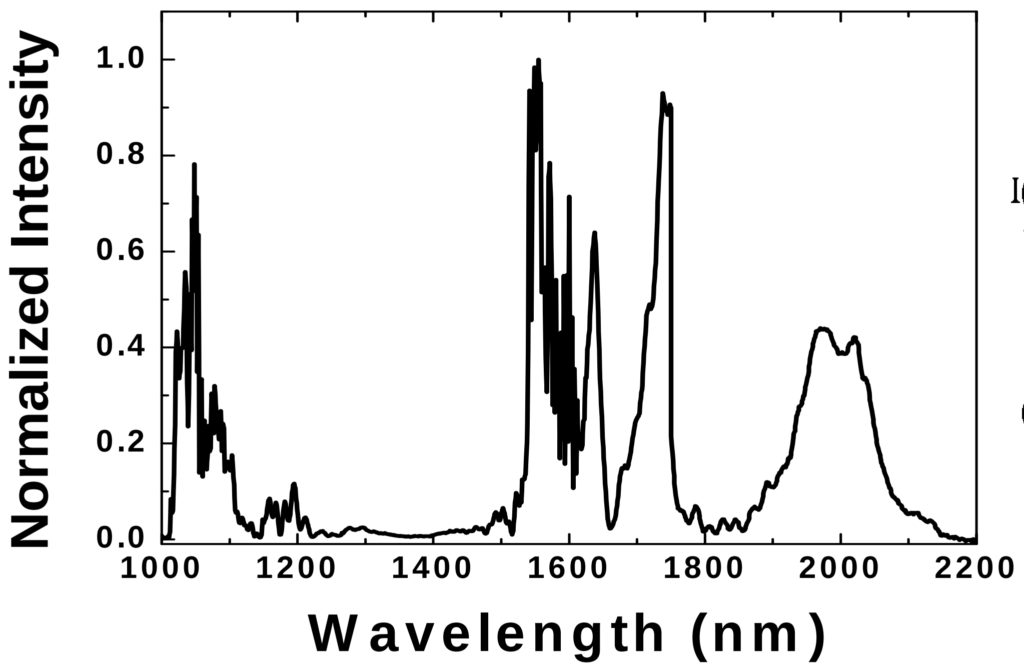
Udem, Reichert, Holwartz, Hänsch, *Phys. Rev. Lett.*, vol. 82 (1999)

Jones et al., *Science*, vol. 288 (2000)

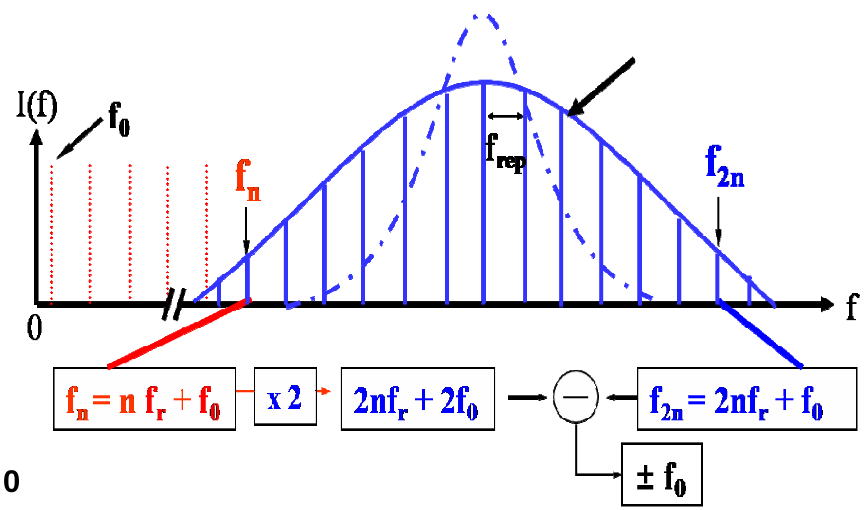
Udem, Holwartz, Hänsch, *Nature*, vol. 416 (2002)

Supercontinuum generation for self referencing f_0

Supercontinuum spectrum

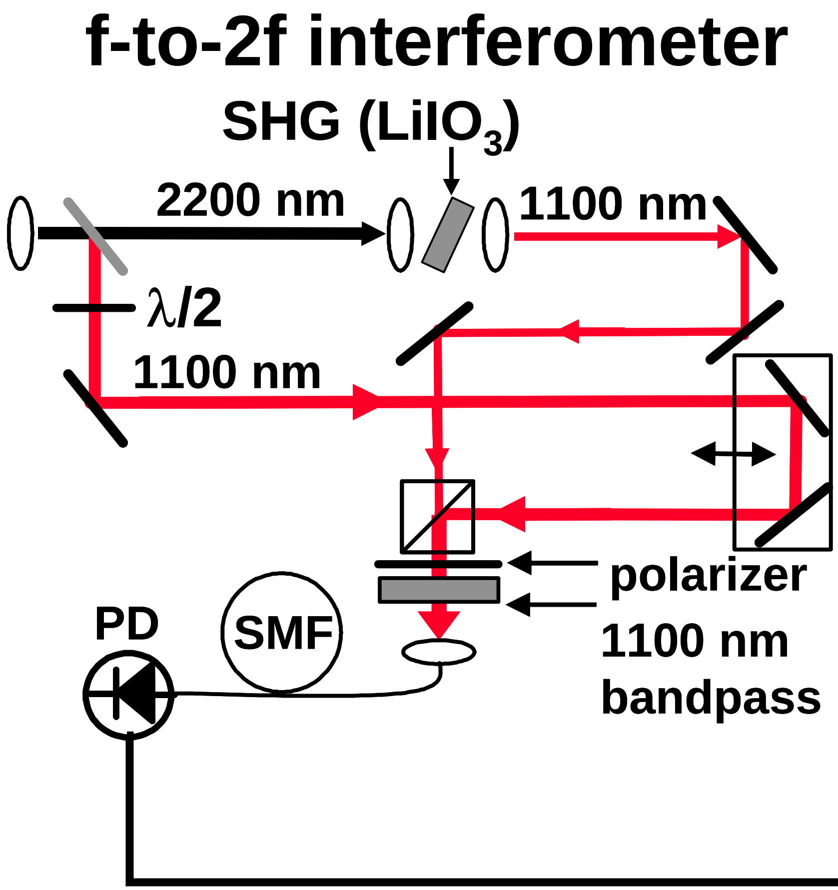


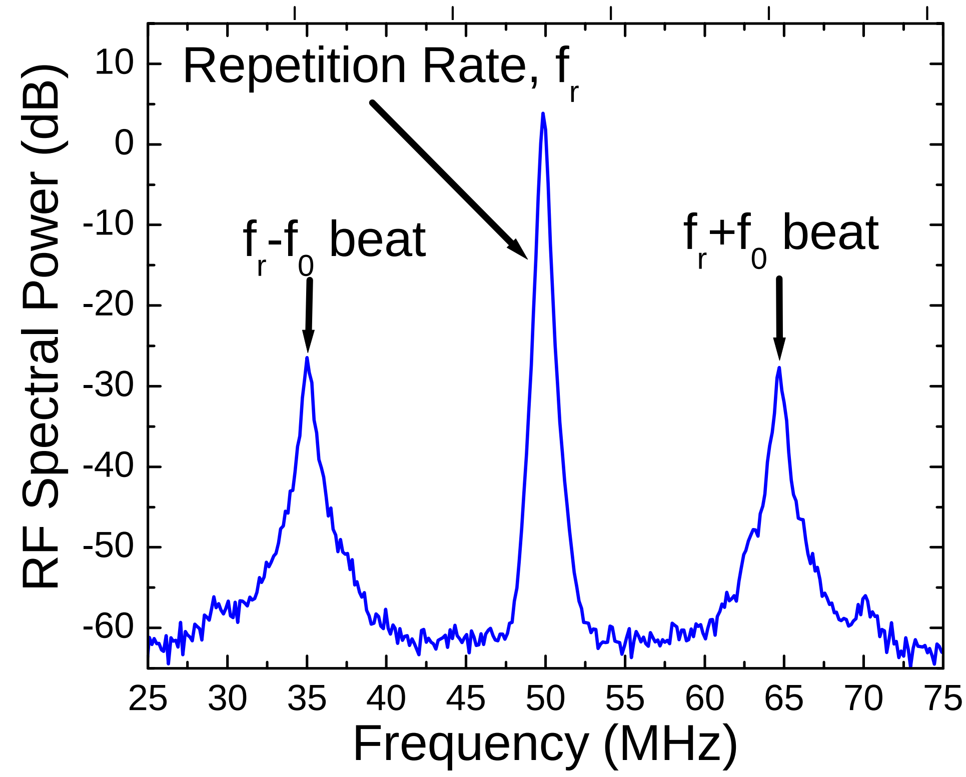
f-to-2f Interferometer



- D. J. Jones, S. A. Diddams, J. K. Ranka, A. Stentz, R. S. Windeler, J. L. Hall, and S. T. Cundiff, "Carrier-envelope phase control of femtosecond mode-locked lasers and direct optical frequency synthesis," Science 288, 635-9 (2000).

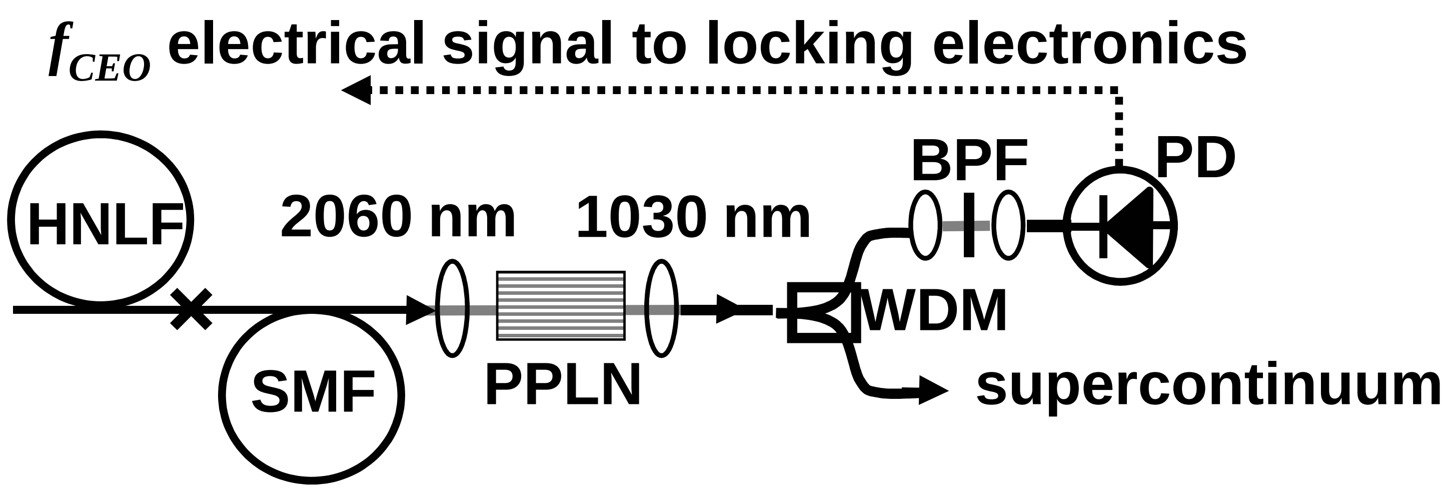
f-to-2f Interferometer



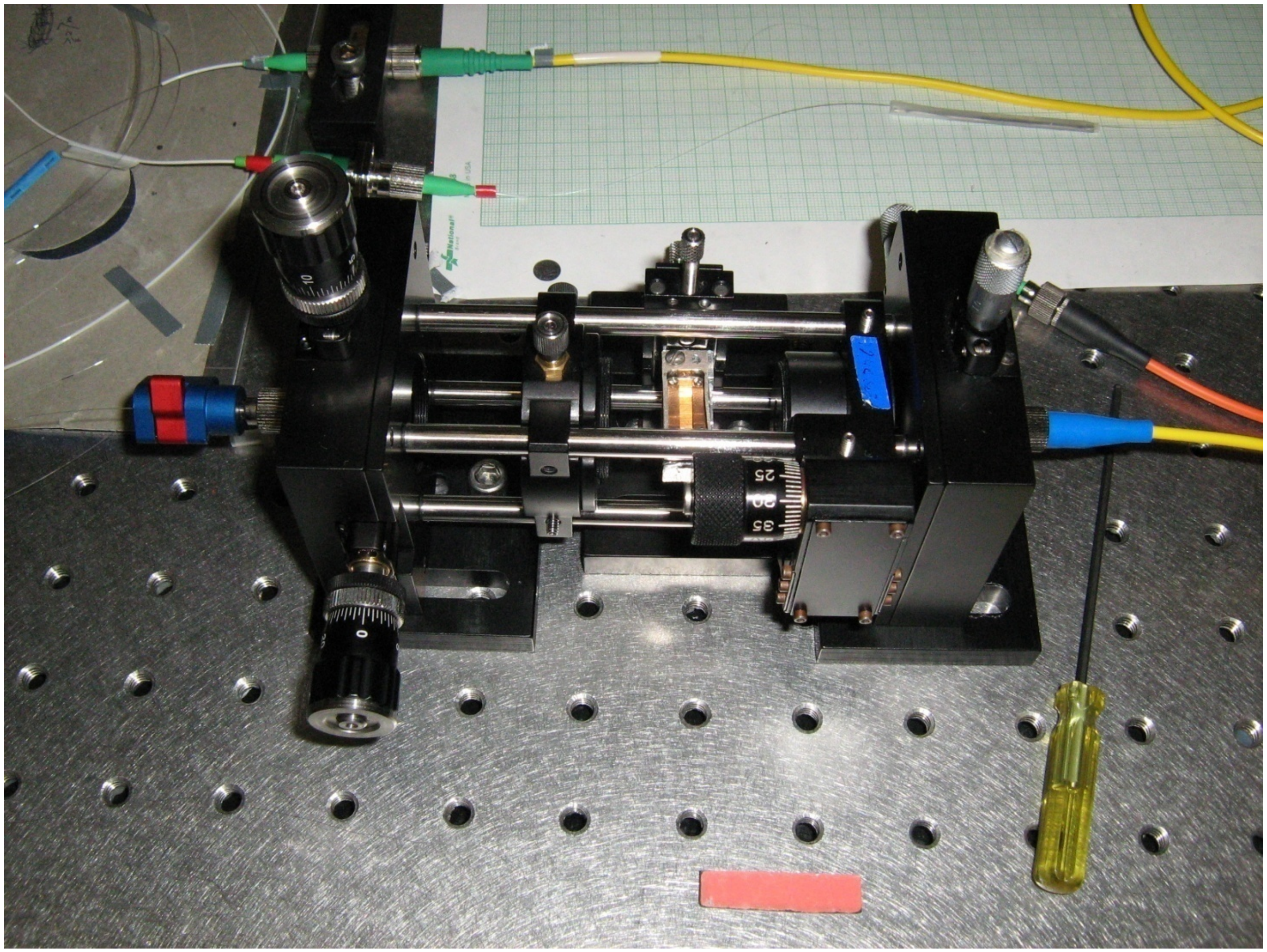


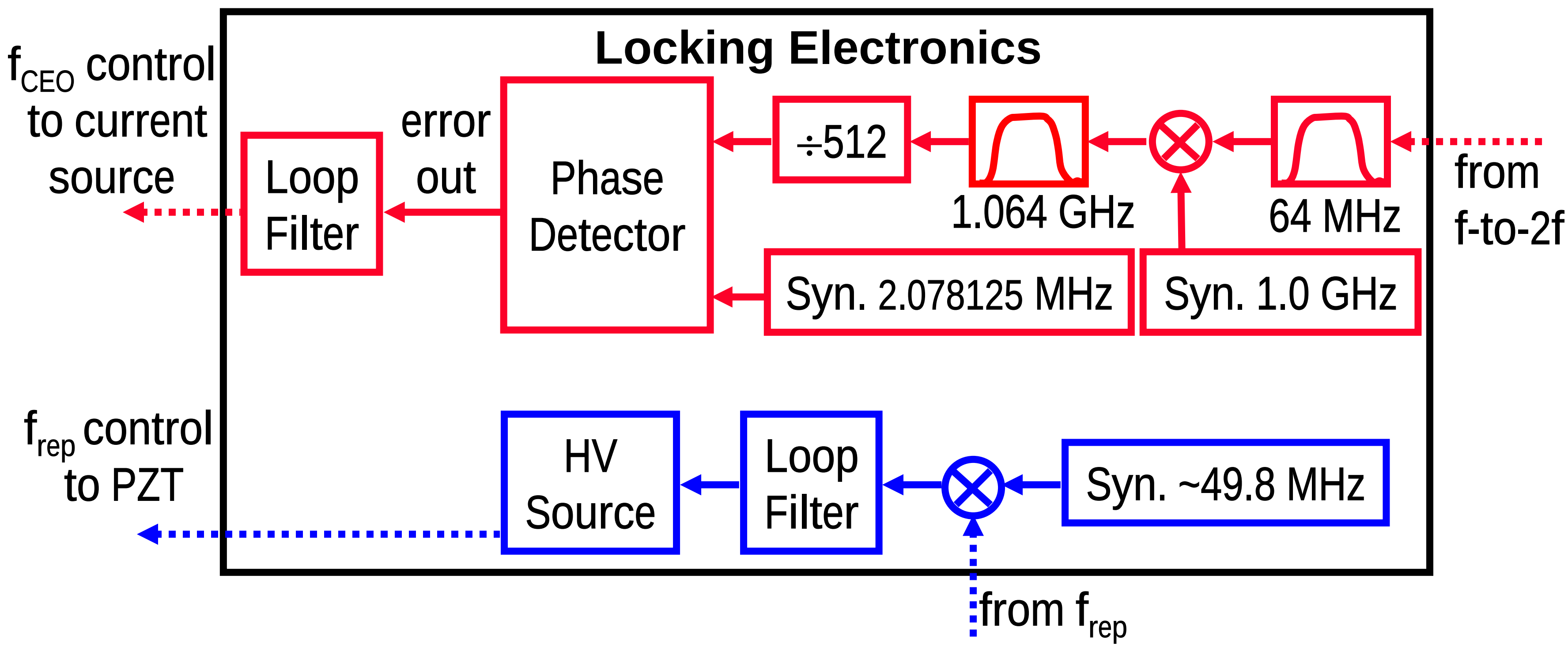
An octave of supercontinuum allow the generation of beat frequencies with a SNR of 30 dB

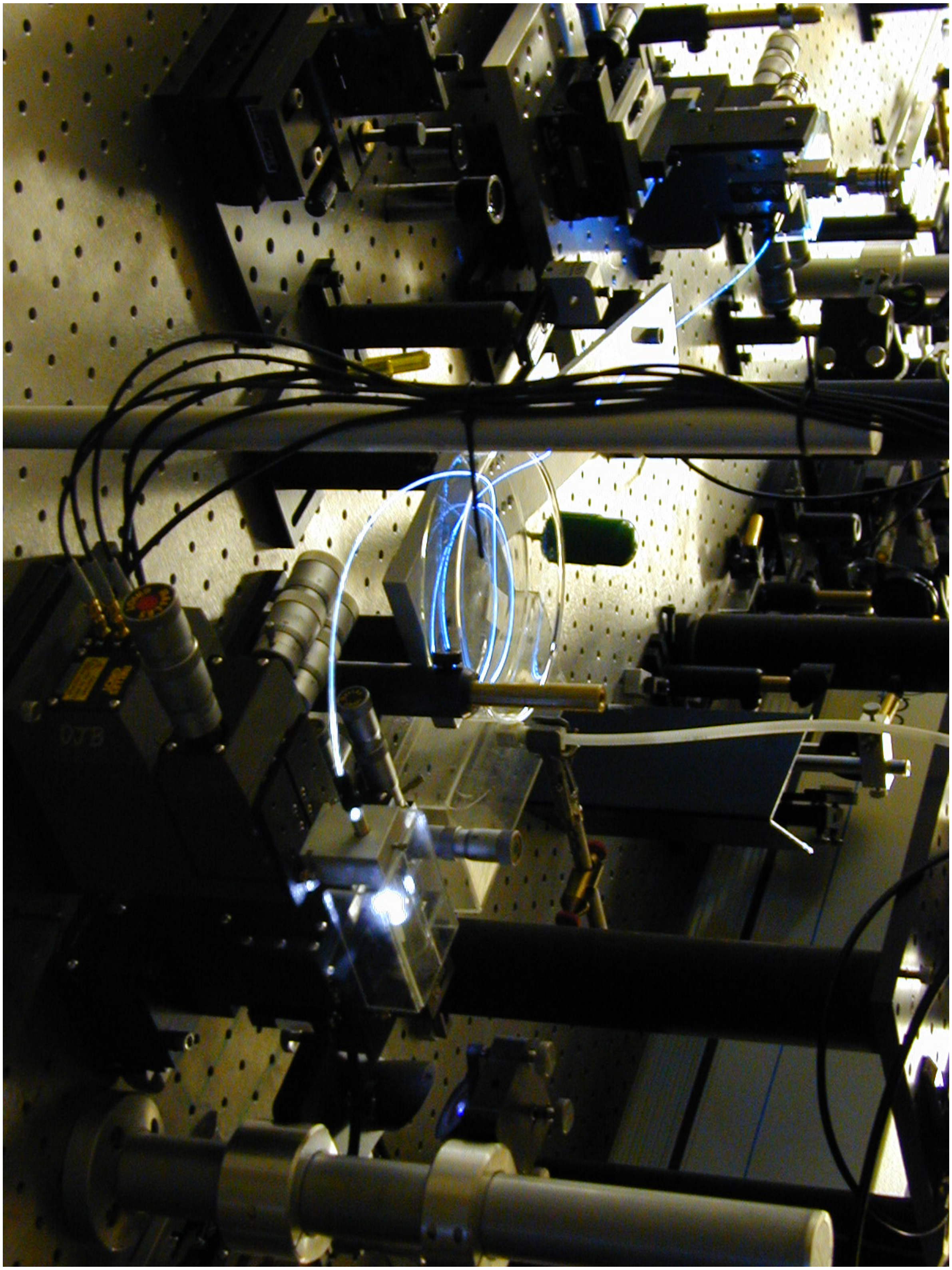
Co-linear all fiber geometry



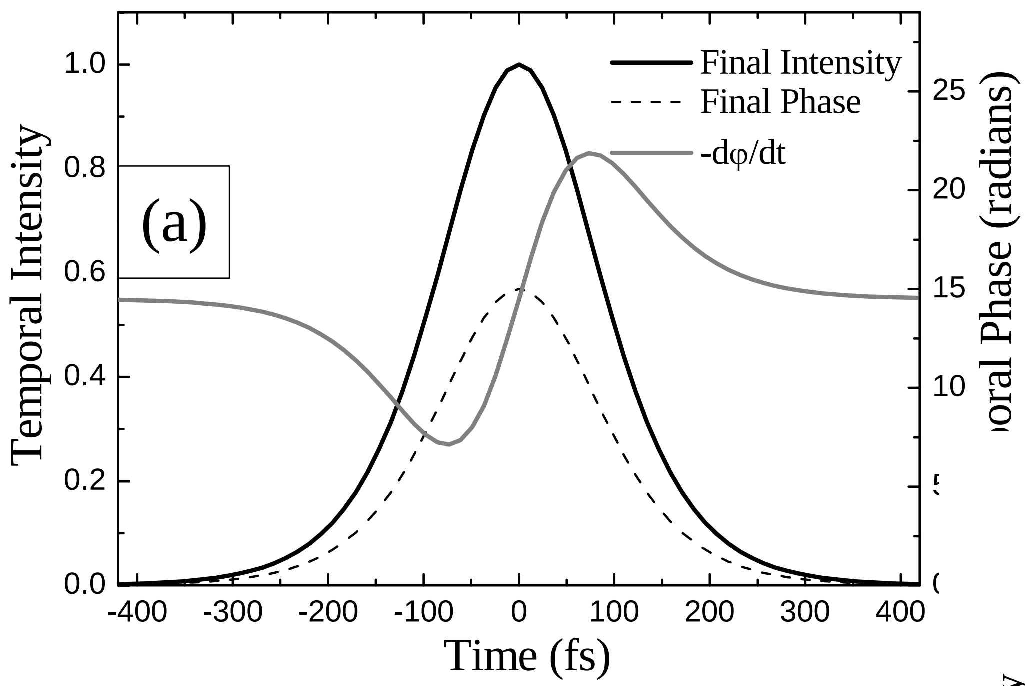
Colinear f-to-2f Inteferometer



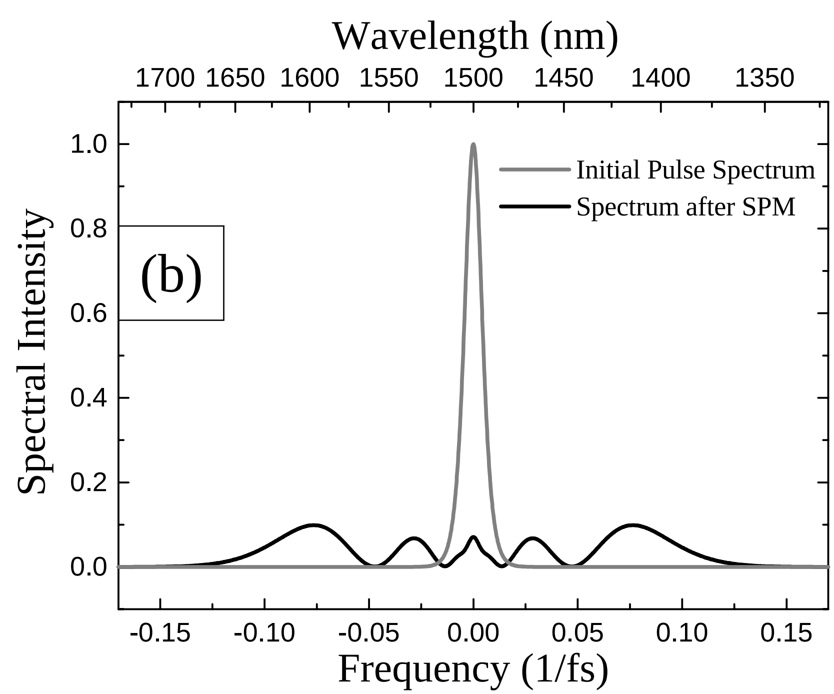

Locking Electronics
fCEO control
to current
source
Loop
Filter
error
out
Phase
Detector
÷512
1.064 GHz
64 MHz
from
f-to-2f
Syn. 2.078125 MHz
Syn. 1.0 GHz
frep control
to PZT
HV
Source
Loop
Filter
Syn. ~49.8 MHz
from frep



Nonlinear propagation in fiber: nonlinearity only



Generate new spectral components



Lecture 11

Third Order Effects

Need to consider $2 * 27 * 3! = 1944$ complex #'s (χ)

Even if we have 1 $\chi^{(3)}$ (Kleinman Symmetry)

$$(E_1 + E_2 + E_3)^3 \sim 216 \text{ terms}$$

So we will first consider phase matched + then unphase matched terms.

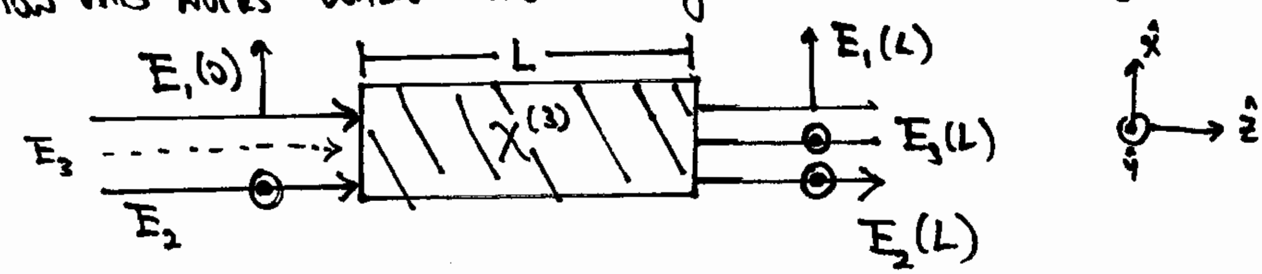
Lecture 18 Four wave mixing + the intensity-dependent index of refraction

Third order nonlinearities involve the interaction of three fields E to generate a nonlinear polarization

$$P_{NL} = \chi^{(3)} E_1(\omega_1) E_2(\omega_2) E_3(\omega_3)$$

We can, like for $\chi^{(2)}$ effects, write down the Maxwell's wave equation and solve for the new electric field. In general this process is known as four wave mixing (FWM) since three input fields induce a nonlinear polarization with induces a fourth field E.

To see how this works consider the case of third harmonic generation:

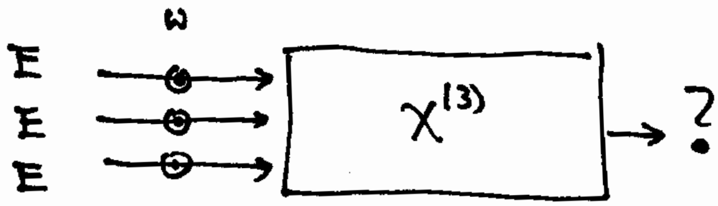


Inputs $E_1(0)$ + $E_2(0)$ which have orthogonal polarizations which both have frequency ω. The generated field $E_3(L)$ has frequency 3ω.

Write down the electric fields

Before we do this....

Let's consider a simplier case of a FWM of three field with same polarization + frequency



$$\omega_1 = \omega_2 = \omega_3 = \omega$$
$$\bar{k}_1 = \bar{k}_2 = \bar{k}_3 = \bar{k}$$

The name for this process is completely degenerate four wave mixing. This will involve the tensor element

$$\chi^{(3)}_{xxxx}(\omega;, \omega - \omega, \omega)$$

Completely degenerate FWM leads to a change of the index of refraction that is dependent on the intensity I. This is also called single field degenerate FWM. The field changes the index of refraction it experiences!! This self modulation leads to two well known effects

- Self phase modulation (SPM)
- Self - Focusing

Let's derive an approximate equation that show the variation of

Fourwave mixing + the intensity dependent index of refraction.

To get the coupled eqs we will assume

- isotropic media
- SVEA
- far from resonance

Thus $\chi^{(3)}$ is frequency independent

Define effective nonlinearity (I = 2

$$\left(\begin{matrix} 4:\ E(t) = \mathcal{E}(\omega)e^{+i\omega t} + c.c. \\ 8:\ \mathcal{E}(t) = \frac{1}{2}(\qquad) \end{matrix}\right) \gamma = \frac{3\mu_0 \omega \chi^{(3)}}{4\,8\,n_0^2 A_{eff}} = \frac{n_2 \omega}{A_{eff}\, c} \quad \left[\frac{1}{W\,m}\right]$$

$$A_{eff} = \pi r^2 \qquad n_2 = \frac{3}{4 n_0^2 \epsilon_0 c}\chi^{(3)} \quad \left[\frac{m^2}{W}\right]$$

Want to show $\boxed{n = n_0 + n_2 I}$ Intensity dependent index of refraction.

Now, using a similar process for $\chi^{(2)}$ we can get 4 coupled DE's

For $l = 1-4$ $\qquad \Delta\beta = \beta_3 + \beta_4 - \beta_1 - \beta_2$

$$\boxed{\frac{\partial A_l}{\partial z} = i\gamma\left[\underbrace{|A_l|^2 A_l}_{SPM} + \underbrace{2\sum_{j\neq l=1}^{4} |A_j|^2 A_l}_{XPM} + \underbrace{2 A_m A_n A_k^* e^{i\eta\Delta\beta z}}_{FWM}\right]}$$

k, l, m, n

$\eta = 1$ if $l = 1$ or 2 $\quad k = 3-l,\ m = 3,\ n = 4$

$\eta = -1$ if $l = 3$ or 4 $\quad k = 7-l,\ m = 1,\ n = 2$

Self phase modulation

SPM: Automatically phase matched. One wave with itself

$\omega = \omega + \omega - \omega = 0$

XPM: Two waves (X)

FWM: Four distinct waves

Solutions

Can solve for pump depletion + phase matching

got 3 coupled eqs $\Rightarrow$ sol^{n} $sn^2[\quad]$

Lets look at a specific case: completely degenerate FWM (SPM)

$$\vec{k}_1 = \vec{k}_2 = \vec{k}_3 \qquad \vec{E}_1 = \vec{E}_2 = \vec{E}_3 \qquad \omega_1 = \omega_2 = \omega_3$$

$$\boxed{P_{NL} \simeq 3\epsilon_0 \chi^{(3)}_{xxxx} \underbrace{|E|^2 E}_{E E^* E}} \quad \Big\{ \text{really } \chi^{(3)}_{xxxx}(\omega; \omega, -\omega, \omega)$$

Write polarization

$$P = P_L + P_{NL} = \epsilon_0 \left[\chi^{(1)} \vec{E} + 3\chi^{(3)} |E|^2 E \right]$$

So

$$\partial_z^2 \vec{E} - \frac{1}{c^2} \partial_t^2 \vec{E} = \mu_0 \epsilon \left[\chi^{(1)} + 3\chi^{(3)} |E| \right] \partial_t^2 \vec{E}$$

$$\partial_z^2 E - \frac{\left[1 + \chi^{(1)} + 3\chi^{(3)} |E|^2 \right]}{c^2} \partial_t^2 E = 0$$

So $\quad \boxed{n^2 = 1 + \chi^{(1)} + 3\chi^{(3)} |E|^2}$

If we define

$$n = n_0 + \bar{n}_2 \langle \mathcal{E}^2 \rangle$$

with $\mathcal{E}(t) = E(\omega) e^{-i\omega t} + c.c.$

$$\langle \mathcal{E}^2 \rangle = 2 |E(\omega)|^2$$

And $$n = n_0 + 2\bar{n}_2 |E(\omega)|^2$$

So $$\left(n_0 + 2\bar{n}_2 |E(\omega)|^2\right)^2 = 1 + \chi^{(1)} + 3\chi^{(3)} |E|^2$$

Expand to order $|E|^2$

$$n_0 + 2\bar{n}_2 |E(\omega)| = 1 + \chi^{(1)} + 3\chi^{(3)} |E(\omega)|^2$$

where $$\boxed{\bar{n}_2 = \frac{3\chi^{(3)}}{4n_0} \qquad n_0 = \sqrt{1 + \chi^{(1)}}}$$

Lets define n_2 in terms of intensity $I = 2n_0 \epsilon_0 c |E(\omega)|^2$

$\Longrightarrow$

$2\bar{n}_2 |E(\omega)|^2 = n_2 I$ $\quad n_2 = \dfrac{3}{4n_0^2 \epsilon_0 c} \chi^{(3)} = \dfrac{\bar{n}_2}{n_0 \epsilon_0 c}$

$$\boxed{n_2 = \frac{3\chi^{(3)}}{4n_0^2 \epsilon_0 c}} \qquad \left[\frac{m^2}{W}\right]$$

Four Wave Mixing Terms

Brian Washburn version 1 9/4/07

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Off[General::spell];
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Here I explicitly expand all the interaction terms for Four Wave Mixing. Each real instantaneous field will be defined as (for i=1,2,3)

$\mathcal{E}_i = \frac{1}{2} E_i + \frac{1}{2} E_i^*$

The nonlinear polarization will be

$\mathcal{P} = \chi^{(3)} \epsilon_0 (\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3)^3$

where I can write the nonlineari polarization as

$\mathcal{P} = \frac{1}{2} P + \frac{1}{2} P^*$

In general we will be doing expansion of a polynomial of three terms cubed.

```
Expand[(b + d + g)^3]
```

$b^3 + 3 b^2 d + 3 b d^2 + d^3 + 3 b^2 g + 6 b d g + 3 d^2 g + 3 b g^2 + 3 d g^2 + g^3$

Where each term in the polynomial has the form $b = \frac{1}{2} bo + \frac{1}{2} bo^*$

■ FWM Case

Consider the case of FWM with three frequencies $\omega_1, \omega_2, \omega_3$. Write out the electric field for this interaction. Here A is the real instantaneous electric field, a is the complex field constant and ac is the complex conjugate of a. Note, I would use E for the electric field, but in *Mathematica* E is the exponential.

$$A_1 = \frac{1}{2} a_1 \text{Exp}[i k_1 z] \text{Exp}[-i \omega_1 t] + \frac{1}{2} ac_1 \text{Exp}[-i k_1 z] \text{Exp}[i \omega_1 t]$$
$$A_2 = \frac{1}{2} a_2 \text{Exp}[i k_2 z] \text{Exp}[-i \omega_2 t] + \frac{1}{2} ac_2 \text{Exp}[-i k_2 z] \text{Exp}[i \omega_2 t]$$
$$A_3 = \frac{1}{2} a_3 \text{Exp}[i k_3 z] \text{Exp}[-i \omega_3 t] + \frac{1}{2} ac_3 \text{Exp}[i k_3 z] \text{Exp}[-i \omega_3 t]$$

Boyd defines this without the 1/2

$$\frac{1}{2} e^{i z k_1 - i t \omega_1} a_1 + \frac{1}{2} e^{-i z k_1 + i t \omega_1} ac_1$$

$$\frac{1}{2} e^{i z k_2 - i t \omega_2} a_2 + \frac{1}{2} e^{-i z k_2 + i t \omega_2} ac_2$$

$$\frac{1}{2} e^{i z k_3 - i t \omega_3} a_3 + \frac{1}{2} e^{i z k_3 - i t \omega_3} ac_3$$

$$A_1 = \frac{1}{2} a_1 \,\mathrm{Exp}[i\, k\, z]\, \mathrm{Exp}[-i\, \omega\, t] + \frac{1}{2} ac_1 \,\mathrm{Exp}[-i\, k\, z]\, \mathrm{Exp}[i\, \omega\, t]$$

$$A_2 = \frac{1}{2} a_2 \,\mathrm{Exp}[i\, k\, z]\, \mathrm{Exp}[-i\, \omega\, t] + \frac{1}{2} ac_2 \,\mathrm{Exp}[-i\, k\, z]\, \mathrm{Exp}[i\, \omega\, t]$$

$$A_3 = \frac{1}{2} a_3 \,\mathrm{Exp}[i\, k_3\, z]\, \mathrm{Exp}[-i\, 3\, \omega\, t] + \frac{1}{2} ac_3 \,\mathrm{Exp}[i\, k_3\, z]\, \mathrm{Exp}[-i\, 3\, \omega\, t]$$

$$\frac{1}{2} e^{i k z - i t \omega} a_1 + \frac{1}{2} e^{-i k z + i t \omega} ac_1$$

$$\frac{1}{2} e^{i k z - i t \omega} a_2 + \frac{1}{2} e^{-i k z + i t \omega} ac_2$$

$$\frac{1}{2} e^{-3 i t \omega + i z k_3} a_3 + \frac{1}{2} e^{-3 i t \omega + i z k_3} ac_3$$

Expand the fields

Expand the fields

$$\textbf{Expand}\left[\chi_{xxxx}\,\epsilon_0\left(\frac{1}{3}\,A_1+\frac{1}{3}\,A_1+\frac{1}{3}\,A_1\right)^3\right]$$

$$\frac{3}{8}\,a^2\,ac\,e^{i\,k\,z-i\,t\,\omega}\,\epsilon_0\,\chi_{xxxx}+\frac{3}{8}\,a\,ac^2\,e^{-i\,k\,z+i\,t\,\omega}\,\epsilon_0\,\chi_{xxxx}+$$
$$\frac{1}{8}\,a^3\,e^{3\,i\,k\,z-3\,i\,t\,\omega}\,\epsilon_0\,\chi_{xxxx}+\frac{1}{8}\,ac^3\,e^{-3\,i\,k\,z+3\,i\,t\,\omega}\,\epsilon_0\,\chi_{xxxx}$$

Now I use a factor of 1/3 to account for the degeneracy of three fields, the three fields are not physically distinquishable. For the complex form of the polarization we got

$$P_{NL}=\frac{3}{4}\chi_{xxxx}\,\epsilon_0 a\;ac\;a$$

where the real instantaneous form of the polarization is

$$\mathcal{P}=\frac{1}{2}P_{NL}+\frac{1}{2}P_{NL}^{*}$$

So you see I get the term 3/8 as expected.

Lecture 19 More on Self phase modulation + $\chi^{(3)}$

Many materials give rise to third order effects
Only materials that have a strong electronic component will offer a fast response

Nonlinear Response $\Rightarrow$ $R(t) = \left(\underbrace{\delta(t)}_{\text{fast}} + \underbrace{h_R(t)}_{\text{slow}} \right)$

SPM, FWM   Raman effects

"Fast" $\Rightarrow$ Electronic contributions : $\sim 10^{-15}$ s

"Slow" $\Rightarrow$ Thermal / vibrational contributions : $\sim 10^{-12}$ s

Third order materials

- Gases
  - Noble gases
  - CS_2
- Liquids
  - Tea
- Isotropic Solid
  - glasses

Leads to nonlinear index of Refraction

$$\boxed{n_2 = \frac{3}{8n_0} \operatorname{Re}\{\chi^{(3)}_{xxxx}\}}$$

FWM $\Rightarrow$ Due to fast response of the $\chi^{(3)}$ medium

Intensity dependent index of Refraction

From Last time we derived:

$$\boxed{n = n_0 + n_2 |E|^2}$$

where

$$n_2 = \frac{3}{8n_0} \mathrm{Re}\{\chi^{(3)}_{xxxx}\}$$

n_2 has units of m^2/V^2

Rewrite this equation in terms of Intensity where

$$I = 2^{-1} \varepsilon_0 c n |E|^2$$

So

$$\boxed{n = n_0 + n_2^I I}$$

Where $n_2^I = \dfrac{2n_2}{\varepsilon_0 c n_0}$ in units of m^2/W

common notation replaces $\underline{\underline{n_2^I}}$ with $\underline{\underline{n_2}}$ (Redefine)

For fused silica

$$\boxed{n_2 = 3 \times 10^{-20}\, m^2/W}$$

$$n_2 \equiv \left(\frac{2}{\varepsilon_0 c n_0}\right)\left(\frac{3}{8n_0} \chi^{(3)}_{xxxx}\right)$$

Self Phase Modulation $\Rightarrow$ Completely degenerate FWM

$$\omega = \omega - \omega + \omega$$

$$P_{NL} = \frac{3}{4}\epsilon_0 \chi^{(3)}_{xxxx}(\omega; \omega, -\omega, \omega)\, E E^* E \qquad (\text{Boyd} * 4)$$

and

$$\mathcal{P}_{NL} = \frac{1}{2}\left(P_{NL}\, e^{-i\omega t}\right) + c.c.$$

$$\mathcal{E} = \frac{1}{2} E\, e^{-i\omega_0 t} + c.c.$$

Sub into wave eq to find $E(z,t)$ generated

$$\partial_z^2 \vec{\mathcal{E}} - \mu_0 \epsilon_0 \partial_t^2 \mathcal{E} = \mu_0 \partial_t^2 \mathcal{P}$$

If we use the slowly varying envelope approximation

$$|k\, \partial_z \mathcal{E}| >> |\partial_z^2 E|$$

we can rewrite the wave eq as

$$\partial_z E = i\frac{3\mu_0 \epsilon_0 c\, \omega}{8 n_0} \chi_{xxxx}^{(3)} |E|^2 E$$

Define ~~$E(t,z) = \sqrt{\frac{P_0}{\pi r^2}}\, U(t,z) \sqrt{\frac{2}{\epsilon_0 c n}}$~~ $U(z,t) \equiv$ normalized function

$$\boxed{E(z,t) = \sqrt{\frac{P_0}{\pi r^2}} \sqrt{\frac{2}{\epsilon_0 c n_0}}\, U(z,t)}$$

So we have (Remember $\epsilon_0\mu_0 = \frac{1}{c^2}$)

$$\cancel{\sqrt{\frac{P_0}{\pi r^2}}}\,\partial_z U = i\,\frac{2\cdot 3\mu_0\epsilon_0 c\,\omega}{c n_0\, 8 n_0 \epsilon_0}\,\chi^{(3)}_{xxxx}\,|U|^2 U\,\cancel{\sqrt{\frac{P_0}{\pi r^2}}}\left(\frac{P_0}{\pi r^2}\right)$$

So $$\partial_z U = \frac{2}{\epsilon_0 c n_0}\left[\left(\frac{3\chi^{(3)}}{8n_0}\right)\frac{\omega}{\pi r^2 c}\right] P_0\, |U(t)|^2\, U(t)$$

But $n_2 = \left(\frac{3\chi}{8n_0}\right)\left(\frac{2}{\epsilon_0 c n}\right)$ Define the effective nonlinearity γ

in m^2/W

$$\gamma \equiv \frac{n_2\,\omega}{(\pi r^2)c} =$$

Thus

$$\boxed{\partial_z U(t,z) = i\gamma P_0\, |U(t,z)|^2\, U(t,z)}$$

Solution

$$\boxed{\begin{aligned} U(z,t) &= U(0,t)\exp\left(i\,\phi_{NL}(z,t)\right) \\ \phi_{NL}(z,t) &= \gamma P_0\, z\, |U(0,t)|^2 \end{aligned}}$$

Define nonlinear length $L_{NL} = \frac{1}{\gamma P_0}$

So $\phi\,(z,t) = z/\ |U(0,t)|^2$

Start with

$$\partial_z^2 \mathcal{E}_T - \mu_0 \epsilon_0 \partial_t^2 \mathcal{E}_T = \mu_0 \partial_t^2 \mathcal{P}_{NL}$$

(Nonlinear wave eq)

$$\mathcal{E}_T = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3$$

$$\partial_z^2 \mathcal{E}_T = \partial_z^2 \mathcal{E}_1 + \partial_z^2 \mathcal{E}_2 + \partial_z^2 \mathcal{E}_3$$

$$\partial_z^2 \mathcal{E}_1 = \partial_z \partial_z \left(E_{01} \exp(ik(\omega)z) \exp(-i\omega t) + c.c. \right)$$

$$= \partial_z \left(\partial_z E_{01} \exp(\,)\exp(\,) + ik E_{01} \exp(\,)\exp(\,) + c.c. \right)$$

$$= \partial_z \left(\partial_z E_{01} \underbrace{-i\, 2k\, E_{01}}_{\text{use this}} - k^2 E_{01} \right) \exp\left(i(kz - \omega t)\right) + c.c.$$

Self phase modulation: Completely degenerate FWM (Boyd Notation)

$$P_{NL} = 3\epsilon_0 \chi^{(3)}_{xxxx}(\omega; \omega, -\omega, \omega)\, E E^* E$$

$$\mathcal{P}_{NL} = P_{NL} e^{-i\omega t} + c.c. \qquad \mathcal{E}_i = E_i e^{-i\omega t} + c.c.$$

$$\mathcal{E}_T = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3$$

$$\partial_z^2 \mathcal{E}_T - \mu_0 \epsilon_0 \partial_t^2 \mathcal{E}_T = \mu_0 \partial_t^2 \mathcal{P}_{NL}$$

Use SVEA $\left(\partial_z^2 \mathcal{E} = \partial_z^2 E - i\, 2k\, \partial_z E - k E_{01}\right)\exp(i(kz-\omega t)) + c.c.$

$$\boxed{\partial_z E = i \frac{3\mu_0 \epsilon_0 c\, \omega}{2 n_0} \chi^{(3)}_{xxxx} |E|^2 E} \qquad (13.3.1)$$

From SVEA + ∂_z^2

Define $U(z,t) \equiv$ Normalized $\left(I = 2 n \epsilon_0 c\, |E|^2\right)$

$$E(t) = \sqrt{\frac{P_0}{\pi r^2}} \sqrt{\frac{1}{2 n \epsilon_0 c}}\, U(z,t)$$

$$\cancel{\sqrt{\frac{P_0}{\pi r^2}}}\, \cancel{\sqrt{\frac{1}{2 n_0 \epsilon_0 c}}}\, \partial_z U = i \frac{3\mu_0 \epsilon_0 c\, \omega}{2 n_0} \chi^{(3)}_{xxxx} |U|^2 U \cancel{\sqrt{\frac{P_0}{\pi r^2}}} \frac{P_0}{\pi r^2} \cancel{\sqrt{\frac{1}{2 n_0 \epsilon_0 c}}} \frac{1}{2 n_0 \epsilon_0 c}$$

$$\partial_z U = i \frac{3\mu_0 \omega}{4 n_0^2} \chi^{(3)} |U|^2 U \frac{1}{\pi r^2}$$

$$\partial_z U = i\left(\overbrace{\left(\frac{3\chi^{(3)}}{4n_0^2\epsilon_0 c}\right)}^{n_2}\left(\frac{\omega}{c}\right)\left(\frac{1}{\pi r^2}\right)\right) P_0 \, |U|^2 U$$

Where $\boxed{n_2 = \dfrac{3\chi^{(3)}}{4n_0^2\epsilon_0 c}}$ $\quad \dfrac{m^2}{W}$

Define $\gamma \equiv \dfrac{n_2\,\omega}{(\pi r^2)c}$ $\quad \dfrac{1}{W\,m}$

Then

$$\boxed{\partial_z U(z,t) = i\gamma P_0 \, |U(z,t)|^2 \, U(z,t)}$$

Solution

$$\boxed{\begin{aligned} U(z,t) &= U(0,t)\ \exp(\phi_{NL}(z,t)) \\ \phi_{NL}(z,t) &= \gamma \cdot P_0 z \, |U(0,t)|^2 \end{aligned}}$$

Define nonlinear length $\boxed{L_{NL} = \dfrac{1}{\gamma P_0}}$

$$\boxed{\phi_{NL}(z,t) = \frac{z}{L_{NL}} \, |U(0,t)|^2}$$

- Nonlinear length

$$\boxed{L_{NL} = \frac{1}{\gamma P_0}}$$ in units of meters

distance to travel to experience 1 radian nonlinear phase shift.

- Units for n_2 : Technically m^2/V^2 $\boxed{\chi^{(3)} \rightarrow m^2/V^2}$ General units $\left(\chi^{(n)} \rightarrow \frac{m^{n-1}}{V^{n-1}}\right)$

Common to quote n_2 in m^2/W $\boxed{n_2 \rightarrow \frac{2n_2}{\varepsilon_0 c n_0}}$

For fused silica $n_2 = 3\times10^{-20}\ m^2/W$

- Units for the effective nonlinearity

$$\gamma \rightarrow \frac{1}{W\,m}$$

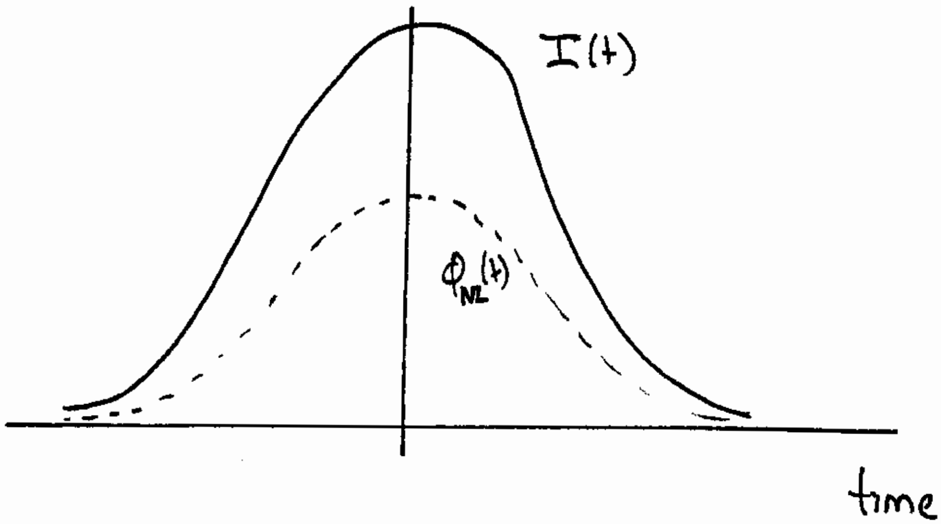
- Dispersion length

$$\boxed{L_D = \frac{T_0^2}{|\beta_2|}}$$ 1/e half width

Where $$T_0 = \frac{\Delta t}{2\ln(1+\sqrt{2})}$$ FWHM

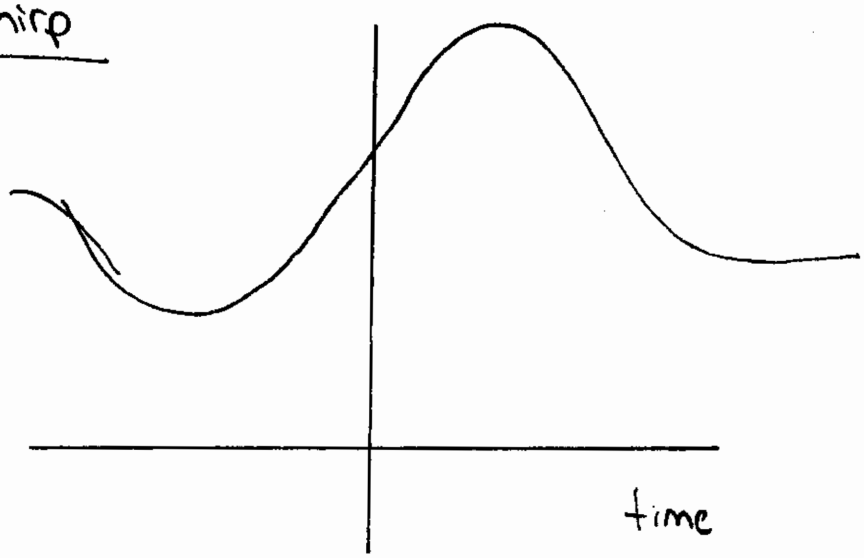
SPM For Pulses

$$E(t) = E_0 \,\mathrm{sech}\,(t/\Delta t)$$



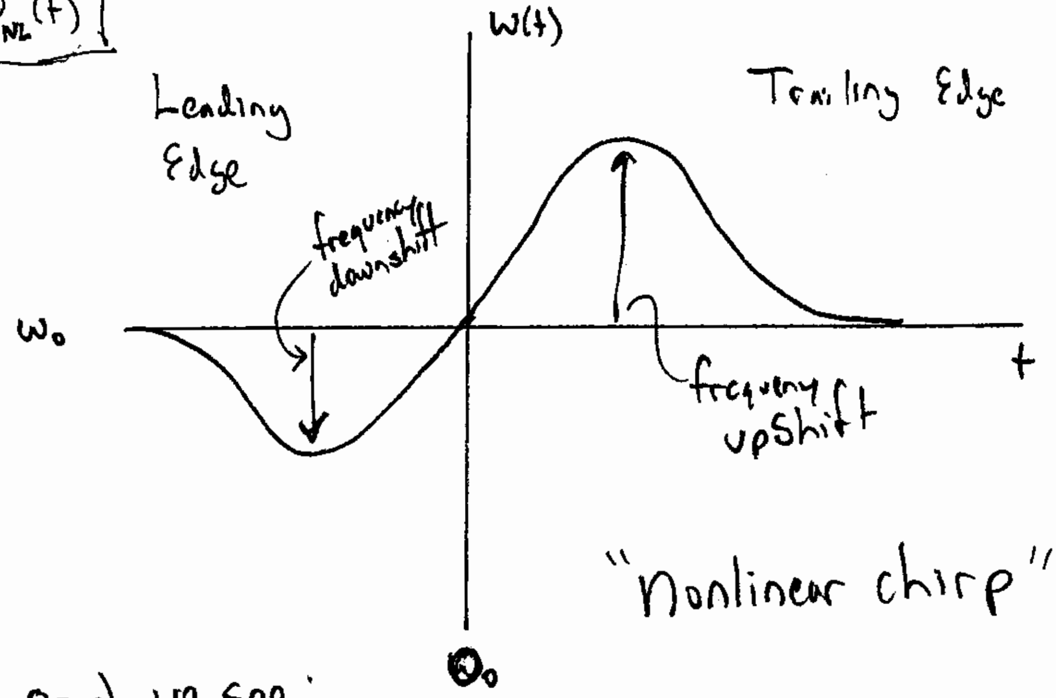
$$\left\{\begin{array}{l} I(t) \sim \mathrm{sech}^2(t) \\ \phi_{NL}(t) \sim \gamma P_0 \,\mathrm{sech}^2(t) \\ \text{Chirp} \sim \omega_0 - \partial_t \phi_{NL} \end{array}\right.$$

Chirp



How to think about this chirp!?

$\boxed{\omega_0 - \partial_t \phi_{NL}(t)}$



From the graph we see:

- Spectral components in leading edge experience frequency down shift (to lower ω) $\Delta\omega < 0$.
- Spectral components in trailing edge experience frequency up shift (to higher ω) $\Delta\omega > 0$.

So the effect of SPM depends on how a pulse is initially chirped by say by dispersion. Consider what happens for negatively + positively chirped pulses.

- Negatively chirped pulses

  SPM causes a frequency downshift of the blue components, an upshift of the red components

  $\Rightarrow$ Spectral Compression

- Positively Chirped Pulses

  SPM causes a frequency downshift of the red components, ... $\Rightarrow$ Spectral

Lecture 20

Nonlinear Fiber Optics : Phase matching for partially degenerate FWM

As mentioned, n_2 is a very small value, on the order of $10^{-20}\ m^2/W$. However, the phase shift due to a third order nonlinearity can be quite large

$$\phi_{NL}(t) = |U(0,t)|^2 \ z/L_{NL}$$

where $L_{NL} = \frac{1}{\gamma P_0}$ + $\gamma = \frac{n_2 \omega}{\pi r^2 c}$

We can make ~~L_{NL} small~~ ϕ_{NL} large by having:

1) Large Peak Power P_0
2) Long distance z
3) Small area πr^2

~~4)~~

even if n_2 is a small number

This is the reason why to use an optical fiber for $\chi^{(3)}$ ...onlinear medium. An optical fiber offers:

1) A small area πr^2

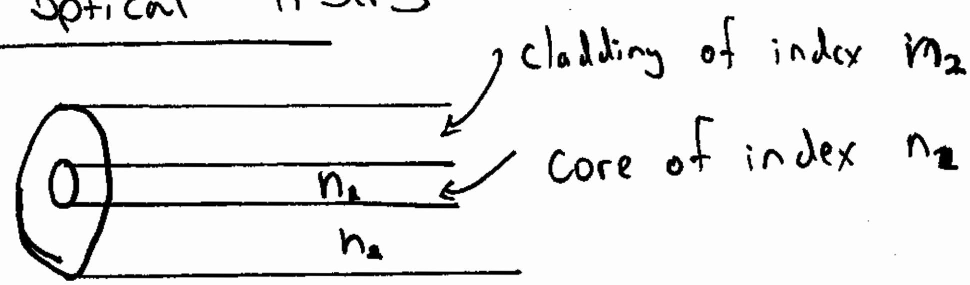
2) A long interaction length z without significant loss. Fiber loss 0.2 dB/km @ 1550nm.

3) High Peak power P_0 if pulses are used.

Introduction to optical fibers

Cladding radius $\sim 62\ \mu m$

Core radius $\sim 5 \mu m$



Fiber guide light using total internal reflection so

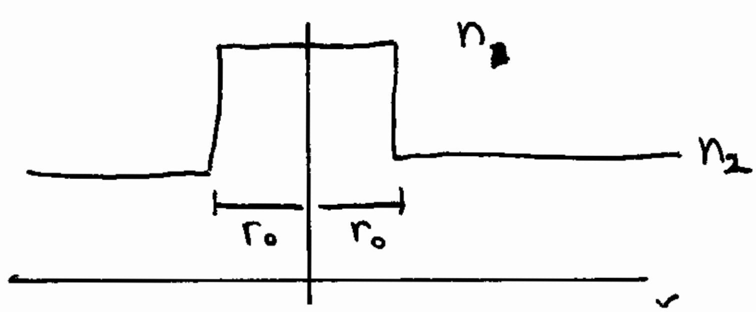
$$n_1 > n_2$$

this is not the nonlinear index!!!

Typically $\quad \Delta n \equiv \dfrac{n_1^2 - n_2^2}{2n_1^2} \simeq \dfrac{n_1 - n_2}{n_1} \simeq 0.008$

So there is a small index difference between core + cladding.

The core is typically fused silica doped with Ge, the cladding is just fused silica. This is called a step index fiber



V Parameter

A normalized frequency that takes into account the fiber's structural parameters and frequency

$$V = \frac{n_1(\omega)\,\omega\, r_0}{c}\sqrt{n_1^2 - n_1^2}$$

$$\boxed{V(\omega) = \frac{n_1(\omega)\,\omega\, r_0}{c}\sqrt{2\Delta n}}$$

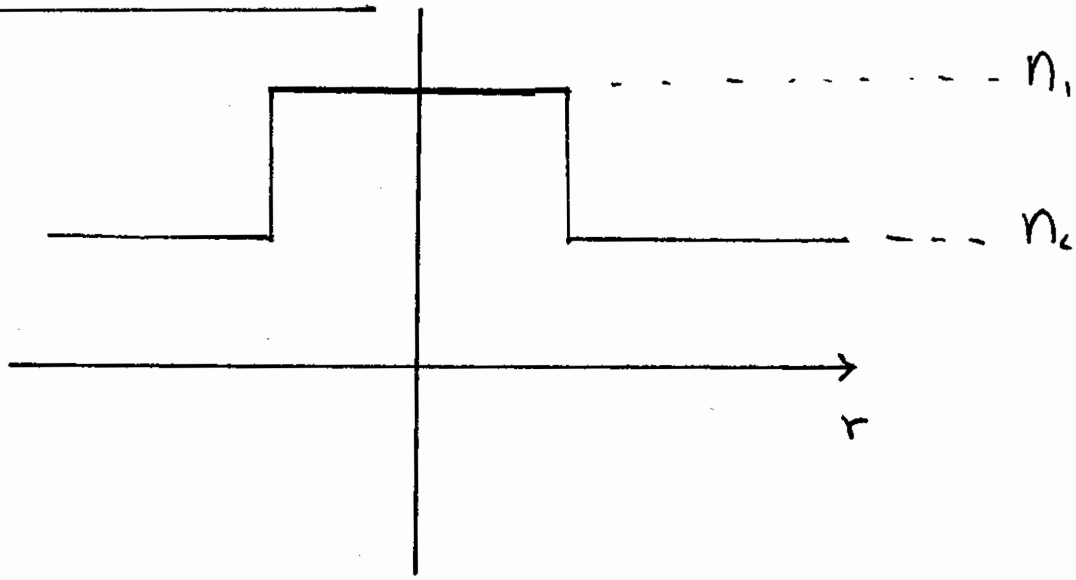
The V parameter is used to determine the cut off wavelength and mode structure of a fiber.

Normalized propagation constant $b(V)$

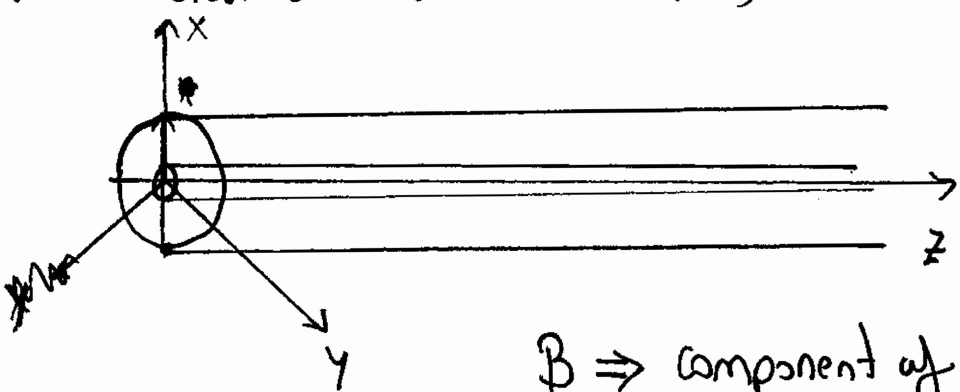
$$b = \left(\frac{\beta^2 - n_2^2\omega^2/c^2}{n_1^2\omega^2/c^2 - n_2^2\omega^2/c^2}\right) = 1 - \frac{r_0^2\left(n_1^2\omega^2/c^2 - \beta^2\right)}{V^2}$$

$$\simeq \frac{n_{eff} - n_2}{n_1 - n_2} \Rightarrow$$ Relative change of index due to waveguide

More on fiber Modes



Solve for the electric field $\vec{E}(x\,y\,z)$ in fiber



$\beta \Rightarrow$ component of $\vec{k}$ along $\hat{z}$

Within the core $E_z \sim e^{i\beta z}$ ~~exponential~~ sinusodal $\Rightarrow$ traveling wave

outside core $E_z \sim e^{-\beta z}$ exponential $\Rightarrow$ evanescent wave

Solutions $E(x\,y\,z) \Rightarrow$ solutions are a set $\Rightarrow$ modes

$l=0$	Lowest mode	HE_{11},	(LP_{01})
$l=1$	Higher order modes	TE_{01}, TM_{01}, HE_{21}	(LP_{11})
$l=2$	"	EH_{11} HE_{31}	(LP_{21})

General Electric fields in a fiber

$\underline{E_z}$

$$r<r_0 \quad E_z \sim J_q(ur/r_0)\sin(q\phi)$$

$$r>r_0 \quad E_z \sim K_q(ur/r_0)\sin(q\phi)$$

where $J_q \equiv$ Bessels function of 1st kind (osuillatory)

$K_q \equiv$ modified Bessels function of 1st kind (exponential)

So within the core we have osuillatory solutions and in the cladding we have exponential (~~ev~~ evenscent) solutions.

Cut off Wavelength λ_c

$$\lambda_c = \frac{2\pi a}{V_c}\sqrt{n_1^2 - n_2^2} \qquad \left\{ \begin{array}{l} \text{Single mode} \\ V_c = 2.405 \end{array} \right.$$

The boundary conditions set up a cutoff wavelength where the mode will propagate.

- The lowest order mode does not have cutoff wavelength. (HE_{11})
- The next higher mode has a cutoff wavelength (TE_{01} TM_{01} HE_{21})
- So for <u>single mode operation</u> we only want the lowest order mode to propagate in the fiber

For single mode operation we want $\lambda > \lambda_c$

Cut off condition

$$\frac{V\,J_{l-1}(V)}{J_{l-1}(V)} = 0 \Rightarrow \boxed{J_{l-1}(V) = 0}$$

For single mode $\quad J_0(V_c) = 0 \qquad V_c = 2.405$

$l = 1 \qquad 1-1=0$

Find cutoff for $l=1$ modes)

The step index creates a set of boundary conditions for the optical electric field in the fiber.

Just like in a metallic waveguide, one gets modes in the optical fiber.

Each fiber, for a given r_o, exhibits a cut off wavelength λ_c. Only one mode will propagate if its wavelength is $\lambda > \lambda_c$

$$\left\{ \begin{array}{l} \text{Typically} \quad \lambda_c \simeq 1270 \text{ nm} \quad \text{for } r_o = 5\,\mu\text{m} \\ \text{The smaller the radius, the shorter the } \lambda_c. \end{array} \right\}$$

Single mode fibers are optical fibers used for $\lambda > \lambda_c$

~~Dispersion in optical fibers~~

~~Not only does the optical fiber exhibit the dispersion of the fused silica (i.e. material dispersion), the wave guiding modifies the index as well~~

$$\cancel{D(\lambda) = D_m(\lambda) + D_w(\lambda)}$$

~~material dispersion~~ ~~waveguide dispersion~~

Dispersion in optical fibers

An optical fiber has dispersion due to 1) material (glass) dispersion and 2) due to the structure of the wave guide.

$$\text{Total Dispersion} = (\text{Material Dispersion}) + (\text{Waveguide Dispersion})$$

The advantage of a fiber is that the total dispersion can be altered by just changing the wave guide structure.

Like for a bulk material, we describe the wave propagation constant $\beta(\omega)$ for the fiber.

For a bulk material we characterized the dispersion using

$$\beta(\omega) = \beta_0 + \beta_1(\omega-\omega_0) + \frac{1}{2}\beta_2(\omega-\omega_0)^2 + \cdots$$

For bulk $$\beta_2(\omega) = \frac{1}{c}\left(2\frac{dn}{d\omega} + \omega\frac{d^2n}{d\omega^2}\right) \simeq \frac{\lambda^3}{2\pi c^2}\frac{d^2n}{d\lambda^2}$$

(Group velocity dispersion)

For a fiber we still will use $\beta(\omega)$ but we cannot use the above expression for β_2 since it does not take into account the change in dispersion due to the waveguide.

For a fiber the mode propagation constant $\beta(\omega)$ can be written as

$$\boxed{\beta(\omega) = \frac{n_2(\omega)\,\omega}{c}\sqrt{1 + 2\Delta n\, b(\omega)}}$$

Where $$b(\omega) \equiv \frac{n_{eff} - n_2}{n_1 - n_2}$$

$b(\omega)$ is a normalized propagation constant for the fiber. It is related to the effective guide index n_{eff} due to the wave guide

Another usefull normalized parameter is the V parameter

$$V(\omega) = \frac{r_0\,\omega}{c}\, n_2(\omega)\sqrt{2\Delta n} \equiv \frac{r_0\,\omega}{c}\sqrt{n_1^2(\omega) - n_2^2(\omega)}$$

Cuttoff Frequency & V

Notice that V is unitless and it depends on the core radius. The single mode condition is just

$$\frac{V J_{\ell-1}(V)}{J_\ell(V)} = 0 \qquad J_\ell \equiv \text{Bessel function}$$

$$J_0(V) = 0$$

$$\text{OR} \quad \boxed{V_0 = 2.405}$$

This is the cuttoff frequency for <u>every</u> step index fiber. This is why we use a normalized frequency.

For $V < V_0$ only the HE_{11} mode will propagate. (LP_{01})

Normalized Propagation Constant

A good approximate form for $b(\omega)$ is given by

$$\boxed{b(\omega) = 1 - \left(\frac{1+\sqrt{2}}{1+\sqrt[4]{4+V(\omega)}}\right)^2}$$

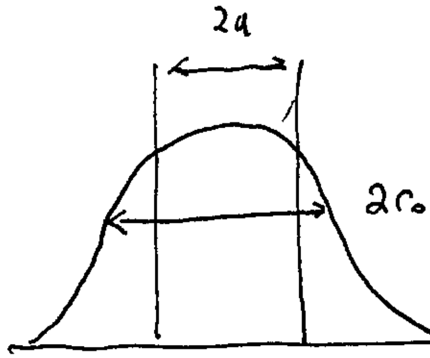
use this for the mini project.

(see Gloge, Applied Optics vol 10 p 2258, 1971)

Mode field Radius r_0

$$\boxed{\frac{r_0}{a} \simeq 0.65 + \frac{1.619}{V^{3/2}} + \frac{2.879}{V^6}}$$

core radius



$r_0 \sim 1/e^2$ half width of intensity,

or $1/e$ half width of Field

$$E(r) \sim E_0 \exp(-r^2/r_0^2)$$

Effective area in γ

$$\boxed{\gamma = \frac{n_2 \omega}{(\pi r_0)^2 c}}$$

mode field radius

Partially Degenerate FWM in optical fibers

Partially Degenerate FWM $\Rightarrow$ $2\omega_p = \omega_s + \omega_i$

where $2\omega_p > \omega_i > \omega_s$

$$\left(\begin{array}{ll} i \equiv \text{idler} & s \equiv \text{signal} \\ p \equiv \text{pump} & \end{array} \right)$$

Notice this "looks" like DFG!

Unlike SPM, this process needs to be phase matched

$$\Delta k = 0$$

However, the waveguide of the fiber adds additional terms to the phase mismatch Δk

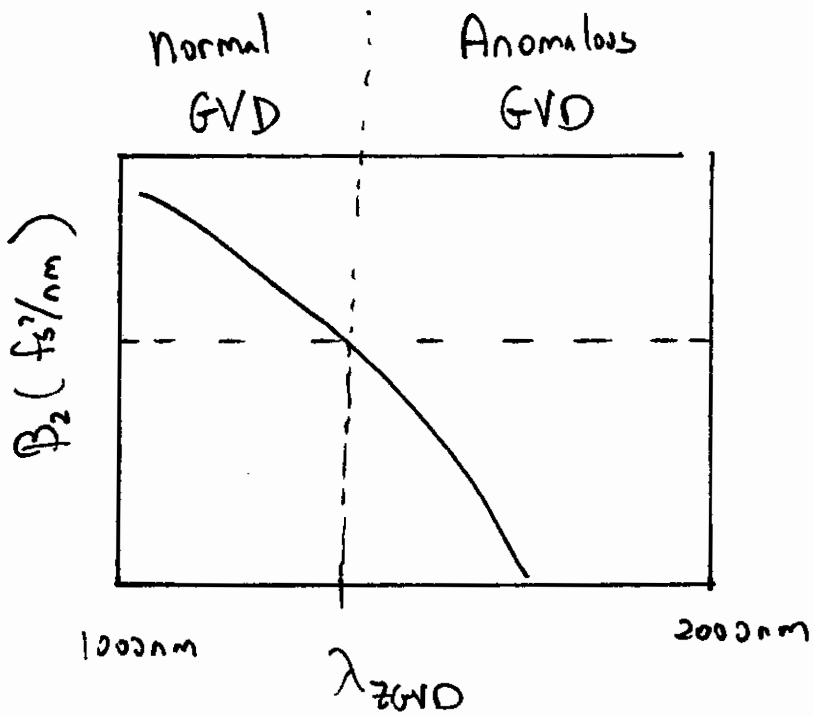
$$\underset{(\Delta\beta)}{\Delta k} = \underbrace{\Delta k_m}_{\text{material}} + \underbrace{\Delta k_w}_{\text{waveguide}} + \underbrace{\Delta k_{NL}}_{\text{nonlinear contribution}}$$

where $$\Delta k_m = \frac{1}{c}\left(n(\omega_s)\omega_s + n(\omega_i)\omega_i - 2n(\omega_p)\omega_p \right)$$

$$\Delta k_W = \frac{\Delta n}{c}\left(b(\omega_s)\omega_s + b(\omega_i)\omega_i - 2b(\omega_p)\omega_p \right)$$

$$\Delta k_{NL} = 2\gamma P_0 \qquad \left(\gamma = \frac{n_2\omega}{c\pi r^2} \text{ as before} \right)$$

Sketch $\beta_2(\omega)$



$\lambda_{ZGVD} \sim 1322\,nm$ for $r_0 \simeq 5\,\mu m$

Where we defined

$$\beta_2 > 0 \Rightarrow \text{Normal GVD}$$

$$\beta_2 < 0 \Rightarrow \text{Anomalous GVD}$$

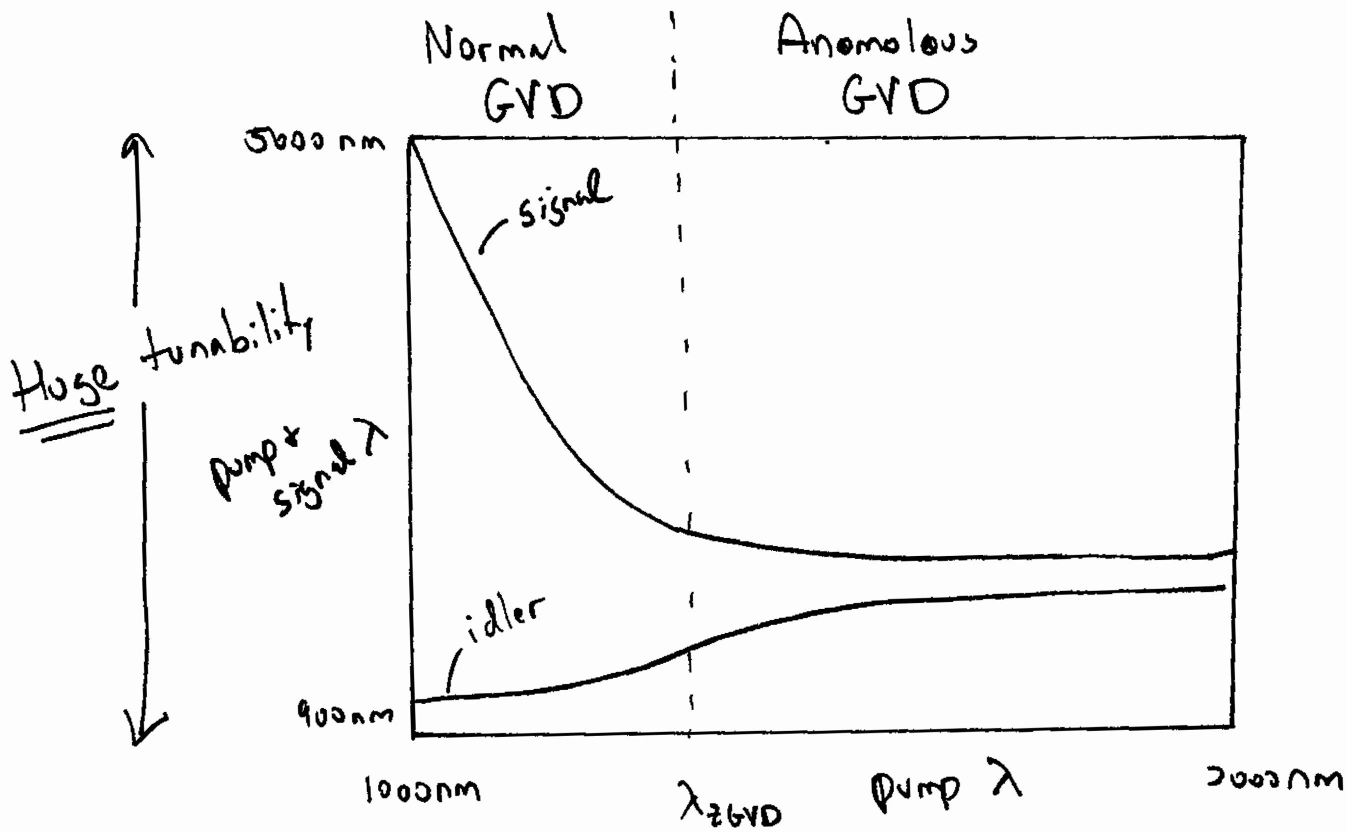
<u>Define dispersion length</u>

$$\boxed{L_D = \frac{T_0^2}{|\beta_2|}}$$ $T_0 \equiv 1/e$ half width

$$T_0 = \frac{\Delta t}{2\ln(1+\sqrt{2})}$$ $\Delta t \equiv$ FWHM for sech $(t/\Delta t)$ fields

Notice that is really looks familiar to OPO / OPA !

In fact, one can make a fiber-based OPO in this manner. This OPO will have a huge tuning range.



The behavior of the signal + idler is different about the fiber's zero group velocity dispersion wavelength λ_{ZGVD}

$$\left(\text{where} \quad \beta_2(\omega_z) = 0 \qquad \text{and} \quad \lambda_{ZGVD} \equiv \frac{2\pi c}{\omega_z} \right)$$

Remember $\quad \beta_2(\omega_z) \equiv \left. \dfrac{\partial^2 \beta(\omega)}{\partial \omega^2} \right|_{\omega = \omega_z}$

Gain Due to OPA process

$$G(z) = \left| \frac{r}{g} \sinh(gz) \right|^2$$

$$r = 2\gamma A_1(0) A_2(0)$$

$$g^2 = r^2 - (K/2)^2$$

$$K = \Delta\beta + \Delta\beta_{NL} \quad (\Delta\beta)$$

D parameter

$$D(\lambda) = -\frac{2\pi c}{\lambda^2} \frac{d\beta^2}{d\omega}$$

$$= \frac{2\pi c}{\lambda^2} \beta_2$$

Small signal gain

$$G_s = \frac{P_s}{P_s(0)} = 1 + \left(\frac{\gamma P_p}{g}\sinh(gL)\right)^2$$

$$\boxed{g = \sqrt{-\Delta\beta\left(\Delta\beta/4 + \gamma P_p\right)}}$$

[illegible]

$$\Delta\beta = \beta_s + \beta_i - 2\beta_p$$

Agrawal $\quad g = \sqrt{(\gamma P_0 r)^2 - ((\Delta k + 2\gamma P_0)/2)^2}$

$$r = 2P_1P_2/P_0 = \frac{2P_1P_2}{P_1+P_2}$$

$$\boxed{\cancel{g = \sqrt{(\gamma P_0)^2 - ((\Delta k + 2\gamma P_0)/2)^2}}} \quad g = \sqrt{(\gamma P_0)^2 - (\Delta k + 2\gamma P_0/2)^2}$$

$$\gamma P_0 - \left(\frac{\Delta k^2}{4} + 2\gamma P_0 + \gamma P_0^2\right)$$

$$\boxed{g = \sqrt{-\Delta k\left(\frac{\Delta k}{4} + \gamma P_0\right)}}$$

$$\boxed{G_A = \frac{P_3(L)}{P_3(0)} = \left(1 + \frac{(\Delta k + 2\gamma P_0)^2}{4g^2}\right)\sinh^2(gL)}$$

$\boxed{\Delta k = k_s + k_i - 2k_p}$ $\quad \boxed{\Delta k < 0}$

Completely degenerate FWM

$$\bar{k}_1 = \bar{k}_2 = \bar{k}_3 \qquad E_1 = E_2 = E_3$$

$$\omega_1 = \omega_2 = \omega_3$$

$$P_{NL} \cong \epsilon_0 \chi^{(3)}_{xxxx} |E|^2 E$$

Remember:

$$\mu = K_m \mu_0$$

$$\varepsilon = K_e \epsilon_0$$

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \qquad v = \frac{1}{\sqrt{\mu \varepsilon}} = c/n$$

Into wave eq

$$P = P_L + P_{NL} = \epsilon_0 \left[\chi^{(1)} E + \chi^{(3)} |E|^2 E \right]$$

Assume $\partial_t^2 |E|^2 E \simeq |E|^2 \partial_t^2 E$ (SVEA)

$$\partial_z^2 E - \frac{1}{c^2} \partial_t^2 E = \mu_0 \epsilon_0 \left[\chi^{(1)} + \chi^{(3)} |E|^2 \right] \partial_t^2 E$$

$\wedge$

or

$$\boxed{\partial_z^2 E - \frac{\left[1 + \chi^{(1)} + \chi^{(3)} |E|^2 \right]}{c^2} \partial_t^2 E = 0}$$

Wave eq for a linear material

$$\partial_z^2 E - \frac{n^2}{c^2} \partial_t^2 E = 0$$

or

$$\partial_z^2 E - \mu\epsilon \, \partial_t^2 E = 0$$

Looks like

$$\partial_z^2 E - \frac{n^2}{c} \partial_t^2 E = 0 \quad \text{where } n = \sqrt{1 + \chi^{(1)} + \chi^{(3)} |E|^2}$$

Remember $n = \sqrt{1 + \chi^{(1)}}$

Usually the linear index of refraction looks like

$$n_0 = \sqrt{1 + \chi^{(1)}}$$

$$n = n_0 \sqrt{1 + \chi^{(3)} |E|^2 / n_0^2}$$

assume the nonlinear term is $\ll n_0$

$$n \simeq n_0 \left(1 + \chi^{(3)} |E|^2 / 2n_0^2 \right)$$

$$n \simeq n_0 + \chi^{(3)} |E|^2 / 2n_0$$

$$\boxed{n \cong n_0 + n_2 I} \qquad I \sim |E|^2$$

nonlinear index of refraction

Really $\boxed{n_2 = \dfrac{3 \operatorname{Re}\{\chi^{(3)}_{xxxx}\}}{8 n_0}}$ $\qquad n_2^I = \dfrac{3\chi}{8 n_0} \left(\dfrac{2}{\varepsilon_0 c n_0} \right)$

Nonlinear Index of Refraction

All three inputs copolarized $\Rightarrow \chi^{(3)}_{xxxx}$

$$P_{NL}^{\omega} = \frac{3}{4}\epsilon_0 \chi_{xxxx}(\omega; \omega, -\omega, \omega) |E_0|^2 E_{01}\, e^{i k(\omega) z}$$

Put into wave eq.

$$-i\, 2k \frac{dE_{01}}{dz} - k^2 E_{01} + \omega^2 \mu_0 \epsilon_0 n_0^2 E_{01} = -\frac{3}{4}\omega^2 \mu_0 \epsilon_0 \chi_{xxxx} |E_{01}|^2 E_{01}$$

assume $\frac{dE_{01}}{dz} = 0$ Solve for k^2

$$k^2 = \omega^2 \mu_0 \epsilon_0 \left(n_0^2 + \frac{3}{4}\chi_{xxxx} |E_{01}|^2 \right)$$

Rewrite as . . . .

$$n = n_0 \left(1 + \frac{3\chi_{xxxx}}{4 n_0^2} |E_{01}|^2 \right)^{1/2}$$

$$n \simeq n_0 + n_2 I_{01}$$

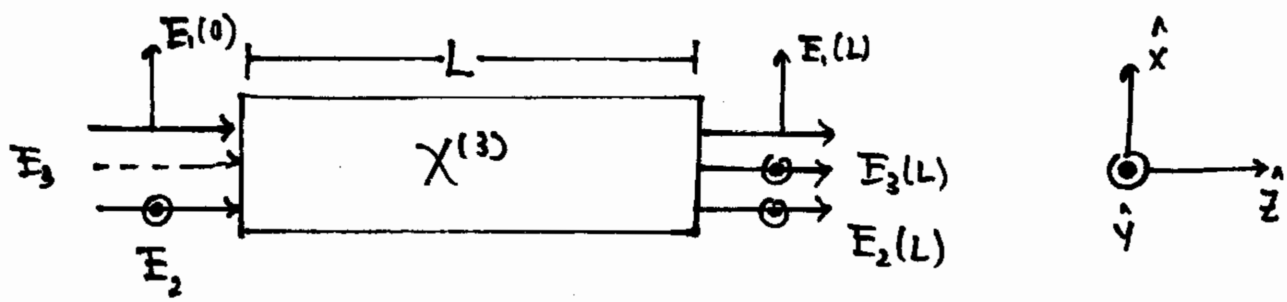
where $\boxed{n_2 = \frac{3\chi^{(3)}_{xxxx}}{8 n_0}}$

Notice we never talked about phase matching.

Since this process is completely degenerate, it is automatically phase matched. In general FWM processes must be phase matched.

Third harmonic generation

We expect this to be a phase matched process. We will go back to the more general situation



Inputs $E_1(0)$ + $E_2(0)$ have frequency ω and are orthogonally polarized. The field E_3 has frequency 3ω.

Write down the field in real instantaneous forms

$$\vec{\mathcal{E}}_1 = \left[\tfrac{1}{2}E_{01}\ \exp(ik(\omega)z)\exp(-i\omega t) + \text{c.c.}\right]\hat{x}$$

$$\vec{\mathcal{E}}_2 = \left[\tfrac{1}{2}E_{02}\ \exp(ik(\omega)z)\exp(-i\omega t) + \text{c.c.}\right]\hat{y}$$

and

$$\vec{\mathcal{E}}_3 = \left[\tfrac{1}{2}E_{03}\ \exp(ik(3\omega)z)\exp(-i3\omega t) + \text{c.c.}\right]\hat{y}$$

where $\quad k(\omega) = \dfrac{n(\omega)\,\omega}{c} \quad + \quad k(3\omega) = \dfrac{n(3\omega)\,3\omega}{c}$

Make the assumption that $E_{02} < E_{01}$ + $E_{03} < E_{01} \Rightarrow$ Strong pump approximation

Find the nonlinear polarization

$$\vec{P}_{NL} = \epsilon_0 \chi^{(3)} \left(\vec{\mathcal{E}}_1 + \vec{\mathcal{E}}_2 + \vec{\mathcal{E}}_3\right)^3$$

$$= \epsilon_0 \chi^{(3)} \left[\, \mathcal{E}_1^3 + \mathcal{E}_2^3 + \mathcal{E}_3^2 + 6\,\mathcal{E}_1\mathcal{E}_2\mathcal{E}_3 + 3\,\mathcal{E}_1^2\mathcal{E}_2 + 3\,\mathcal{E}_1\mathcal{E}_2^2 + 3\,\mathcal{E}_1^2\mathcal{E}_2 + 3\,\mathcal{E}_2^2\mathcal{E}_3 + 3\,\mathcal{E}_1\mathcal{E}_3^2 + 3\,\mathcal{E}_2\mathcal{E}_3^2 \,\right]$$

What a mess! To keep things simple since $E_{02} < E_{01}$ + $E_{03} < E_{01}$ we will only keep terms of the produce of $E_{01} \times E_{02}$ or $E_{01} * E_{03}$

the resulting P_{NL} terms will oscillate at ω + 3ω

$$P_{NL} = P_{NL}^{\omega} + P_{NL}^{3\omega} \begin{cases} P_{NL}^{\omega}(\omega) = P_{NL}^{\omega}\, e^{-i\omega t} + P_{NL}^{*}\, e^{+i\omega t} \\ P_{NL}^{3\omega} = P_{NL}^{3\omega}\, e^{-i3\omega t} + P_{NL}^{3\omega}\, e^{+i\omega t} \end{cases}$$

Where

$$P_{NL}^{\omega} = \frac{3}{4}\epsilon_0 \chi_{yxxy}(\omega;\omega,-\omega,\omega)\left[2|E_{01}|^2E_{02} + E_{01}^2E_{02}^*\right]e^{ik(\omega)z}$$

$$+\frac{3}{4}\epsilon_0 \chi_{yxxy}(\omega;-\omega-\omega,3\omega)(E_{01}^*)^2E_{03}\quad \exp(+i(k(3\omega)-2k(\omega))z)$$

$$P_{NL}^{3\omega} = \frac{6}{4}\epsilon_0 \chi_{yxxy}(3\omega;\omega,-\omega,3\omega)|E_{01}|^2E_{03}\,e^{ik(3\omega)z}$$

$$+\frac{3}{4}\epsilon_0 \chi_{yxxy}(3\omega;\omega,\omega,\omega)E_{01}^2E_{02}\quad \exp(+i3k(\omega)z)$$

the wave eq gives

$$\boxed{\partial_z^2(\bar{\xi}_1+\bar{\xi}_2+\bar{\xi}_3) - \mu_0\epsilon_0 n^2\partial_t^2(\bar{\xi}_1+\bar{\xi}_2+\bar{\xi}_3) = \mu_0\partial_t^2\vec{\bar{P}}_{NL}}$$

Here we will invoke the slowly varying envelope approximation again

$$|k\,\partial_z E_{0i}| \gg |\partial_z^2 E_{0i}|$$

We will then get

Remember the slowly varying envelope approximation

$$|k\,\partial_z E| \gg |\partial_z^2 E|$$

It can be rewritten as

$$\left|\lambda\,\partial_z\left(\partial_z E_{0i}\right)\right| \ll \left|\partial_z E_{0i}\right|$$

– The change in the slope of the field envelope over distance λ is much less than the magnitude of the slope itself.

This expression is valid for pulses except where

$$\boxed{\Delta t \simeq \frac{2\pi}{\omega_0}} \Longrightarrow \text{at } 800\text{ nm} \quad \frac{2\pi}{\omega_0} \simeq 2\pi\frac{\lambda_0}{2\pi c} \approx \frac{\lambda_0}{c} \simeq \frac{800\text{ nm}}{300\ \frac{\text{nm}}{\text{fs}}}$$

$$= 2.67\text{ fs}$$

$$\left[(k(\omega))^2 \left(E_{01} + E_{02} \right) + i\, 2k(\omega)\, \partial_z E_{02} \right] \exp(ik(\omega)z - i\omega t)$$

$$+ \left(k^2(3\omega) E_{03} + i\, 2k(3\omega)\, \partial_z E_{03} \right) \exp(i\, k(3\omega) z - i3\omega t)$$

$$- \mu_0 \epsilon_0\, n^2(\omega)\, \omega^2 \left(E_{01} + E_{02} \right) \exp(-i\omega t + k(\omega) z)$$

$$- \mu_0 \epsilon_0\, n^2(3\omega)\, (3\omega)^2 \left(E_{03} \right) \exp(-i3\omega t + k(3\omega) z)$$

$$= \mu_0 \omega^2 P_{NL}^{\omega} \exp(-i\omega t) + \mu_0 (3\omega)^2 P_{NL}^{3\omega} \exp(-i3\omega t)$$

by separating the above eq in ω + 3ω, we get two coupled DEs

$$\frac{dE_{02}}{dz} = \frac{-i\,3\omega}{8\, n(\omega) c} \Big[\chi_{yxxy}(\omega; \omega, -\omega, \omega) \left[2 |E_{01}|^2 E_{02} + E_{01}^2 E_{02}^* \right] + \chi_{yxxy}(\omega; -\omega, \omega, 3\omega)\, E_{01}^{*2} E_{03}\, e^{-i\Delta k z} \Big]$$

$$\frac{dE_{03}}{dz} = \frac{-i\,3\omega}{8\, n(3\omega) c} \Big[6\, \chi_{yxxy}(3\omega; \omega, -\omega, 3\omega) |E_{01}|^2 E_{03} + 3\, \chi_{yxxy}(3\omega; \omega, \omega, \omega)\, E_{01}^2 E_{02}\, e^{+i\Delta k z} \Big]$$

$$\frac{dE_{01}}{dz} = 0$$

where $\Delta k = 3\,k(\omega) - k(3\omega)$

$$\left[(k(\omega))^2 \left(E_{01} + E_{02} \right) + i\, 2k(\omega)\, \partial_z E_{02}\right] exp(i k(\omega) z - i\omega t)$$

$$+ \left(k^2(3\omega) E_{03} + i\, 2k(3\omega)\, \partial_z E_{03} \right) exp(i\, k(3\omega) z - i3\omega t)$$

$$- \mu_0 \epsilon_0\, n^2(\omega)\, \omega^2 \left(E_{01} + E_{02} \right) exp(-i\omega t + k(\omega) z)$$

$$- \mu_0 \epsilon_0\, n^2(3\omega)\, (3\omega)^2 \left(E_{03} \right) exp(-i3\omega t + k(3\omega) z)$$

$$= \mu_0 \omega^2 P_{NL}^{\omega}\, exp(-i\omega t) + \mu_0 (3\omega)^2 P_{NL}^{3\omega}\, exp(-i3\omega t)$$

By separating the above eq in ω + 3ω, we get two coupled DEs

$$\frac{dE_{02}}{dz} = \frac{-i\,3\omega}{8\, n(\omega) c} \Big[\chi_{yxxy}(\omega; \omega, -\omega, \omega) \left[2|E_{01}|^2 E_{02} + E_{01}^2 E_{02}^* \right]$$
$$+ \chi_{yxxy}(\omega; -\omega, \omega, 3\omega)\, E_{01}^{*2} E_{03}\, e^{-i\Delta k z} \Big]$$

$$\frac{dE_{03}}{dz} = \frac{-i\,3\omega}{8\, n(3\omega) c} \Big[6\, \chi_{yxxy}(3\omega; \omega, -\omega, 3\omega) |E_{01}|^2 E_{03}$$
$$+ 3\, \chi_{yxxy}(3\omega; \omega, \omega, \omega)\, E_{01}^2 E_{02}\, e^{+i\Delta k z} \Big]$$

$$\frac{dE_{01}}{dz} = 0$$

Where $\Delta k = 3k(\omega) - k(3\omega)$

If we also assume

$$\partial_z E_{02} \simeq 0$$

look familiar

$$\frac{dE_{03}}{dz} = \left(\qquad \qquad \right) \exp(+i \Delta k z)$$

or

$$E_{03}(L) = \frac{-i\, 9\omega \chi_{yxxy}}{8\, n(3\omega)\, c} E_{01}^2 E_{02} L \frac{\sin(\Delta k L/2)}{(\Delta k L/2)} e^{+i\Delta k L/2}$$

$$I_{03} = 2\varepsilon_0 n_3 c |E_{03}|^2 \sim \operatorname{sinc}^2(\Delta k L/2)\, L^2$$

phase match process

where $\Delta k = 3k(\omega) - k(3\omega)$

Two important features

1) $I_{03} \sim L^2$

2) $\operatorname{sinc}^2(\ )$

just like $\chi^{(2)}$!

Lecture 21 Pulse Propagation in Optical Fibers

Review We discussed how the geometry of an optical fiber alters the net dispersion.

$$\beta(\omega) = \frac{n_1(\omega)\,\omega}{c}\sqrt{1 + 2\Delta n\, b(\omega)}$$

$b(\omega) \equiv$ normalized propagation constant

$\Delta n \equiv$ index difference (~ 0.008)

Since Δn is small we can write

$$\beta(\omega) \simeq \frac{n_1(\omega)\,\omega}{c} + 2\Delta n\, b(\omega)$$

$$\beta_{total} \simeq \beta_{material} + \beta_{waveguide}$$

Role of Dispersion on Nonlinear effects

We wish to consider pulse propagation in a $\chi^{(3)}$ material.

In the absence of dispersion, the only effect is SPM

$$E(t,z) = E(t,0)\exp(i\,\phi_{NL}(t))$$

$$\phi_{NL}(t) = |U(t,0)|^2\, z/L_{NL} \qquad \left(L_{NL} = \frac{1}{\gamma P_0}\right)$$

This expression was derived assuming small (or zero) material dispersion. What do we get if we assume dispersion.

How to characterize dispersion : GVD

$\beta_2 \equiv$ GVD

$\beta_2 > 0$ Normal

$\beta_2 < 0$ Anomalous

Dispersion Length (GVD)

$$\boxed{L_D \equiv \frac{T_0}{|\beta_2|}}$$

$T_0 \equiv 1/e$ halfwidth

$$\left(T_0 = \frac{\Delta t}{2 \ln(1+\sqrt{2})} \qquad \Delta t \equiv \text{FWHM for sech( )} \right)$$

A Gaussian pulse will increase its width by $\sqrt{2}$ by propagation L_D.

Compare L_D + L_{NL}

$L_{NL} >> L_D$ Nonlinear material / consider only SPM

$L_D >> L_{NL}$ Dispersive material / consider only GVD

The nonlinear length does not take into account any higher order nonlinearities.

3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER
COMET

Derivation of NLSE

$$\nabla^2 \vec{E} - \frac{1}{c^2} \partial_t^2 \vec{E} = \mu_0 \partial_t^2 \vec{P}$$

Assumptions

1) P_{NL} is small

2) $\vec{E}$ is linearly polarized

3) $\Delta\omega / \omega_0 \approx 1$

5) SVEA

$$P_{NL} = 3\chi^{(3)} |E|^2 E = \varepsilon_0 \varepsilon_{NL} E \qquad \text{(Boyd notation)}$$

3/4 Agrawal

Work in freq. domain

$$\nabla^2 E(\omega) + \overbrace{\left(1 + \chi^{(1)} + 3\chi^{(3)} |E|^2\right)}^{\varepsilon(\omega)} k_0^2 \vec{E} = 0$$

$$\varepsilon(\omega) = 1 + \chi^{(1)} + 3\chi^{(3)} |E|^2$$

Write

$$n = n_0 + n_2 |E|^2 \qquad \alpha = \alpha_0 + \alpha_2 |E|^2 \approx \alpha_0$$

Use separation of variables

slowly varying

$$\vec{E}(\vec{r}, \omega - \omega_0) = F(x, y)\, A(z, \omega - \omega_0) \exp(i \beta_0 z)$$

Get separate Eqs in wave eq.

$$\partial_x^2 F + \partial_y^2 F + \left(\varepsilon(\omega) k_0^2 - \bar{\beta}^2\right) F = 0 \qquad (1)$$

$$2i\beta_0 \, \partial_z A + \left(\bar{\beta}^2 - \beta_0^2\right) A = 0 \qquad (2)$$

Need to solve for $\bar{\beta}$

Approximate $\quad \varepsilon = (n_0 + \Delta n)^2 \simeq n_0^2 + 2 n_0 \Delta n$

$$\simeq n_0^2 + 2 n_0 n_2 |E|^2 + \frac{i\alpha}{2k_0}$$

Solve (1) by 1st order perturbation

Replace ε with n_0^2 $\quad$ Find $F(x,y)$ + $\beta(\omega)$

Include $\quad n_2 |E|^2 + \dfrac{i\alpha}{2k}$ in (1)

Find $\quad \bar{\beta}(\omega) = \beta(\omega) + \Delta\beta$

$$\Delta\beta = \frac{k_0 \int_A \left(n_2|E|^2 + i\alpha/2k_0\right)|F|^2 \, dA}{\int_A |F|^2 \, dA} \qquad \cancel{\Delta\beta = \frac{k_0 \iint \Delta n \, |F(x,y)|^2 \, dA}{\iint |F(x,y)|^2 \, dA}}$$

Back to (2) $\qquad \partial_z A = i\left(\beta(\omega) + \Delta\beta - \beta_0\right) A$

$\beta(\omega) \Rightarrow$ Taylor series

Back transform to time

$$\partial_z A = -\beta_1 \partial_t A - \frac{i}{2}\beta_2 \partial_t^2 A + i\Delta\beta A$$

$$\boxed{\partial_z A + \beta_1 \partial_t A + \frac{i}{2}\partial_t^2 A + \frac{\alpha}{2} A = i\gamma |A|^2 A}$$

In general

$$\boxed{\partial_z E = \overbrace{-\frac{\alpha}{2} E}^{\text{Absorption}} - \overbrace{\left(\sum_{m=2} \beta_m \frac{i^{m-1}}{m!}\partial_t^m\right)}^{\text{dispersion}} E + \overbrace{i\gamma |E|^2 E}^{\text{SPM}}}$$

re write in frame moving with $v_g = \frac{c}{N} = \beta_1^{-1}$

If we add self steepening + SRS

$$\partial_z E = -\frac{\alpha}{2}E - \left(\sum_{m=2}\beta_m \frac{i^{m-1}}{m!}\partial_t^m\right)E + (1-f_R)\left[i\gamma |E|^2 E - \frac{2\gamma}{\omega_0}\partial_t\left(|E|^2 E\right)\right]$$
$$+ i\gamma f_R\left(1 + \frac{i}{\omega}\partial_t\right)\left(E\int h_R(t)\,|E(z,t-t')|^2\,dt'\right)$$

$$\gamma = \frac{n_2\,\omega}{c\,A_{eff}} \qquad A_{eff} = \frac{\left(\int_A |\Psi|^2 dA\right)^2}{\int_A |u|^4 dA}$$

The equation is called the Nonlinear Schrödiner equation because it has the form

$$\partial_z E + \frac{i}{2}\beta_2 \partial_t^2 E - i\, |E|^2 E = 0$$

for $q = q(z,t)$ $\qquad k = -\beta$

$$\boxed{\partial_z q + k\frac{i}{2}\partial_t^2 q - i\,|q^2|\,q = 0}$$

So

$$\boxed{\partial_t q - k\frac{i}{2}\partial_z^2 q = 0}$$

Swap t & z

Wave eq

$$\boxed{\frac{1}{v^2}\partial_t^2 q - \partial_z^2 q = 0}$$

Korteweg - de Vries (KdV)

$$\partial_t q + \partial_x^3 q + 6\, q\, \partial_x q = 0$$

Can we derive a more generic wave eq that is valid for both non linear + dispersive effects?

Start with wave eq

$$\nabla^2 \bar{E} - \frac{1}{c^2} \partial_t^2 \bar{E} = \mu_0 \partial_t^2 \vec{P}$$

Express this eq in Fourier Domain

$$\nabla^2 E(\omega) + \overbrace{\left(1 + \chi^{(1)} + \frac{3}{4}\chi^{(3)}_{xxxx} |E|^2\right)}^{\varepsilon(\omega)} \frac{\omega^2}{c^2} E(\omega) = 0$$

linear dispersion $\qquad \chi^{(3)} \Rightarrow n_2$

$$\varepsilon = (n + \Delta n)^2 \simeq n^2 + 2n\,\Delta n$$

$$\Delta n = n_2 |E|^2 + \frac{i\alpha}{2k_0}$$

Find solution of the form

$$E(r,\omega) = F(x,y)\, A(z,\omega) \exp(i\beta_0 z)$$

For a fiber this represents the fiber modes.

Subsitution gives two coupled Eqs assuming SVEA

call this $n(\omega)$

(1) $$\partial_x^2 F + \partial_y^2 F - \left[\left(\overbrace{1 + \chi^{(1)} + \frac{3}{4}\chi^{(3)}_{xxxx} E}^{\varepsilon(\omega)}\right)\frac{\omega^2}{c^2} - \beta^2(\omega)\right] F(x,y) = 0$$

(2) $$2i\beta_0 \partial_z \tilde{E}(z,\omega) + \left(\underbrace{\beta^2(\omega)}_{(\beta(\omega))} - \beta_0^2\right) \tilde{E}(z,\omega) = 0$$

How to solve this: Use a pertabative solution using terms of

$$\varepsilon(\omega) = 1 + \chi^{(1)} + \frac{3}{4}\chi^{(3)}_{xxxx} |E|^2$$

<u>Procedure</u> for solving $\beta(\omega)$

1. Zeroth order $\quad \varepsilon(\omega) = 1 + \chi^{(1)}$
   - use Eq (1) and solve for $\beta^{(0)}$ + $F(x,y)$
   - Assume $F(x,y)$ is a fiber mode (single mode)

2. 1st order $\quad \varepsilon(\omega) = 1 + \chi^{(1)} + \frac{3}{4}\chi^{(3)}|E|^2$
   - Sub $\varepsilon(\omega)$ into Eq (1) and solve for $\beta^{(1)}$
   - Sub $\beta^{(1)} + \beta^{(0)}$ into Eq (2)

From (2) we get

$$\partial_z E(z,\omega) = i\left[\beta^{(0)} + \beta^{(1)} - \beta_0\right] E(z,\omega)$$

effect of dispersion

Nonlinearity only $\Rightarrow$ $\beta^{(1)} = \gamma |E|^2$

Dispersion only (and absorption) $\Rightarrow$ $\beta^{0}_{0}(\omega) = \beta_0 + \beta_1(\omega-\omega_0)^2 + \frac{1}{2}(\beta_2)(\omega-\omega_0)^3 = \beta(\omega)$

Solution

$$\boxed{\partial_z E = -\underbrace{\left(\sum_{m=1} \beta_m \frac{i^{(m-1)}}{m!} \partial_m\right) E}_{\text{Dispersion}} + \underbrace{i\gamma |E|^2 E}_{\text{SPM}}}$$

The name of this equation is the nonlinear Schrödinger Equation (NLSE)

A more accurate form of this equation which takes into account higher order effects (self steepening + Raman effect)

$$\partial_z E = -\frac{\alpha}{2} E - \left(\sum_{m=2} \beta_m \frac{i^{m-1}}{m!} \partial_t^m\right) E + (1 - f_R)\left[i\gamma |E|^2 E - \frac{\partial\gamma}{\omega_0}\partial_t(|E|^2 E)\right]$$

$$+ i\gamma f_R \left(1 + \frac{i}{\omega_0}\partial_t\right)\left(E \int h_R(t) |E(z, t-t')|^2 dt'\right)$$

Solitons

An analytic solution to the NLSE is of the form

$$\boxed{E(t) \sim \operatorname{sech}(t/\Delta t)\, e^{-i\omega_0 t} N \sqrt{P_1}}$$

where $\boxed{P_1 = \frac{1}{\gamma L_D}}$ $\boxed{N^2 = \frac{L_D}{L_{NL}} \equiv (\text{soliton order})^2}$

Lecture 22 More on pulse propagation in fibers

Review

Need to consider both dispersive (GVD) and nonlinear effects (SPM)

NLSE
$$\boxed{\partial_z E = -\frac{i}{2}\beta_2 \partial_t^2 E + i\gamma|E|^2 E \cancel{=0}}$$

$$\partial_z E = \underbrace{\left(-\frac{\alpha}{2}E\right)}_{\text{Absorption}} - \underbrace{\left(\beta_1 \partial_t E + \frac{i}{2}\beta_2 \partial_t^2 E\right)}_{\text{dispersion (GVD)}} + \underbrace{\left(i\gamma|E|^2 E\right)}_{\text{SPM}} \cancel{=0}$$

An analytic solution is given by

$$\boxed{E(t) \sim N\sqrt{P_1}\ \text{sech}(\eta\, t/\Delta t)}$$

Where $\frac{1}{\gamma L_D} \equiv P_1$ $\qquad N^2 = \frac{L_D}{L_{NL}}$ (Soliton order)

Optical Solitons (Self reinforcing solitary wave)

Balancing GVD + SPM

John Scott Russell (~1834) in water

Water wave ⇒ ~~followed it for 8 miles~~ 2 miles (Union Canal)

See in plasma's + in optics

A 1st order soliton occurs to the balanced effects of GVD + SPM

GVD $\Rightarrow$ Temporal Broadening

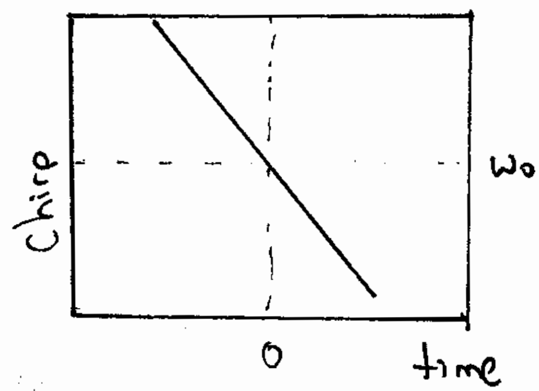
SPM $\Rightarrow$ Spectral Broadening

For $N=1$ $\quad L_N = L_D$ $\quad$ look at chirp $\quad \omega(t) = \omega_0 - \partial_t \phi(t)$

Compare the chirp due to GVD + SPM

Anomalous GVD

- $\beta_2 < 0$

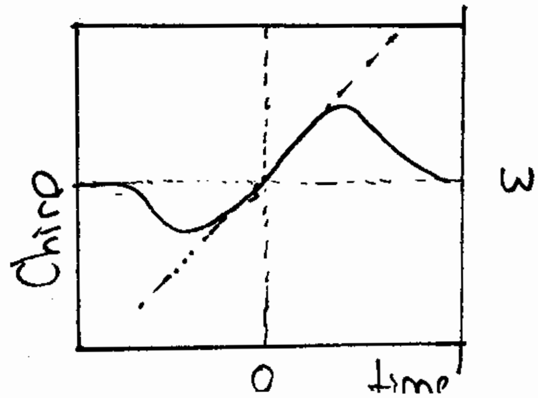


Negative chirp

SPM

$$\phi_{NL}(t) \sim \gamma \, |\operatorname{sech}(t/\Delta t)|^2 L$$

$$\omega(t) = \omega_0 - \partial_t \left(\gamma L \, |\operatorname{sech}(t/\Delta t)|^2 \right)$$



The slope of the chirp due to SPM across the width of the pulse is equal + opposite to that of anomalous GVD.

Three properties of solitons

Permanent form

Localized

Interact with other solitons, emerged unchanged.

Optical Solitons

Experimentally observed in 1973

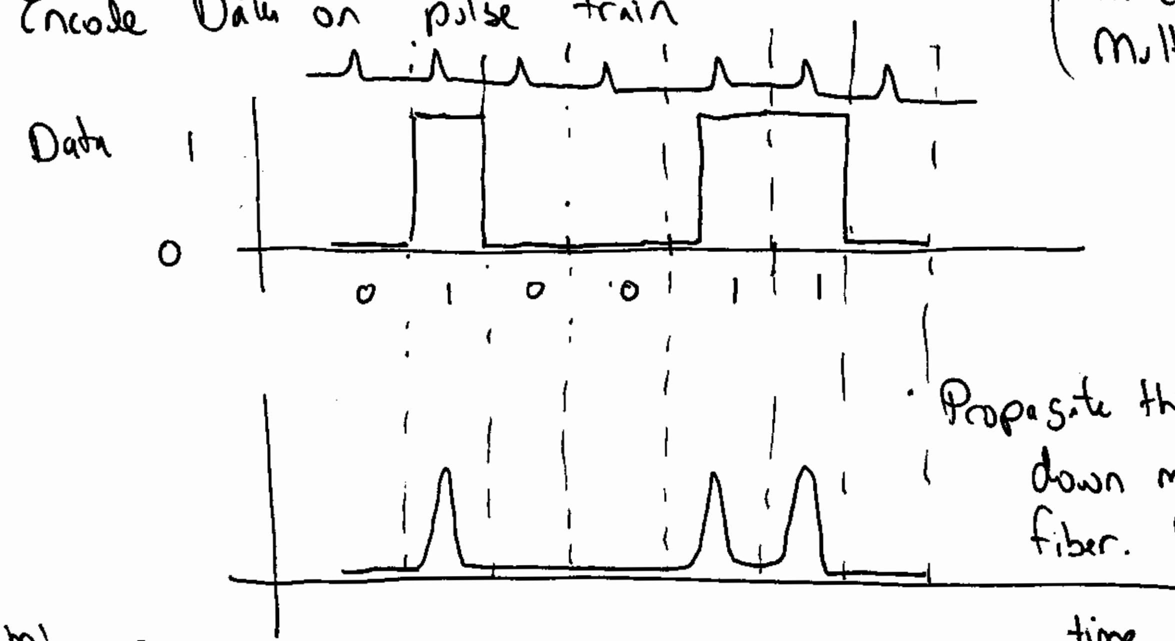
Result from interplay of anomalous GVD + SPM

Fundamental Soliton $N = 1$ Constant Shape

Higher order solitons $N > 1$ Periodic Shape

Applications for solitons ⇒ Optical Communications

Encode Data on pulse train (Time Division Multiplexing)



Propagate this pulse train down many km of fiber. Do not want temporal broadening

Numbers: 20 Gb/s soliton train over 150000 km with

How to Solve the NLSE: Split Step Fourier Method

Break the fiber in n steps of length h

Use operator method

Dispersion operator $\boxed{\hat{D} = -\frac{i}{2}\beta_2\,\partial_t^2}$

Nonlinearity operator $\boxed{\hat{N} = i\gamma|E|^2}$

Write NLSE

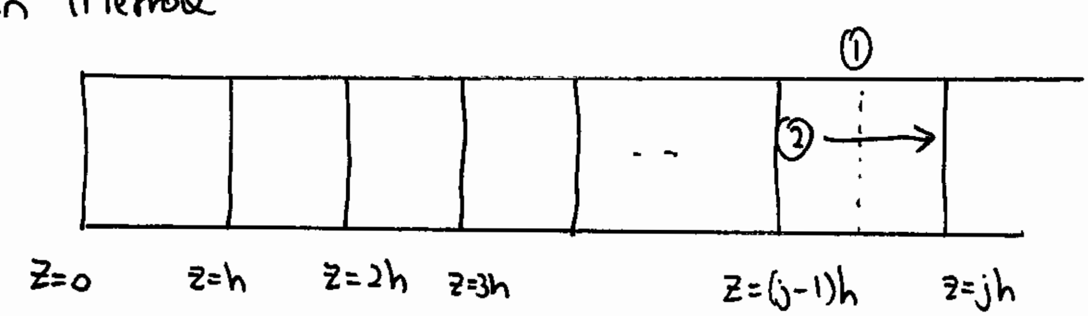
$$\boxed{\partial_z E = (\hat{D} + \hat{N})E}$$

So $$E(jh, t) = \exp(\hat{D} + \hat{N})\, E((j-1)h, t)$$

If dispersion acts independantly of the nonlinearity (assumed for a small step size) then

$$\exp((\hat{D} + \hat{N})h) \simeq \exp(\hat{D}h)\exp(\hat{N}h)$$

Solution Method



In general

$$\boxed{\mathcal{F}\left\{ \partial_t^n f(t) \right\} = (i\omega)^n \, \mathcal{F}\left\{ f(t) \right\}}$$

1. Calculate nonlinearity at midpoint of step

$$\exp(h\hat{N})\, E((j-1)h, t)$$

2. Calculate Dispersion in frequency domain

$$\exp(h\hat{D}(\omega))\, \mathcal{F}\{\exp(h\hat{N})\, E((j-1)h,t)\}$$

Why? The operator $\hat{D}$ is a differential operator

$$\hat{D} \sim \partial_t^2$$

However

$$\mathcal{F}\{\partial_t^2\} = (i\omega)^2$$

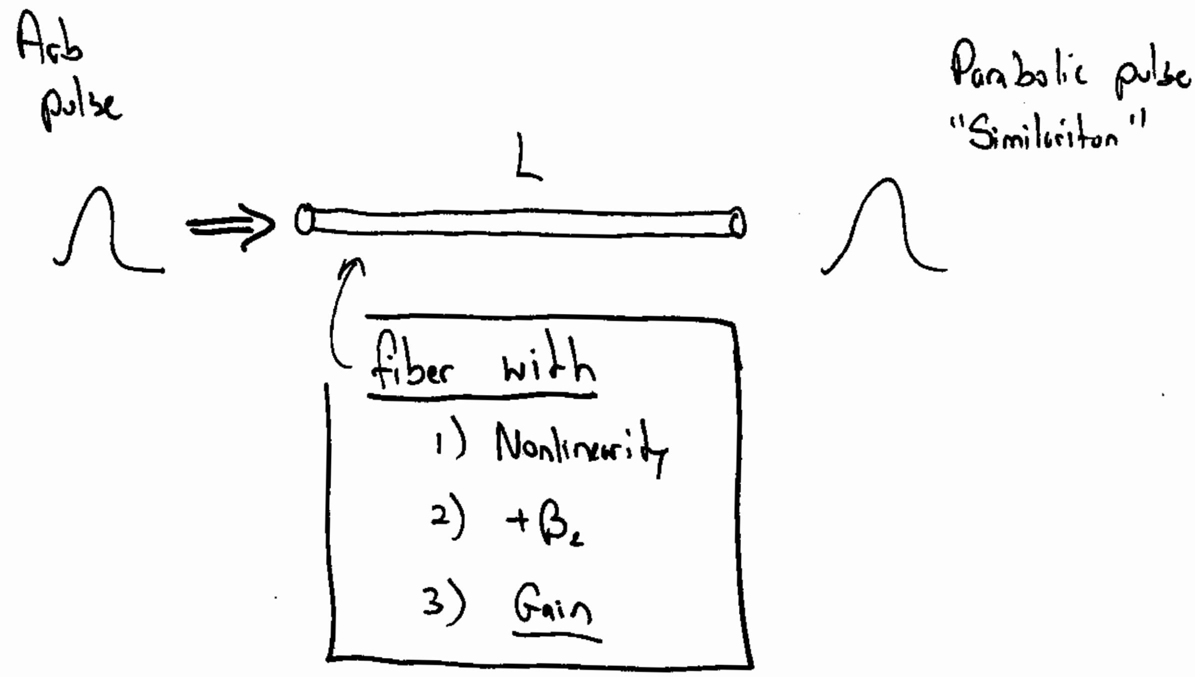
So $\hat{D}$ in the Fourier Domain is a <u>multiplicative</u> <u>operator</u>

3. The solution after the step is

$$\boxed{E(jh, t) = \mathcal{F}^{-1}\{\exp(\hat{D}h)\, \mathcal{F}\{\exp(\hat{N}h)\, E((j-1)h,t)\}\}}$$

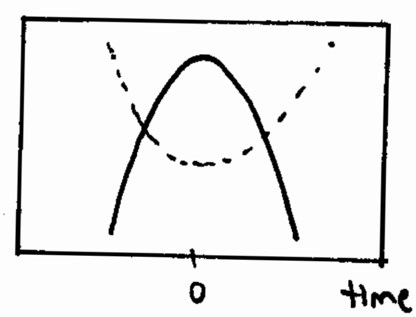
Repeat procedure over all steps to L.

One other "special" Pulse : Similariton



In the limit of infinite length the electric field profile is a parabola! It also has a parabolic phase distortion.

$$\boxed{E(\omega,z) \sim A_0 \sqrt{1 - \frac{\omega^2}{\omega_p^2}} \exp(i\,\omega^2/c)}$$



Intensity + Phase

Why are similarton's important

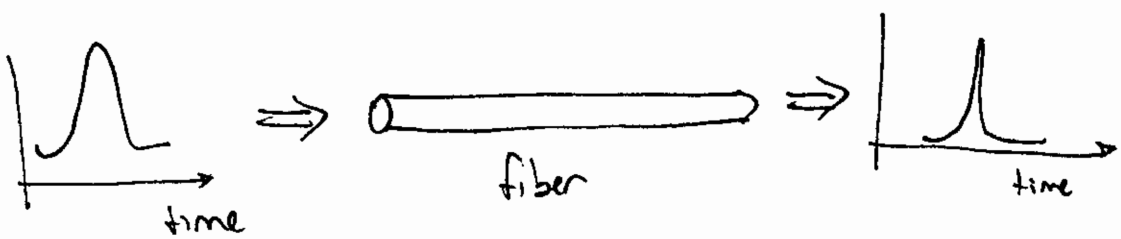
The pulse only has a quadratic phase distortion which can be compressed with GVD only!

Lecture 23 Applications of $\chi^{(3)}$ Effects

- Ultrashort pulse compression
- Self Focusing and Self Filamentation
- Supercontinuum Generation.
- Nonlinear Switching

Ultrashort pulse compression in optical fibers

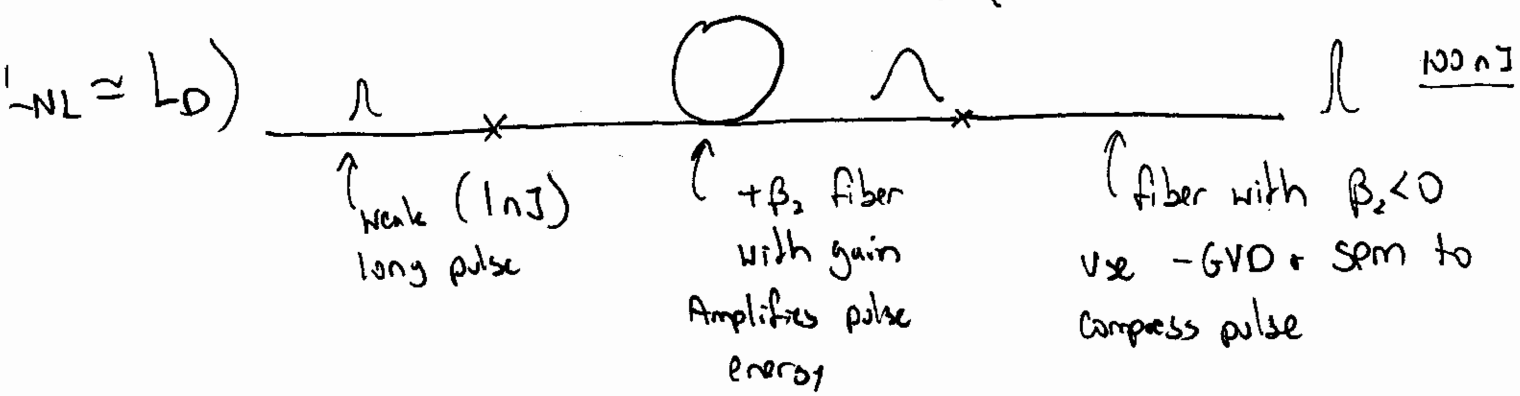
If a fiber exhibits a large nonlinearity with small ~~B2~~ GVD then the nonlinear spectral broadening can be used for pulse compression.



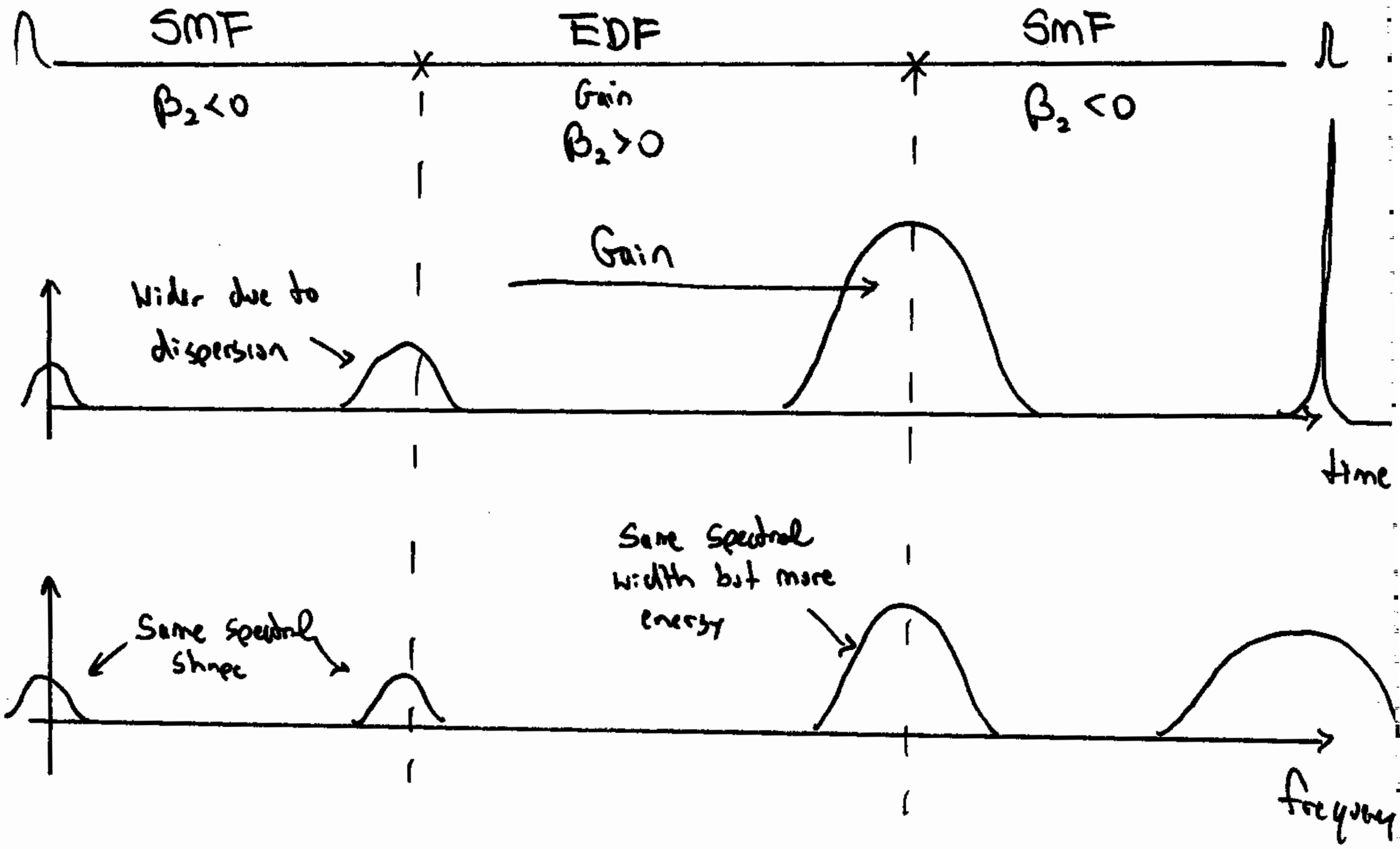
The process works better when the group velocity dispersion is near zero and anomalous.

Anomalous Dispersion Compression Scheme in optical fibers (Compression + Amplification)

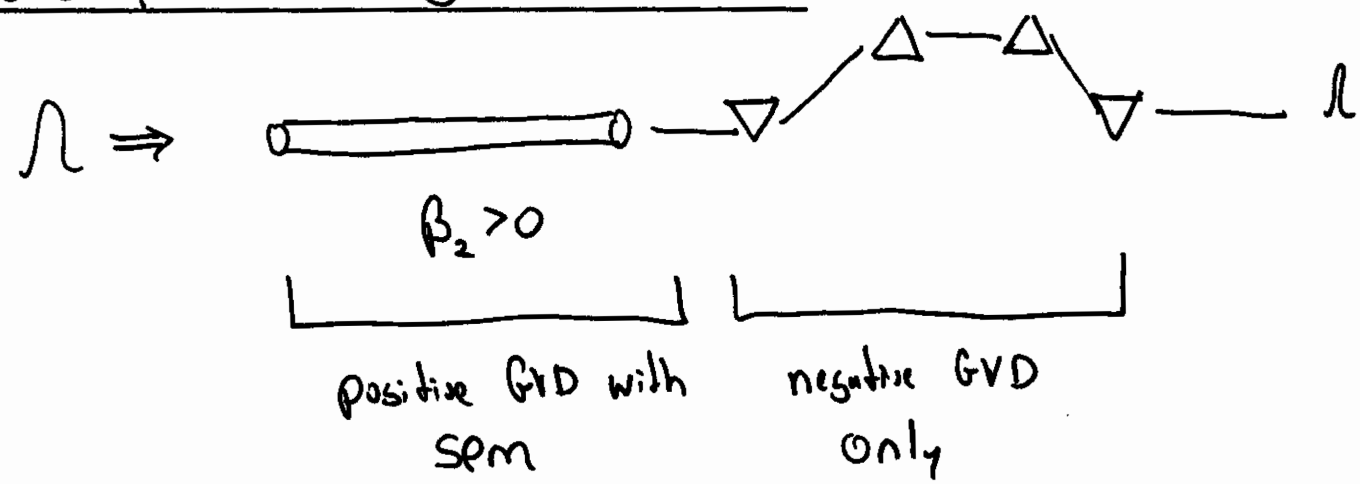
Compression from 200 fs to 50 fs



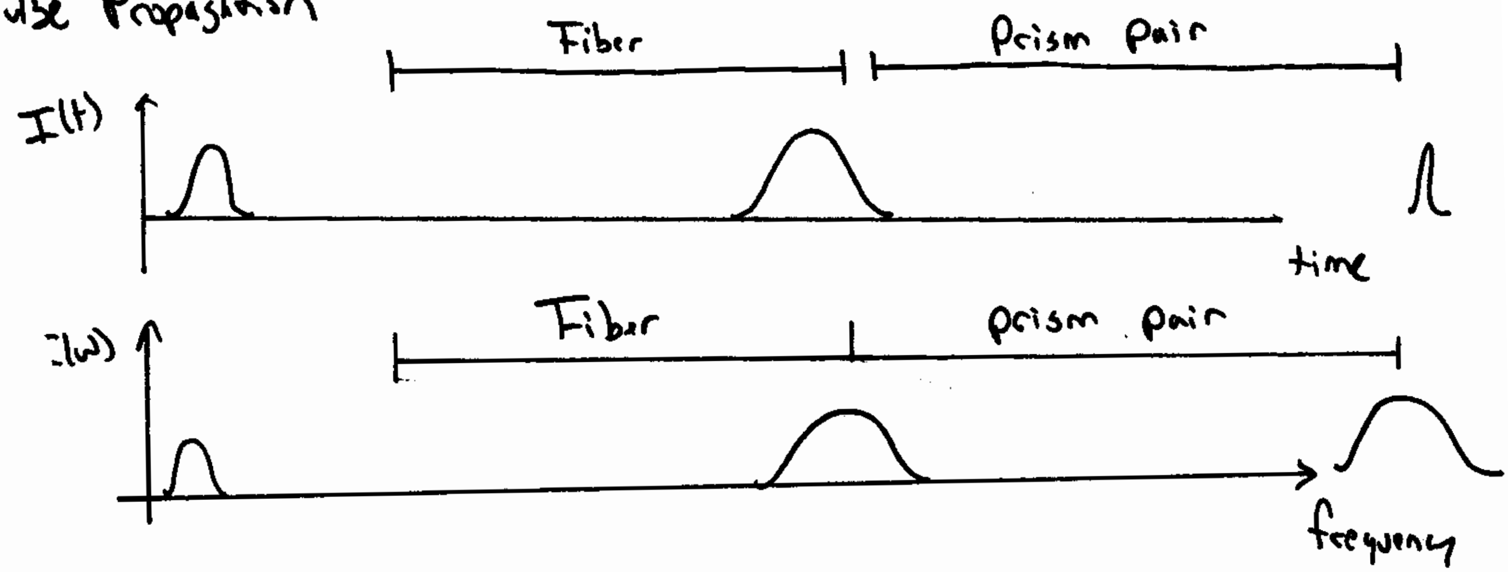
Pulse compression using negative GVD



Pulse Compression using positive GVD



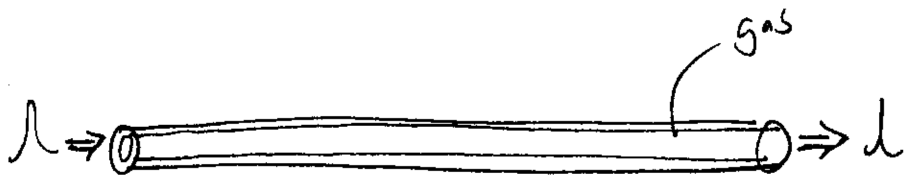
Pulse Propagation



Ultrashort pulse compression in Noble Gases.

Specific Noble gases (Ne, etc) exhibit a large n_2 with small dispersion. The SPM [illegible] due to the gas in the presence of small GVD will cause spectral broadening / temporal compression.

$L_{NL} \ll L_D$



Short pulse
~1mJ

Shorter pulse!

Compression from ~20fs to < 5fs

Using the interaction of SPM + GVD for pulse compression

~~Self Focusing~~

~~spatial analog to self~~

Supercontinuum Generation

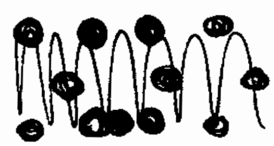
Generation of extremely broadband light ($\Delta\omega \simeq \omega_0$)

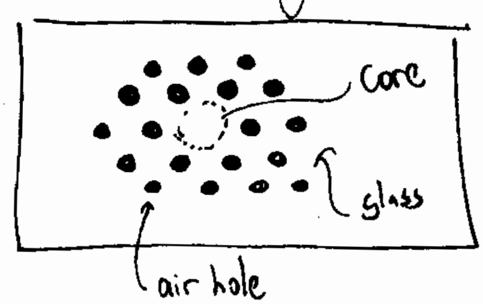
How? Typically two methods

1) ~ 1 mJ pulses in quartz or sapphire plate
2) ~ 1 nJ pulses in a "special" optical fiber

"special fiber"
- microstructured optical fiber
- photonic crystal fiber
- photonic bandgap fiber
- holey fibers

These are optical fibers that have a very small core surrounded by a cladding of air holes + glass.





The presence of the air holes in the cladding reduces the effective cladding index thus increasing Δn. This makes the mode field diameter smaller thus a larger effective nonlinearity γ.

$$\gamma = \frac{n_2 \omega}{c(\pi r_0)^2}$$

Heriot-Watt University

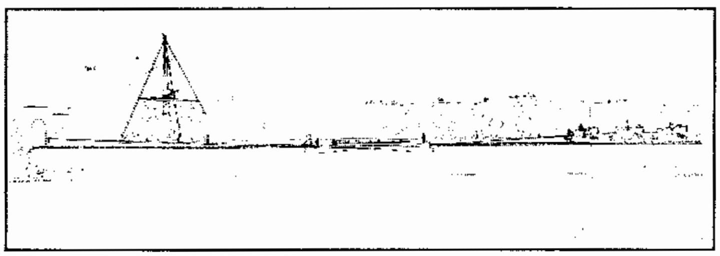
Department of Mathematics

John Scott Russell and the solitary wave

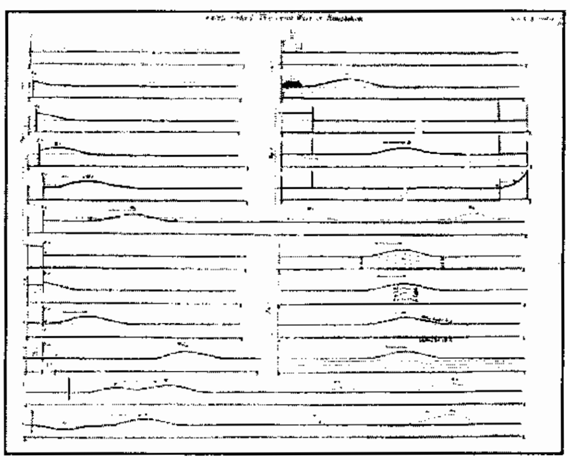
Over one hundred and fifty years ago, while conducting experiments to determine the most efficient design for canal boats, a young Scottish engineer named John Scott Russell (1808-1882) made a remarkable scientific discovery. As he described it in his "Report on Waves": (Report of the fourteenth meeting of the British Association for the Advancement of Science, York, September 1844 (London 1845). pp 311-390. Plates XLVII-LVII).

> ``I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation".
> (Cet passage en francais)

This event took place on the Union Canal at Hermiston, very close to the Riccarton campus of Heriot-Watt University, Edinburgh.



Following this discovery, Scott Russell built a 30' wave tank in his back garden and made further important observations of the properties of the solitary wave.

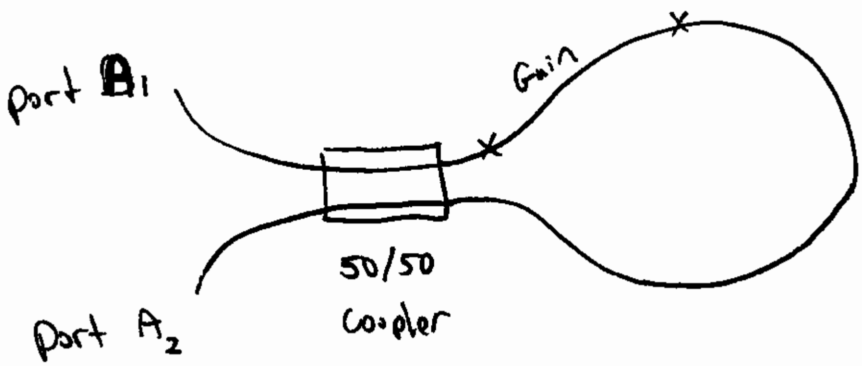


Throughout his life Russell remained convinced that his solitary wave (the ``Wave of Translation") was of fundamental importance, but nineteenth and early twentieth century scientists thought otherwise. His fame has rested on other achievements. To mention some of his many and varied activities, he developed the "wave line" system of hull construction which revolutionized nineteenth century naval architecture, and was awarded the gold medal of the Royal Society of Edinburgh in 1837. He began steam carriage service between Glasgow and Paisley in 1834, and made one of the first experimental observations of the "Doppler shift" of sound frequency as a train passes. He reorganized the Royal Society of Arts, founded the Institution of Naval Architects and in 1849 was elected Fellow of the Royal Society of London. He designed (with Brunel) the "Great Eastern" and built it; he designed the Vienna Rotunda and helped to design Britain's first armoured warship (the "Warrior"). He developed a curriculum for technical education in Britain, and it has recently become known that he attempted to negotiate peace during the American Civil War.

It was not until the mid 1960's when applied scientists began to use modern digital computers to study nonlinear wave propagation that the soundness of Russell's early ideas began to be appreciated. He viewed the solitary wave as a self-sufficient dynamic entity, a "thing" displaying many properties of a particle. From the modern perspective it is used as a constructive element to formulate the complex dynamical behaviour of wave systems throughout science: from hydrodynamics to nonlinear optics, from plasmas to shock waves, from tornados to the Great Red Spot of Jupiter, from the elementary particles of matter to the elementary particles of thought.

For a more detailed and technical account of the solitary wave, see for example R K Bullough, *"The Wave" "par excellence", the solitary, progressive great wave of equilibrium of the fluid - an early history of the solitary wave*, in **Solitons**, ed. M Lakshmanan, Springer Series in Nonlinear Dynamics, 1988, 150-281, or "The Spirited Horse,

Nonlinear Switching Loop Mirrors

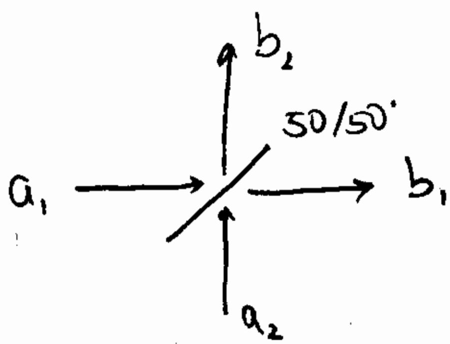


Loop mirror

Fast switch

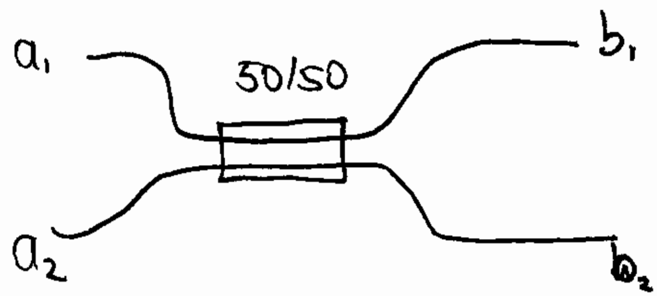
Sagnac Interferometer

Difference between beam splitter + directional coupler



$\boxed{\text{beam splitter}}$

π phase shift between b_1 + b_2



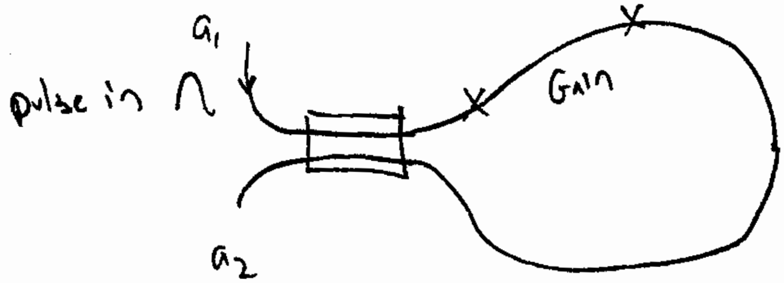
$\boxed{\text{directional coupler}}$

(uses evanescent wave coupling)

$\frac{\pi}{2}$ phase shift between b_1 + b_2 for 50/50.

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} t & e^{i\pi/2}\sqrt{1-t^2} \\ e^{i\pi/2}\sqrt{1-t^2} & t \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

The nonlinear phase shift is asymmetric



$G \equiv$ Gain

The directional coupler will split ~~the~~ the input pulse into two:

1) Clockwise Pulse : Pulse that goes clockwise get first amplified and then experiences a large nonlinear phase shift

2) Counter Clockwise Pulse : Pulse that goes counter clockwise goes thru fiber and get a small nonlinear phase then goes thru amplifier

Difference in phase shift between two pulses.

$$\Delta\phi \sim n_2 (G-1) I(t) L$$

If the pulses are weak then $\Delta\phi = 0$ and the pulse will exit the loop out port a_1

If the pulses are strong then $\Delta\phi \simeq \pi$ and the pulse will exit the loop out port a_2.

Lecture 25 Stimulated Raman Scattering

Spontaneous Raman effect (1928)

Scattering of light by vibrations of the medium

Scattering of photons by optical phonons

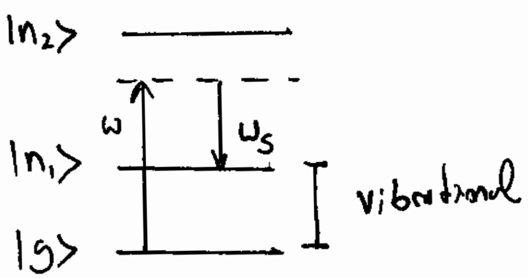
New spectral components

Stokes $\Rightarrow$ longer λ / ~~larger~~ smaller ω

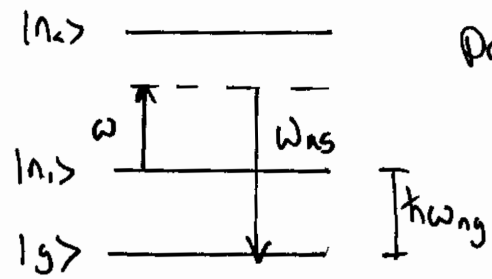
Anti Stokes $\Rightarrow$ shorter λ / larger ω

Stokes components are an order of manitude larger than anti-Stokes.

Two photon process



Stokes



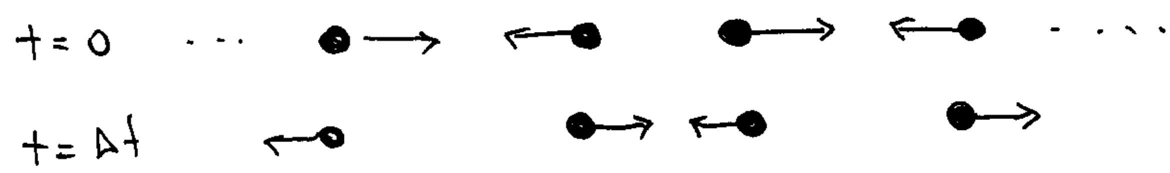
Anti Stokes

Anti-Stokes emission is smaller since at Room temp. mostly the $|g\rangle$ state is populated.

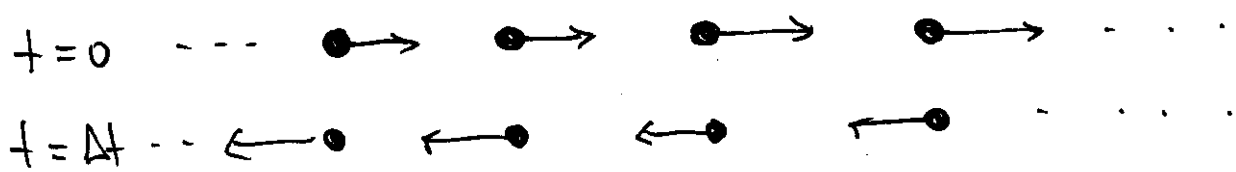
$|n\rangle$ population smaller by $\exp(-\hbar\omega_{ng}/kT)$

— Optical Phonons

quantized lattice vibrations



Versus Acoustic Phonons



(Note : scattering of photons by acoustic phonons is called Brillouin scattering)

~~Stimulated Raman Scattering ⇒ Strong Pumps~~

~~Most stronger process~~

~~Forward scattering process~~

Properties of Spontaneous Raman Scattering

- Stokes intensity grows linearly with length of Raman material
- Process is weak   scattering $\frac{\text{cross section}}{\text{Volume}}$ $\sigma \sim 10^{-6}\,cm^{-1}$

(1 in 10^6 will be scattered)

- Two Photon Process

Stimulated Raman Scattering (SRS)

Four Photon Process

Slow resonance is excited by two optical fields at two frequencies that differ by the molecular resonant frequency. These frequencies interact to produce sum + difference frequencies.

This is a nonlinear process since it depends on the product of fields.

Stokes generation dominates since

- upper levels are not previously excited
- phase matched for colinear propagation

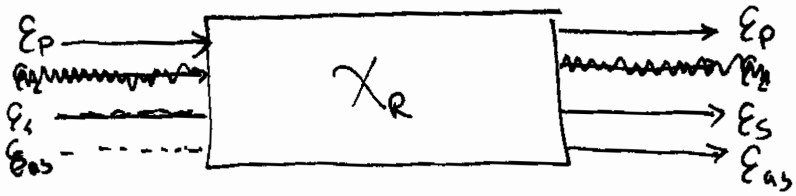
Unlike spontaneous Raman Scattering, SRS is a forward scattering

Process of SRS

- Pump wave generates Stokes via spontaneous Raman Scattering
- The pump is constant so more Stokes photons are generated this also causes a slight excitation of the molecular resonance.
- As the scattered Stokes increases in intensity, the stimulated regime is reached.
- Now the Stokes wave interacts with the pump to further excite resonances, increasing the rate of frequency conversion.

Classical Description of SRS

Treat medium as collections of atoms



(ignore anti stokes for this)

Atoms will obey the differential eq

$$m\ddot{q} + m\xi_R\dot{q} + kq = e\mathcal{E} \qquad (1)$$

where $q \equiv$ atomic displacement $\quad k \equiv$ atomic spring const

$\xi_R \equiv$ Raman damping $\quad \mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 = \mathcal{E}_p + \mathcal{E}_s$

∴ to find the output electric fields

1) First find P + χ_R We can write

$$P_{NL} = \epsilon_0 N\alpha_p\, q(z,t)\, \mathcal{E}(z,t) \qquad (2)$$

$$P_{NL} = \epsilon_0 \chi_R \mathcal{E} \qquad (P = P_{NL} + P_L) \qquad (3)$$

2) Solve for $q(z,t)$ using (1)

3) Find P using (2)

4) Find χ_R using (2) + (3)

$$\chi_R = -\frac{\epsilon_0 N\alpha_p^2}{8m\omega_0\xi_R}\left(\frac{i + \delta_R}{1 + \delta_R^2}\right) \qquad \omega_0 \equiv \sqrt{k/m} \qquad \delta_R \equiv \frac{2}{\xi_R}\left((\omega_2 - \omega_1) - \omega_0\right)$$

5) Find $\mathcal{E}_s$ + $\mathcal{E}_{as}$ by putting $P = P_{NL} + P_L$ into the wave equation.

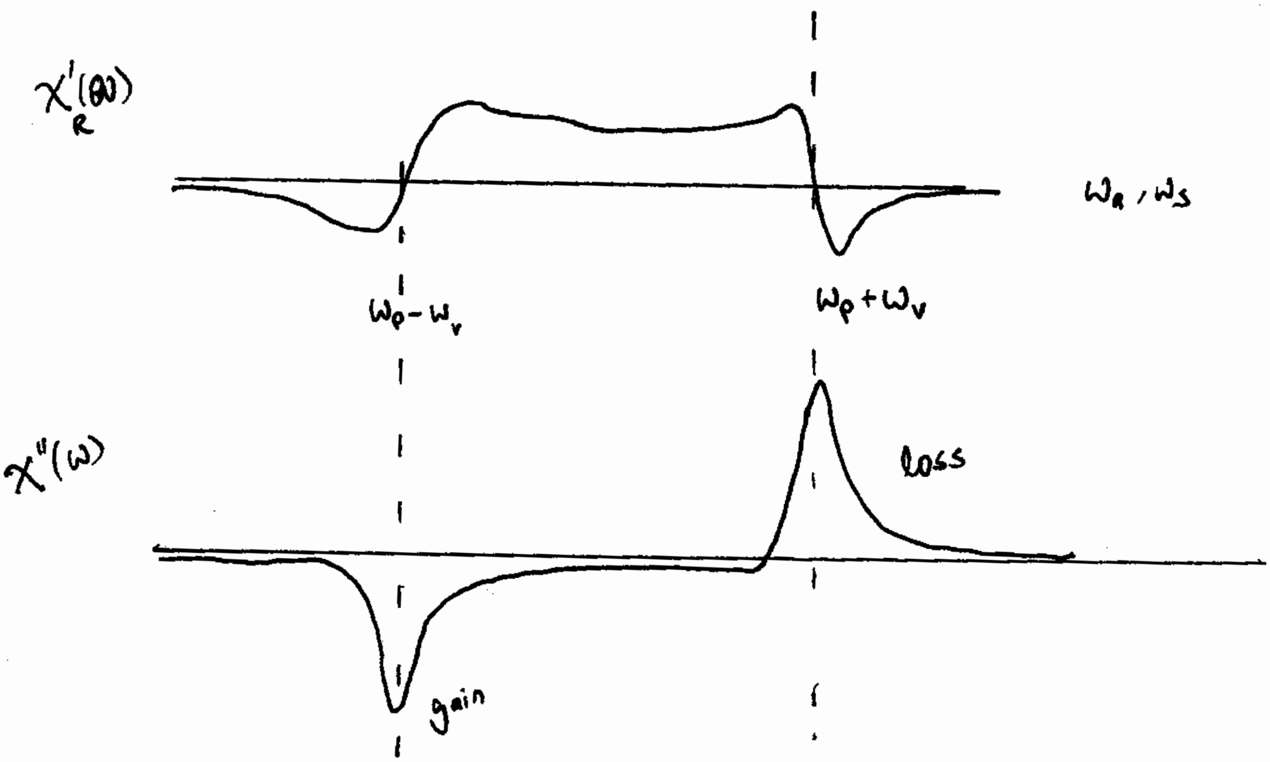
Can rewrite wave eq as

$$\frac{dI_s}{dz} = g_R I_p I_s - \alpha_s I_s$$

$$\frac{dI_p}{dz} = -\frac{\omega_p}{\omega_s} g_R I_p I_s - \alpha_p I_p$$

where $$g_R \equiv \frac{\epsilon_0 N \omega_p \alpha_{pi}^2 \sqrt{\mu_0/\epsilon_0}}{4 m \omega_0 \zeta_e c n^2} \frac{1}{(1+\delta_R^2)}$$

Stokes + Anti Stokes Susceptibilities



Coupled Eq.

Stokes $\quad P_S^{(3)} = 6\epsilon_0 \chi_R^{(3)}(-\omega_s; \omega_p - \omega_p, \omega_s)\,|E_p|^2 E_s$

Pump $\quad P_p^{(3)} = 6\epsilon_0 \chi_R^{(3)}(-\omega_p; \omega_s, -\omega_s, \omega_p)\,|E_s|\, E_p$

Lead to Coupled Eq.

$$\frac{dA_s}{dz} = i\,\frac{3\omega_s}{n_s c}\,\chi_R^{(3)}(\omega_0)\,|A_p|^2 A_s$$

$$\frac{dA_p}{dz} = i\,\frac{3\omega_p}{n_p c}\,\chi_R^{(3)}(\omega_p)\,|A_s|^2 A_p$$

If not pump depleted $\quad |A_p| \simeq$ const. get gain on A_s

Note that ~~A_s~~ Stokes wave does not pick up phase distortions of A_p $\quad \sim |A_p|$

Where does $I_s(0)$ come from? $\Rightarrow$ Spontaneous

$$\cancel{I_s(z) = I_s(0)\exp(g_R I_0)}$$

$$I_s(z) \cong I_0 \exp(g_R I_0 z) \quad \left\} \begin{array}{l} \text{For } \alpha(\omega_p) \simeq 0 \\ \alpha(\omega_s) \simeq 0 \end{array} \right.$$

$L_{eff} \simeq z \quad$ for $\quad \alpha$ small

Threshold for SRS $\Rightarrow$ Gain

$$\frac{dI_s}{dz} = g_R I_p I_s - \alpha_s I_s \qquad \left(g_R = \frac{\epsilon_0 N \chi_0 \omega_1 \alpha_p^2}{4 m \omega_0 \zeta_0 c n^2} \right)$$

$$\frac{dI_p}{dz} = -\frac{\omega_p}{\omega_s} g_R I_p I_s - \alpha_p I_p$$

For small loss $\alpha_p \cong 0$ $\alpha_s \simeq 0$ we can rewrite

$$\frac{d}{dz}\left(I_s + \frac{\omega_s}{\omega_p} I_p \right) = 0$$

Ignoring pump depletion with loss

$$\frac{dI_p}{dt} = 0$$

$$\frac{dI_s}{dz} = g_R I_0 \exp(-\alpha_p z) I_s - \alpha_s I_s$$

Solution

$$\boxed{\begin{aligned} I_s(L) &= I_s(0) \exp(g_R I_0 L_{eff} - \alpha_s L) \\ L_{eff} &= \frac{1}{\alpha_p}\left(1 - \exp(-\alpha_p L) \right) \end{aligned}}$$

Where does $I_s(0)$ come from? $\Rightarrow$ Spontaneous Raman Scattering

Define Raman threshold

~~Input power~~

Input pump power ~~at~~ which the output Stokes power becomes equal to the output pump power.

$$P_s(L) = P_p(L) = \overset{\text{initial pump power}}{P_0} \exp(-\alpha_p L)$$

Approximation

$$\boxed{g_R P_0^{cr} L_{eff} / (Area) \simeq 16}$$

$\left(g_R \simeq 10^{-13}\ m/W \right)$ at $1\mu m$ for fused silica

Raman Shift in fused silica

$$\Delta \nu = 440\ cm^{-1}$$

$$\Delta f = 13.2\ THz$$

at 1550nm

$$\Delta\lambda = \frac{\lambda_0^2}{c} \Delta f = \frac{(1550nm)^2}{(300\ nm/fs)} \left(0.0132\ 1/fs\right)$$

$$\simeq \boxed{105.7\ nm}$$

at 800nm

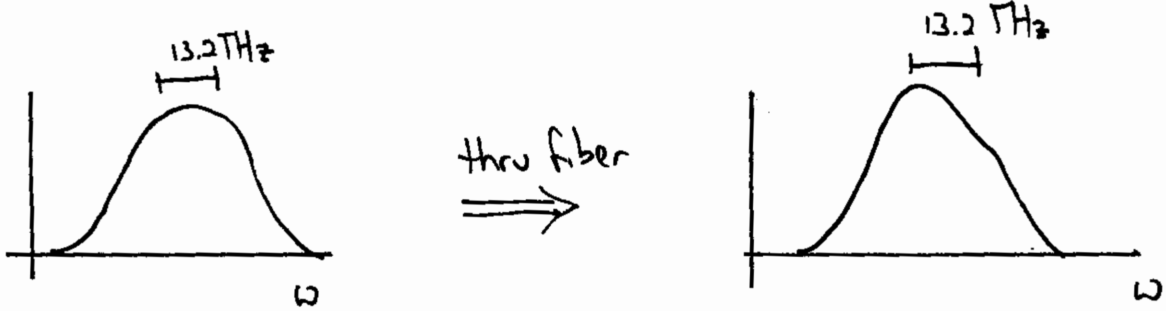
$$\Delta\lambda = \frac{(800\ nm)^2}{300\ nm/fs} (0.0132\ 1/fs) = \boxed{28\ nm}$$

For optical fibers

$$P_0^{cr} \simeq 100\,W \quad \text{for } 10m$$

Interpulse Stimulated Raman Scattering

Short pulses may have a bandwith larger than 13.2 THz. As the pulse propagates, SRS will cause a shift of the spectrum to ~~shorter~~ longer wavelengths.



The process continues thru the length of the fiber. This effect is called the self frequency shift: you can change the center freq. of a pulse via SRS just by propagation in a Raman medium.

Intel + the Silicon "Laser"

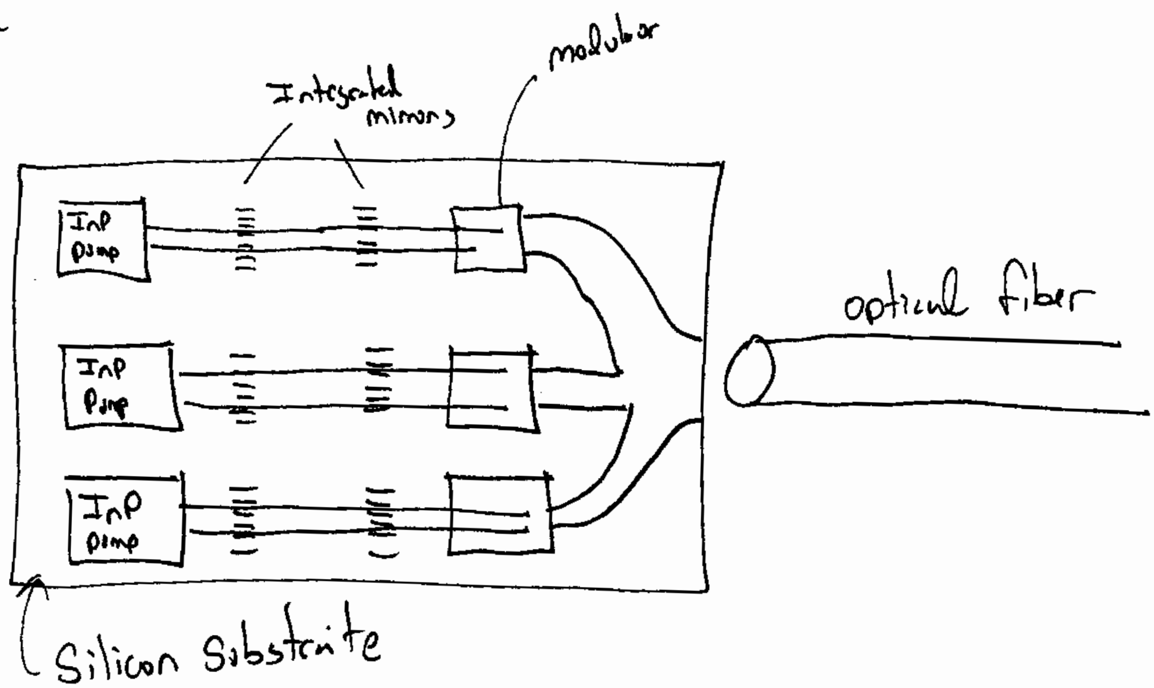
A few years back, Intel announced the demonstration of a laser using silicon. This was a big deal since silicon does not directly lase (it has an indirect bandgap).

~~so making a laser in a material that can rates~~

The silicon laser would allow a laser on your pentium chip, opening the door to computers that use light instead of electrons.

However, the problem here it isn't a laser, but a Raman amplifier. It uses a InP pump laser

Raman effect is 10000 times stronger in Silicon than fused silica

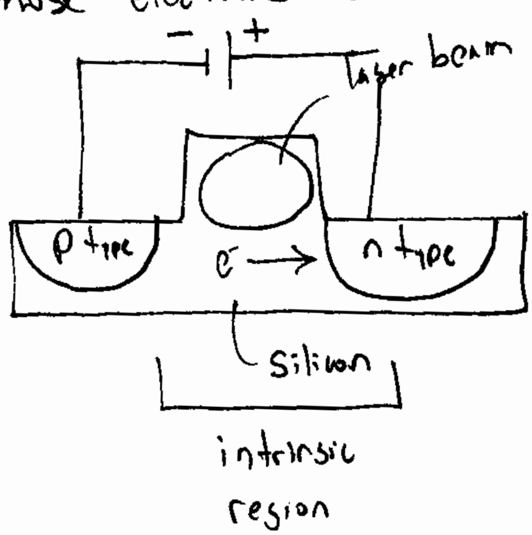


Two Photon absorption

Silicon is transparent to IR light

For high powers two photons cause an atom to free its electron. ~~These free electrons~~ If the intensity is high enough the rate of generating free electrons will exceed the recombination rate. The free electrons will cause the material to have a higher absorption + prevent lasing + Raman Gain

Intel's solution was to use a p-i-n junction to "Sweep" those electrons out



$p \equiv$ p type

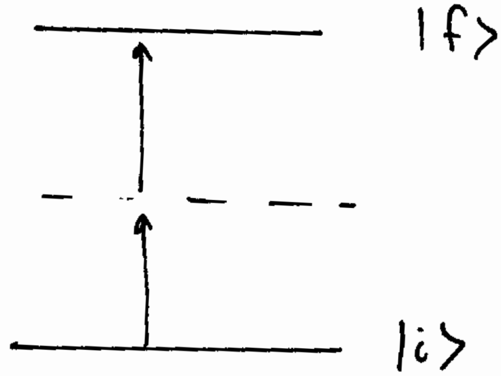
$i \equiv$ intrinsic

$n \equiv$ n. type

Two Photon Absorption

Nonlinear change to the absorption

Two photons simultaneously absorped to excite a state



absorption cross section is smaller than single photon process.

Corresponds to $\chi^{(3)}$ process

Raman Active vibration

$\Rightarrow$ No change dipole moment due to atomic displacements.

Effect of SRS in Fibers

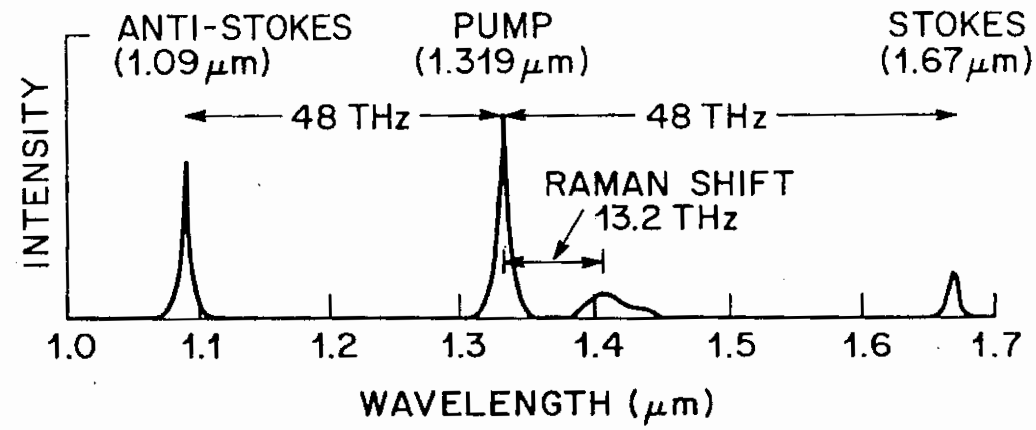


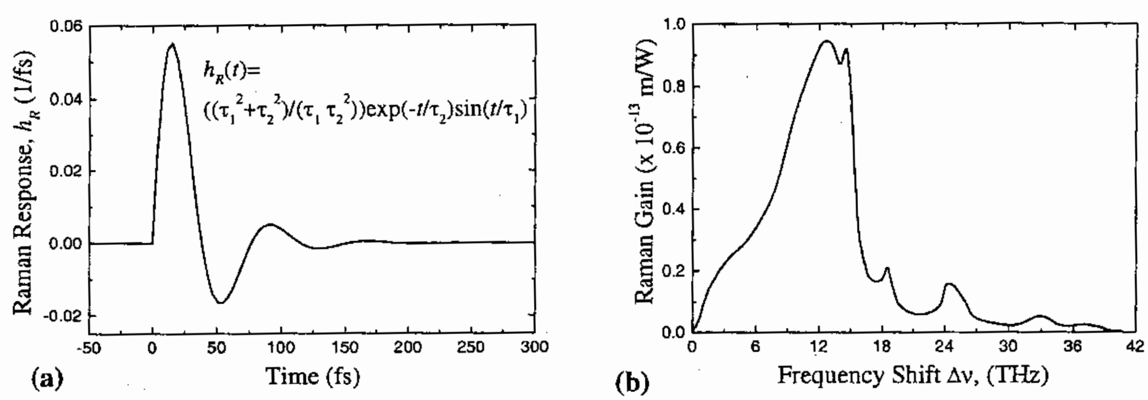
Figure 4-1 FWM Stokes and anti-Stokes components due to propagation in standard SMF near the zero dispersion wavelength (1319 nm). Stimulated Raman scattering also is present which produces spectral components near 1400 nm. Figure reproduced from Ref. [Lin, 1981 #32].

This plot shows Raman Scattering in a optical fiber

The vibrations of fused silica has a resonance at 13.2THz

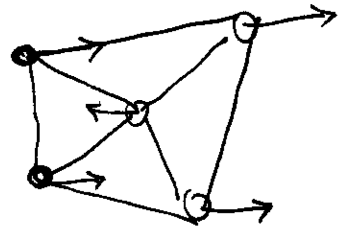
Note that the Stoke + anti stokes components are not

due to SRS but due to partially degenerate FWM.

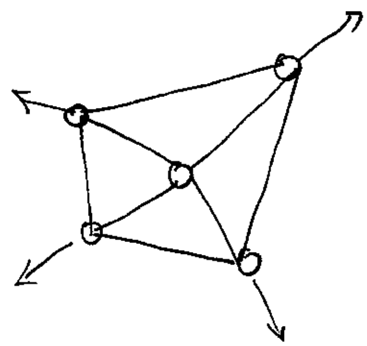


Raman Stretches in fused silica

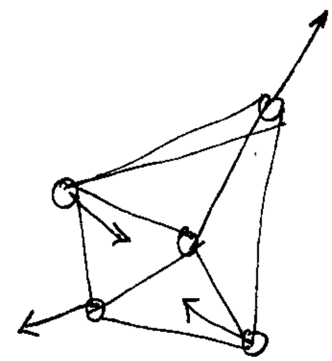
SiO_2 tetrahedra



1056 cm^{-1}



800 cm^{-1}



440 cm^{-1}

Supercontinuum generation in photonic crystal fibers.

Lucent/OFS Microstructure Fiber

— 92 μm —

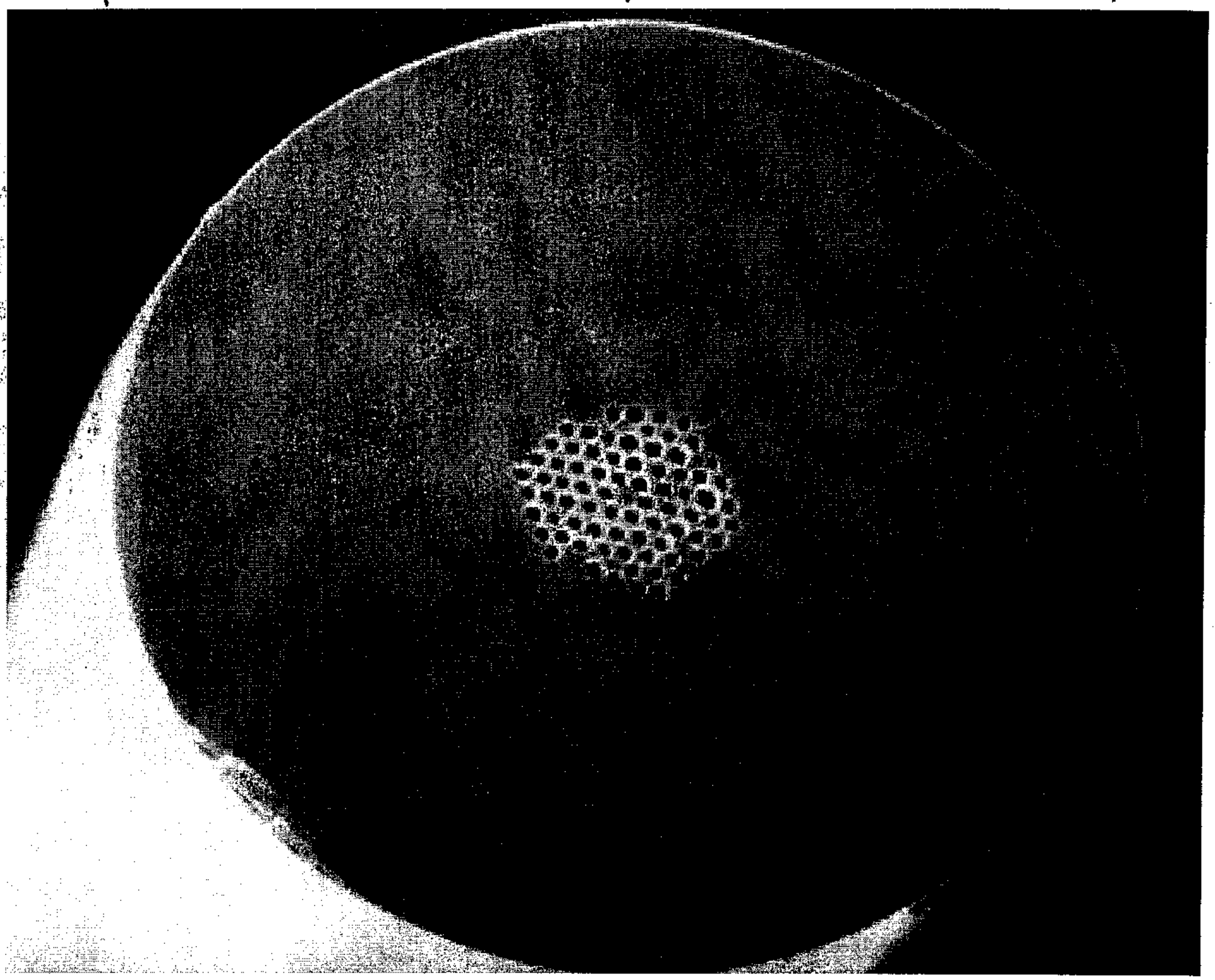
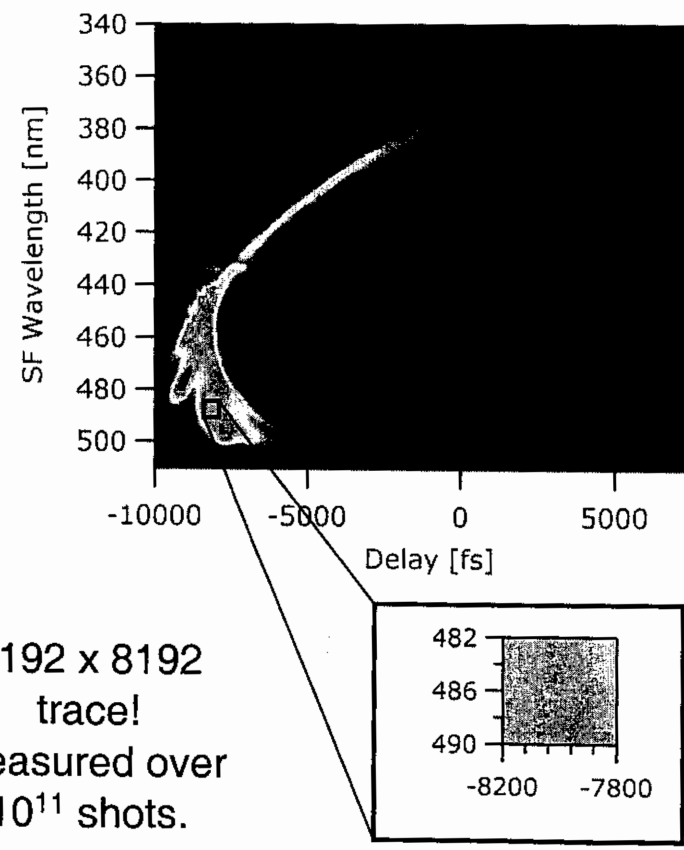
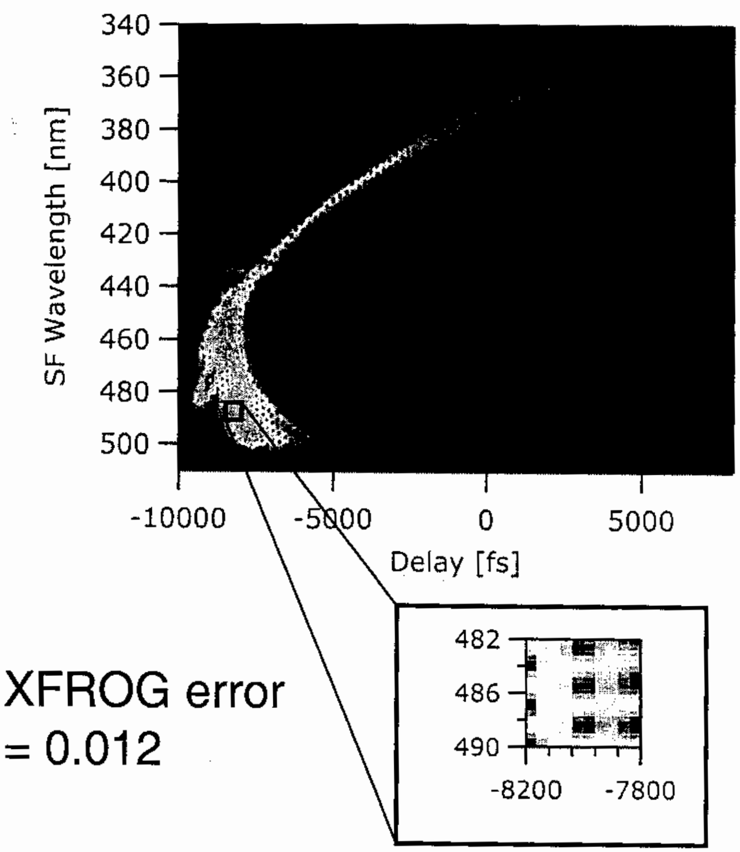


Figure courtesy of OFS

XFROG measurement of the continuum



8192 x 8192 trace!
Measured over 10^{11} shots.



XFROG error = 0.012

While the large-scale structure of each trace is identical, the measured trace lacks the fine-scale structure of the retrieved trace.

From Trebino's Lecture notes

Microstructured optical fibers offer two advantages for supercont[illegible] generation.

1) Large effective nonlinearity $\left(\begin{array}{l} \gamma \simeq 110 \frac{1}{W\,km} \\ \gamma = 1 \frac{1}{W\,km} \end{array} \right)$

2) Small core moves zero GVD wavelength to shorter λ.

Conventional fiber $\{ \lambda_{ZGVD} = 1300nm$, diameter $10\mu m$

microstructure fiber $\left\{ \begin{array}{l} \lambda_{ZGVD} = 900nm \text{ , diameter } 4\mu m \\ \lambda_{ZGVD} = 700\,nm \text{ , diameter } 2\mu m \end{array} \right.$

Also microstructure fibers offer negative GVD at shorter λ.

For supercontinuum generation to occur

1. Large effective nonlinearity
2. Low loss
3. Low or zero GVD at center wavelength

Unfortunately associated with the spectral broadening is temporal breakup of the pulse

5nJ into 15cm at 800nm $\Rightarrow \left\{ \begin{array}{l} \text{Broadening from 500-1200nm} \\ \text{Temporal Breakup over 3ps!} \end{array} \right.$

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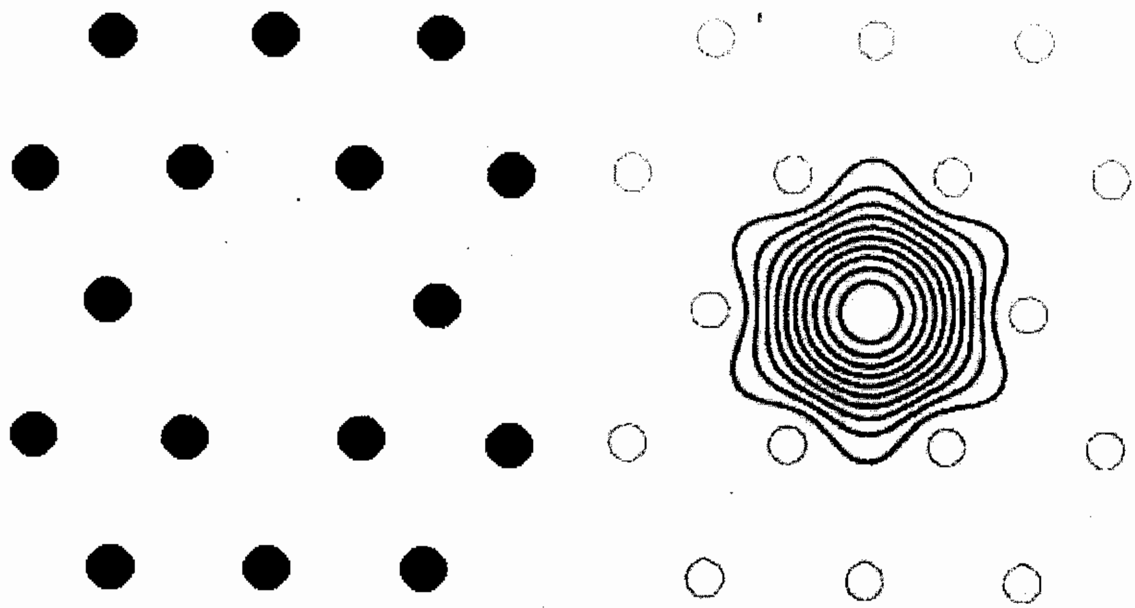
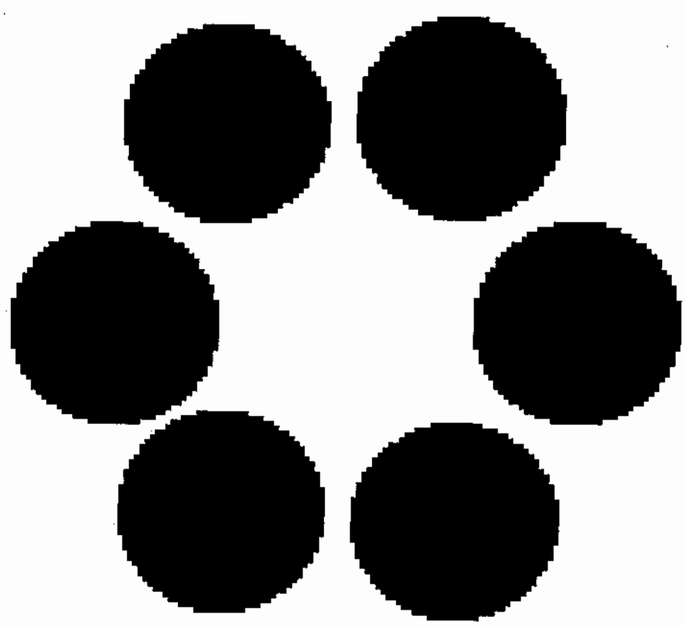




Why is Supercontinuum generation important?

1) Remember f_0 detection? Use supercontinuum generation to set f_n and f_{2n}

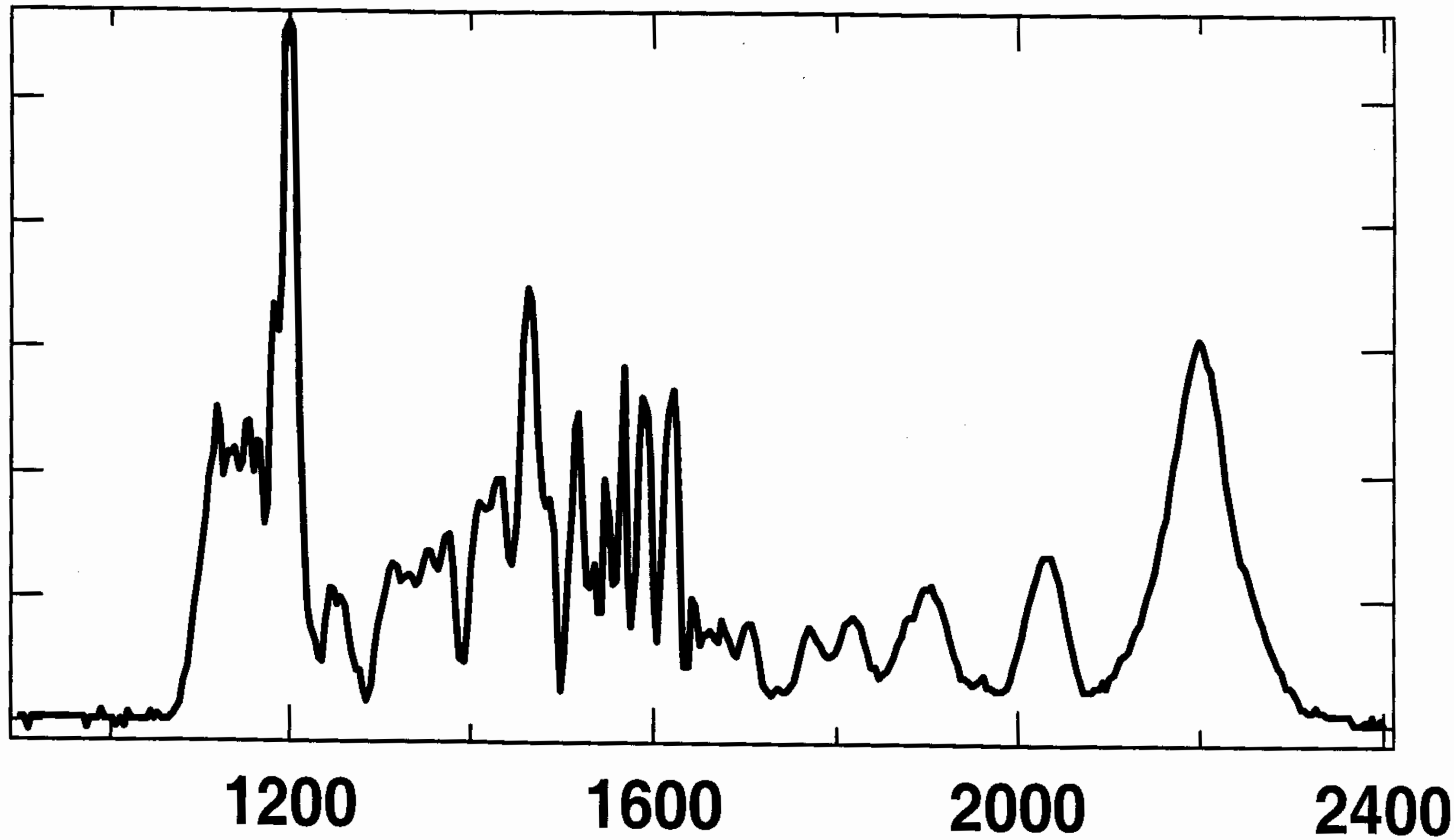
2) Wide spectral shape that is coherent.

How to explain supercontinuum generation

- Complicated non linear process : (Involves SPM, Stimulated Raman Sat? and Self steepening

Soliton fission

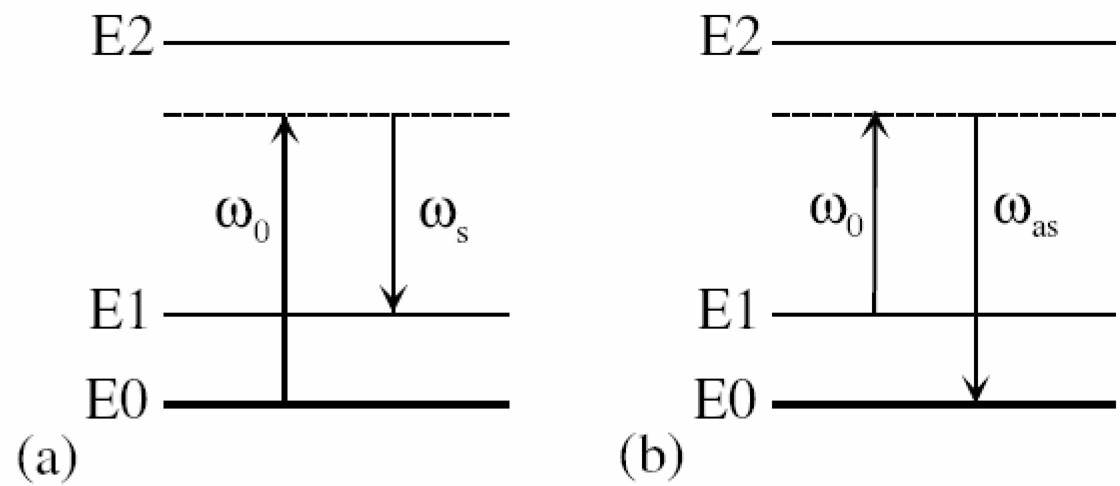
{ A n^{th} order soliton will break up into n $\underline{1st}$ order solitons


1200
1600
2000
2400

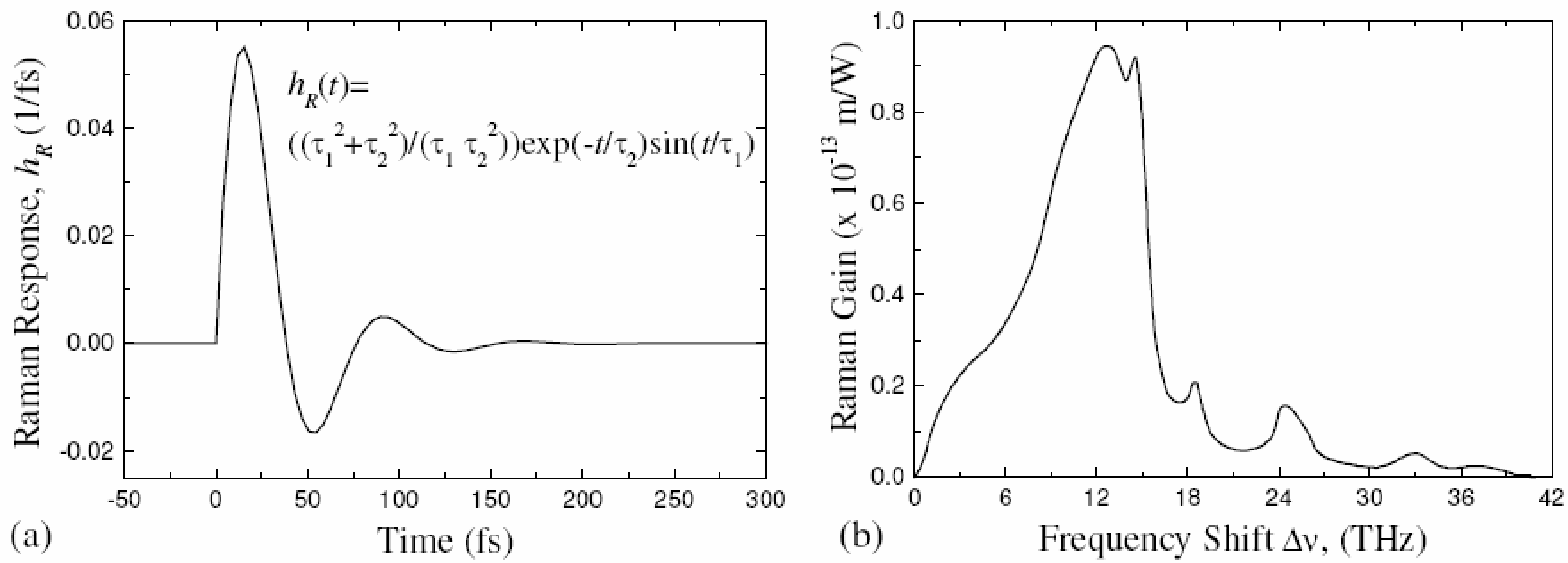
$$\frac{\partial E(z,t)}{\partial z} = -\overbrace{\frac{\alpha}{2}E}^{\text{Absorption}} - \overbrace{\left(\sum_{m=2}\beta_m \frac{i^{m-1}}{m!}\frac{\partial^m}{\partial t^m}\right)E}^{\text{Dispersion}} + (1-f_R)\left\{\overbrace{i\gamma|E|^2 E}^{\text{SPM}} - \overbrace{\frac{2\gamma}{\omega_0}\frac{\partial}{\partial t}\left(|E|^2 E\right)}^{\text{Self Steepening}}\right\}$$

$$+\overbrace{i\gamma f_R\left(1+\frac{i}{\omega_0}\frac{\partial}{\partial t}\right)\left(E\int_0^\infty h_R(t')\left|E(z,t-t')\right|^2 dt'\right)}^{\text{Raman Effect}}.$$

$$h_R(t) = ((\tau_1^2+\tau_2^2)/(\tau_1\ \tau_2^2))\exp(-t/\tau_2)\sin(t/\tau_1)$$



$$g_R(\omega) = \frac{\omega_0}{c n_0} f_R \chi^{(3)} \,\mathrm{Im}\left[\mathcal{F}\{ h_R(t) \} \right]$$

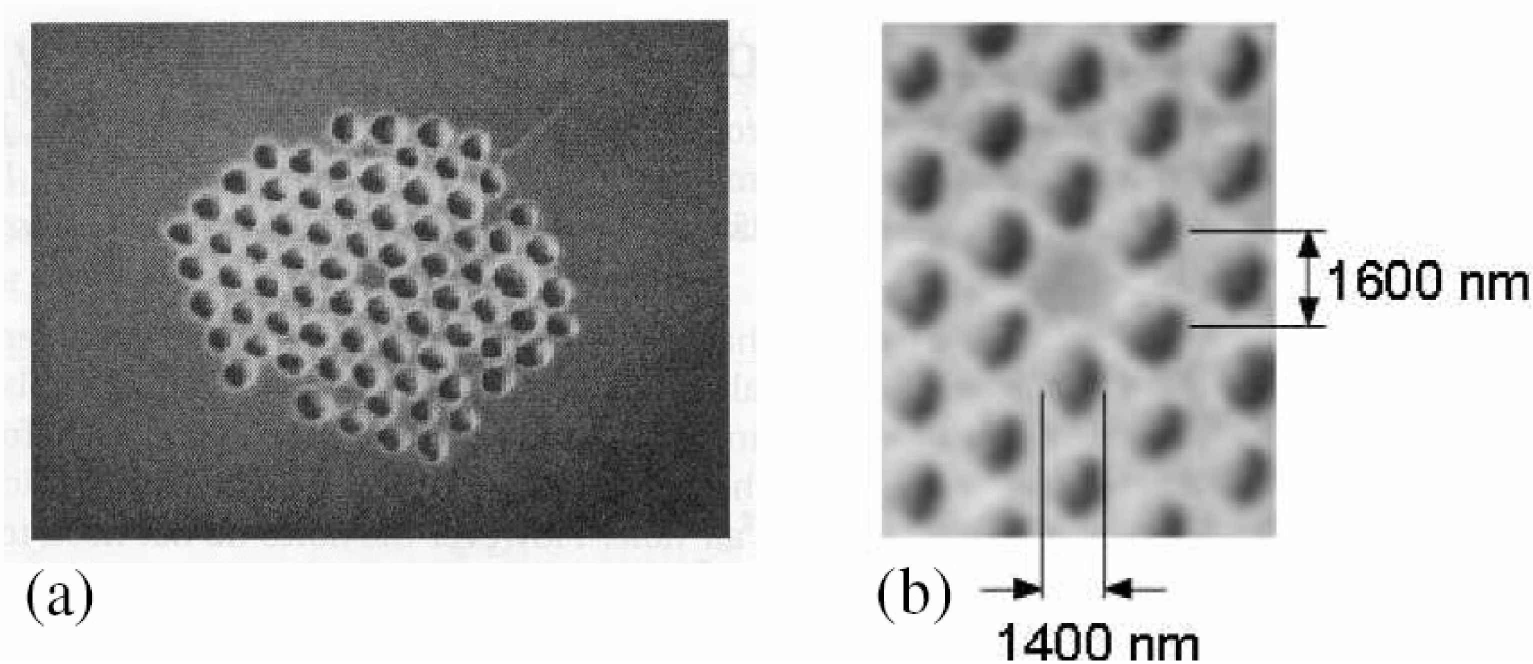

Raman Response, h_R (1/fs)
0.06
0.04
0.02
0.00
-0.02
-50
0
50
100
150
200
250
300
$h_R(t)$=
$((\tau_1^2+\tau_2^2)/(\tau_1\ \tau_2^2))\exp(-t/\tau_2)\sin(t/\tau_1)$
(a)
Time (fs)
Raman Gain (x 10^{-13} m/W)
1.0
0.8
0.6
0.4
0.2
0.0
0
6
12
18
24
30
36
42
(b)
Frequency Shift $\Delta\nu$, (THz)

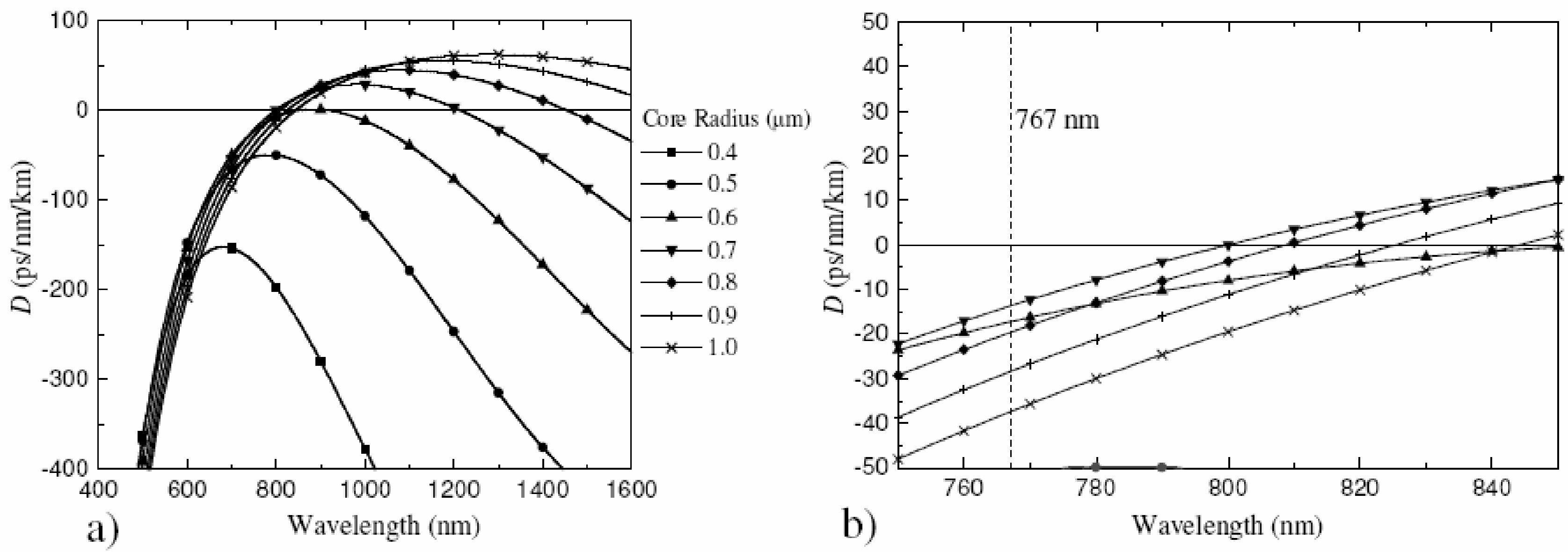
$$\frac{g_R(\Delta\omega_R)P_R}{\alpha\pi r_0^2}\left[1-\exp(-\alpha L)\right]\approx 16$$

$$g_R(\omega)=\frac{\omega_0}{cn_0}f_R\chi^{(3)}\,\mathrm{Im}\left[\mathcal{F}\{h_R(t)\}\right]$$

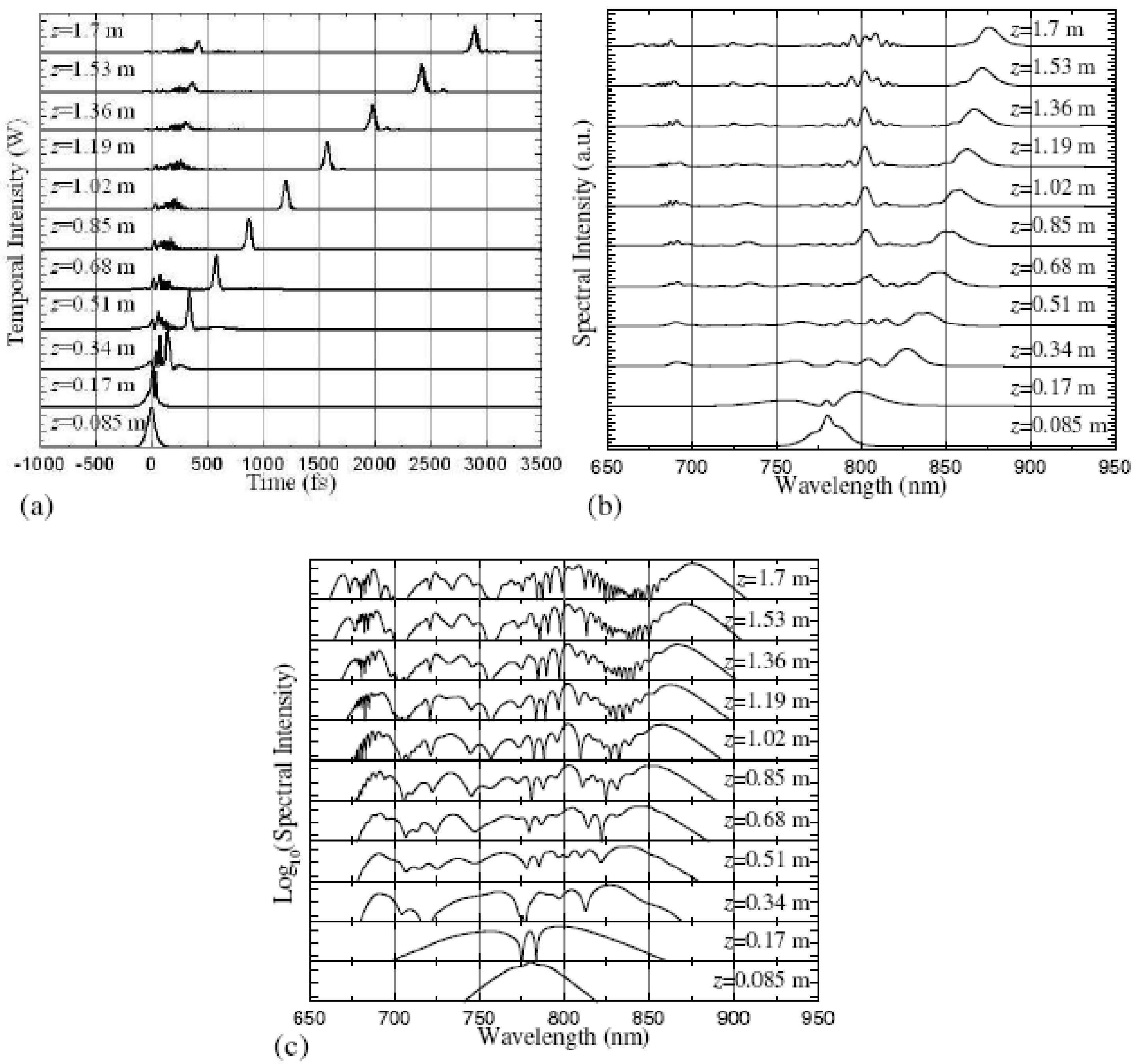
$$\frac{d\omega_{SSFS}}{dz}=-\frac{\lambda_0}{16n_2}\int\Omega^3\frac{g_R\left(-\Omega/2\pi T_0\right)}{\sinh^2(\pi\Omega/2)}d\Omega, \tag{3.57}$$

where $\Omega\equiv\left(\omega-\omega_0\right)T_0$.


(a)
(b)
1600 nm
1400 nm



a)
D (ps/nm/km)
Wavelength (nm)
Core Radius (μm)
0.4
0.5
0.6
0.7
0.8
0.9
1.0
b)
D (ps/nm/km)
Wavelength (nm)
767 nm



Temporal Intensity (W)
Time (fs)
-1000 -500 0 500 1000 1500 2000 2500 3000 3500
z=1.7 m
z=1.53 m
z=1.36 m
z=1.19 m
z=1.02 m
z=0.85 m
z=0.68 m
z=0.51 m
z=0.34 m
z=0.17 m
z=0.085 m
(a)
Spectral Intensity (a.u.)
Wavelength (nm)
650 700 750 800 850 900 950
z=1.7 m
z=1.53 m
z=1.36 m
z=1.19 m
z=1.02 m
z=0.85 m
z=0.68 m
z=0.51 m
z=0.34 m
z=0.17 m
z=0.085 m
(b)
Log10(Spectral Intensity)
Wavelength (nm)
650 700 750 800 850 900 950
z=1.7 m
z=1.53 m
z=1.36 m
z=1.19 m
z=1.02 m
z=0.85 m
z=0.68 m
z=0.51 m
z=0.34 m
z=0.17 m
z=0.085 m
(c)

June 26

Stokes / Anti-Stokes Coupling

Extend the previous derivation: set susceptibility for stokes + Antistokes.

Equations using the SVEA

$$\frac{dA_p}{dz} = 0$$

$$\frac{dA_s}{dz} = -\alpha_s A_s + \kappa_s A_2^* e^{i\Delta k z}$$

$$\frac{dA_{as}}{dz} = -\alpha_{As}^* A_{as}^* + \kappa_{as}^* A_1 e^{-i\Delta k z}$$

Gaussian units (sorry)

$$\alpha_s = \frac{12\pi i \omega_s}{n_s c}\chi_R(\omega)|A_p|$$

$$\kappa_s = \frac{6\pi i \omega_s}{n_s c}\chi_R A_p^2$$

$$\Delta k = 2\bar{k}_p - \bar{k}_s - \bar{k}_{as}$$

$\alpha_s \equiv$ Real part of Raman susceptibility

$K \equiv$

Solutions

$$A_1(z) = \left[(\quad) e^{g_+ z} + (\quad) e^{g_- z} \right] e^{+i\Delta k z/2}$$

$$A_2^*(z) = \left[(\quad) e^{g_+ z} + (\quad) e^{g_- z} \right] e^{-i\Delta k/2}$$

$$g_\pm = \pm\left[\kappa_1\kappa_2 - (\Delta k/2)^2\right]^{1/2} \Leftarrow \text{Represents coupled gain}$$

$$\simeq i\,\frac{\Delta k}{2}\left[1 - i\,\frac{4\chi_R}{\Delta k}\right]$$

$$\left(\begin{array}{l} - \Rightarrow \text{Stokes} \\ + \Rightarrow \text{anti stokes} \end{array} \right)$$

at $\Delta k = 0$ anti-stokes wave is strongly coupled to stokes that prevents it to grow exponentially } Strongly matched coupled

$\Delta k > 0$ Strong stokes growth

$\Delta k < 0$ Strong anti-stokes growth

} Strongly mismatched Stokes + Antistokes are decoupled

Coherent Anti-Stokes Raman Scattering (CARS)

- Stimulated process that produces more photons than spontaneous Raman spectroscopy

- Developed in 1965 by the Ford Motor Co (no joke)

- Vibrationally sensitive nonlinear optical technique

Classical Description

Raman active medium of frequency ω_v.

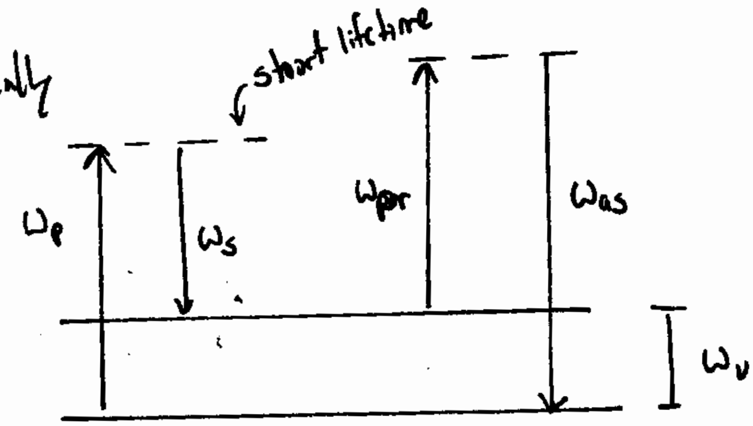
Medium driven by difference $\omega_p - \omega_s \Rightarrow$ tune ω_p

When $\omega_p - \omega_s$ is close to ω_v the material responds with large amplitude.

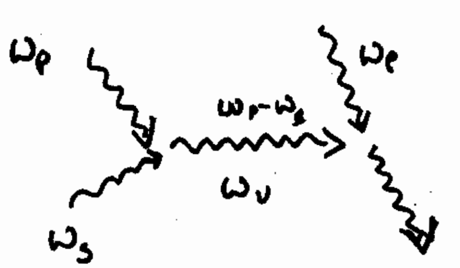
A third probe beam experiences a large change in index due to the large amplitude vibrations

Anti Stoke light is emitted

Quantum mechanically

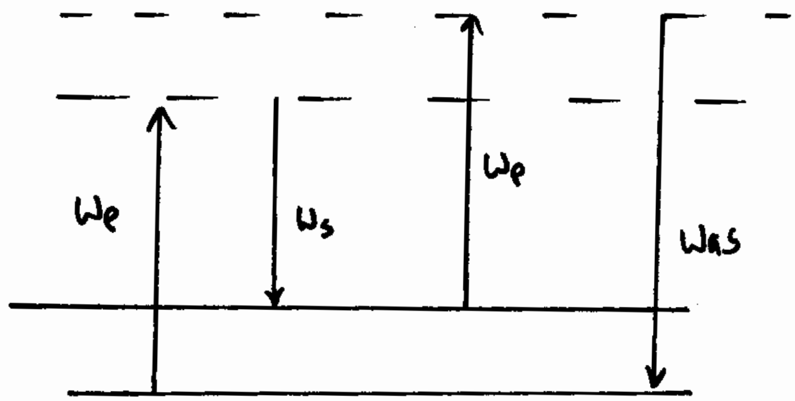


$\omega_p = \omega_{pr}$



$\omega_{as} = 2\omega_p - \omega_s$

Process for gain for Antistokes



Parametric process

Quantum Mechanical

- joint action of pump + stokes establishes a coupling between the ground state + vibrationally excited state

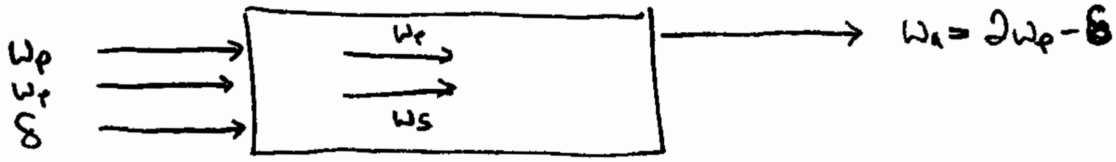
- molecule is in coherent superposition of the two states

- The probe beam investigates the coherence between states
  It promotes to a virtual state

- The molecule falls to the ground state emitting as photon.

- Probe beam interrogates medium superposition of states —

Typically, one seeds this process with a tunable source

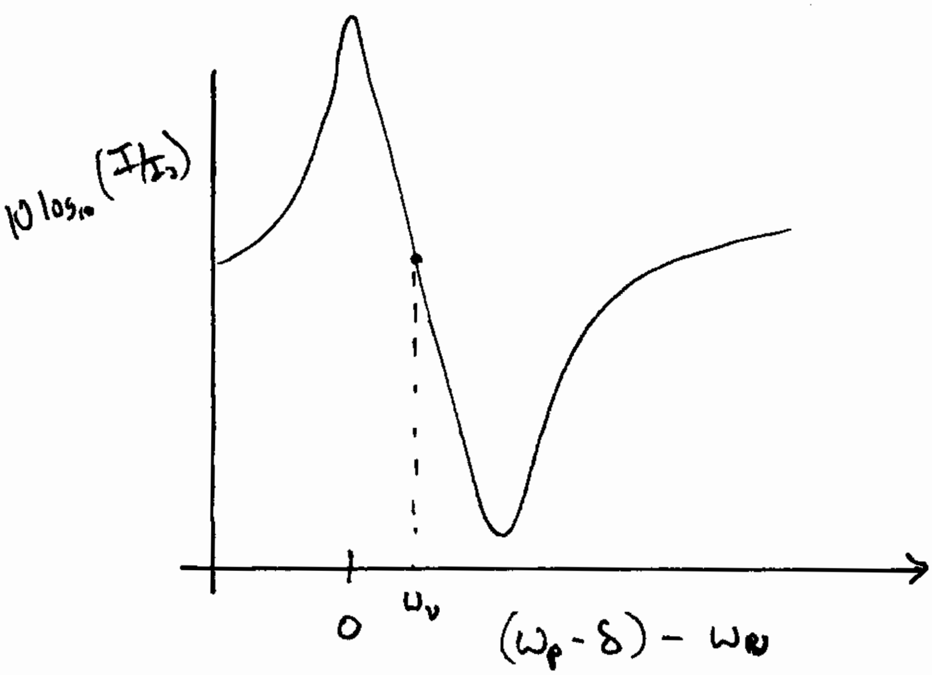


get highest amount of index change when

$$\delta \simeq \omega_s$$

OR

$$\omega_p - \delta \simeq \omega_v$$



Review of self focusing

The intensity dependent index of refraction creates a "lens" in the material

$$n = n_0 + n_2 I_0 \left(1 - 2r^2/w_0^2\right)$$

$$\text{So} \quad \phi(r) = \frac{n\omega}{c} L \simeq (n_0 + n_2 I)\frac{n_0\omega}{c} L - \frac{n_0\omega}{c} L\left(\frac{2r^2}{w_0^2}\right)$$

Thru the results of Fraunhofer diffraction a lens induces the same phase shift to the wave

$$\boxed{\phi(r) \sim -r^2} \qquad \text{lens}$$

The electric field after the lens is

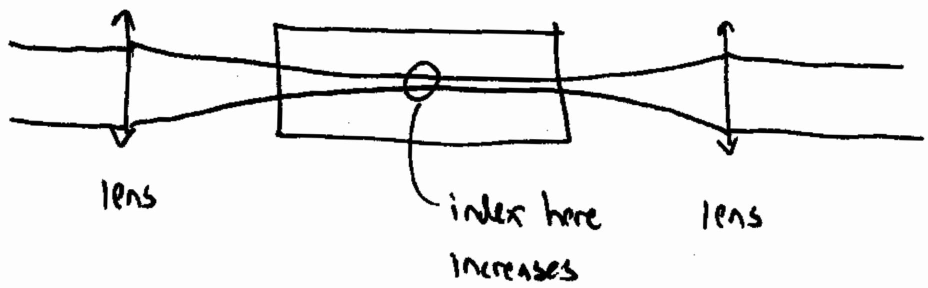
$$E(\vec{r},t)\, \exp(i\phi(r))$$

Domains for the electric field + Analogies

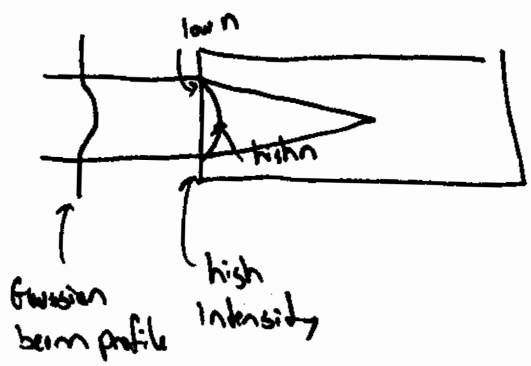
	Time / Frequency		position / k-space
linear	Dispersion	$\Longleftrightarrow$	Diffraction
Linear	GVD	$\Longleftrightarrow$	Lens (quadratic phase distortion)
Nonlinear	SPM	$\Longleftrightarrow$	Self focusing

Lecture 24 Self focusing : Spatial $\chi^{(3)}$ Effects

- Spatial analog of Self phase modulation
- Intense beam of light modifies the medium (index) it experiences



Since $n_2 > 0$ the action of self focusing causes a larger index of refraction for high intensities. This ~~mead~~ creates another lens in the material.



Important
$\Rightarrow$ Critical Power
not intensity

Spatial Solitons (self trapping)

Analog to temporal solitons

Balance of diffractive & nonlinear effects.

Critical Power

$$P_c = \frac{\pi (0.61)^2 \lambda^2}{8 n_0 n_2} = \frac{\lambda^2}{8\pi n_0 n_2} \quad \boxed{P_c > 1\,MW}$$

Whole beam self focusing: Continuous wave

(Phase distortion $\boxed{\phi = \frac{n\omega}{c} L}$)

$$n = n_0 + n_2 I$$

$$n = n_0 + n_2 I_0 \exp(-2r^2/w_0^2)$$

$$n \simeq n_0 + n_2 I_0 (1 - 2r^2/w_0^2)$$

Phase delay due to spatial nonlinearity

$$\phi(r) = n k_0 L = n_0 k_0 L + n_2 k_0 L I_0 (1 - 2r^2/w_0^2)$$

$$\phi(r) \sim -2 n_2 k_0 L I_0 \frac{r^2}{w^2} \Leftarrow$$ (analog between a lens and GVD)

quadratic spatial phase distortion.

This is what a lens does!

Critical Power is defined where $\Rightarrow \boxed{P_c = \lambda^2/8\pi n_0 n_2} \sim 1\ \mathrm{MW}$

$$\boxed{\phi_{\text{self focus}}(r) = \phi_{\text{diffraction}}(r)}$$

$$Z_{SF} = \frac{1}{2} \left(\frac{\pi r_0^2 n_0}{\lambda} \right) \frac{1}{(P/P_c - 1)^{1/2}}$$

Self - de focusing

Higher order nonlinearities $\sim n_4 I^2$

Self trapping

$\Rightarrow$ leads to spatial soliton formation

Critical angle due to total internal reflection

$$\boxed{\theta_{tir} = \cos^{-1}\left[\frac{n_o}{n_o + n_2 I}\right]}$$

becomes equal to diffraction angle

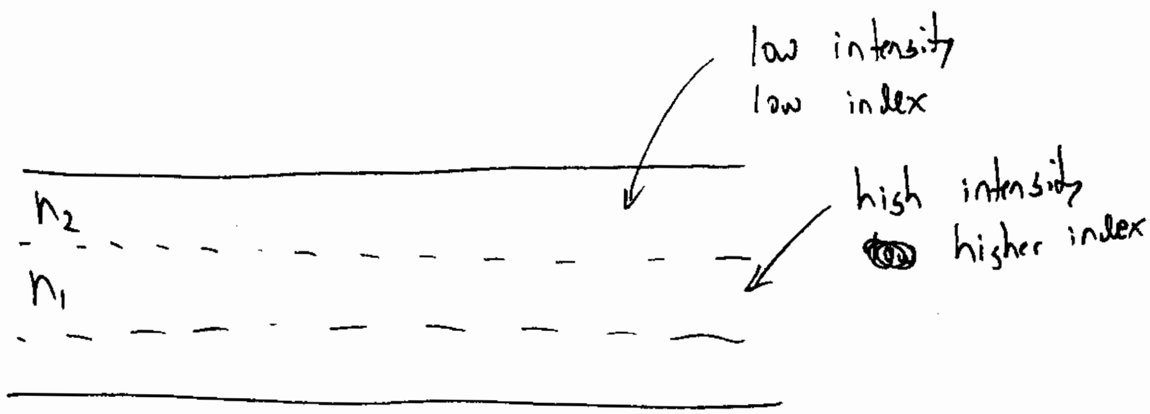
$$\boxed{\theta_d = 1.22\,\frac{\lambda}{4 n_o r_0}}$$

(r_0 = beam radius
d = beam diameter)

from diffraction theory

Setting $\theta_d = \theta_{tir}$

$$\boxed{P_{cr} = \frac{(1.22)^2 \pi \lambda^2}{32\, n_o n_2}}$$



$n_1 \gtrsim n_2 \Rightarrow$ total internal reflection.

Another way to look at this Self trapping

Critical angle due to total internal reflection $\equiv$ diffraction θ

Diffraction angle = self focusing angle.

$$\theta_{diff} = \theta$$

$$\boxed{P_c = \frac{(1.22)^2 \pi \lambda^2}{32\, n_0\, n_2}}$$

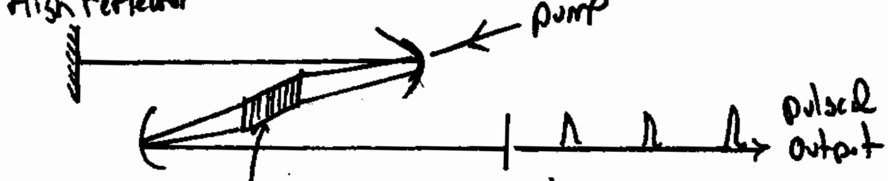
Example of Self focusing $\Rightarrow$ Kerr lens modelocking

Modelocking is a method to get short pulse formation in a laser cavity thru the coherent addition of cavity modes

Add up modes to generate pulse train

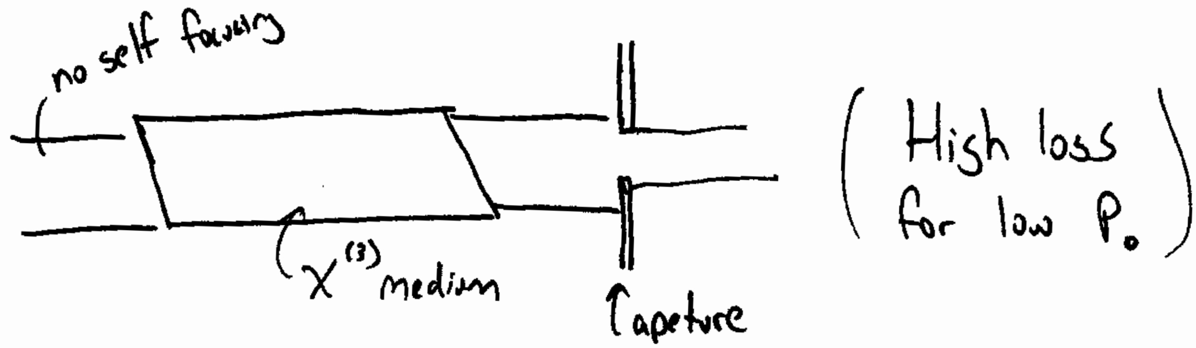
This is typically done by setting up a condition in the cavity that favors high peak powers. (Self amplitude mod).

Kerr-lens $\Rightarrow$ "lens" due to self focusing in a nonlinear crystal.

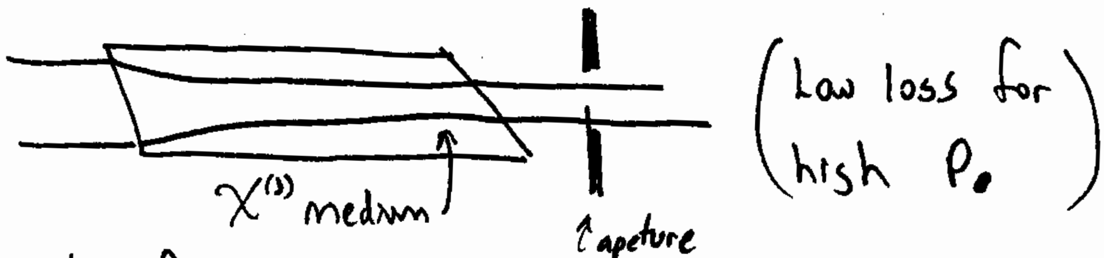


Mode-locked laser

How to use self focusing for mode locking?



Set up a condition where there is low loss for high peak powers

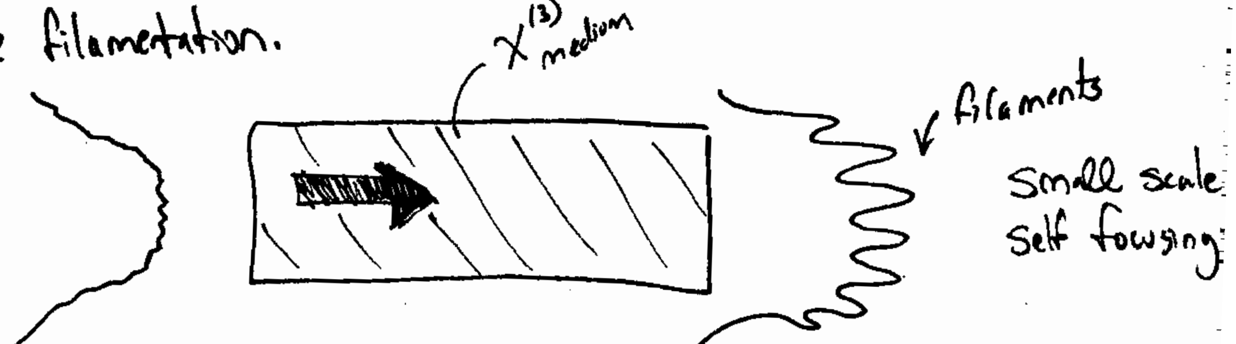


This condition ~~permits~~ favors high peak power pulses.

Self filamentation

During self focusing a single beam will break up into multiple small beams.

The spatial beam profile will have variations due to quantum noise. These variation each will focus causing the filamentation.



Filaments grow exponentially with distance

Self filamentation leads to a beam with random intensity distributions

Unfortunately, the power for self filamentation is on the same order of that of self focusing.

Light Bullets

$\Rightarrow$ 3D spatial solitons

Spatial / temporal coupling

Self Focusing using pulses

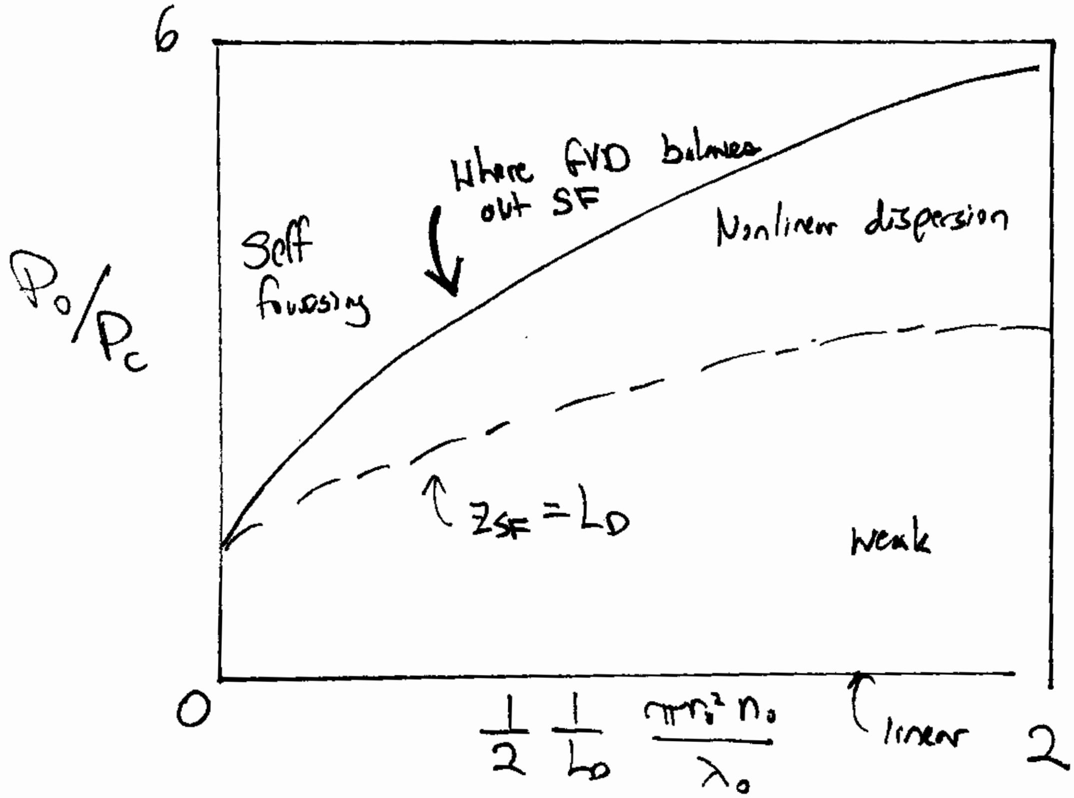
Need to include space - time coupling

Material dispersion ⇒ higher peak powers for self focusing compared to cw case

Need to consider higher order dispersion + nonlinearities

$$\frac{1}{2}\frac{1}{L_D}\left(\frac{\pi r_0^2 n_0}{\lambda_0}\right) \equiv \text{measure of dispersion relative to diffraction}$$

$$\frac{P_0}{P_{cr}} \equiv \text{Strength of pulse with respect to critical power}$$



Notes on SPM in gases

- SPM is more complex because it is coupled with self focusing
- Self focusing effects are detrimental for using SPM for temporal compression in gases.
- Ionization in gases modify the beam propagation + produces asymmetric SPM.

Saturation of the nonlinear response

Self Focusing in Solids

Causes damage tracks (glass)

Fiber fuse effect $\Rightarrow$ Bad

Self Focusing using pulses

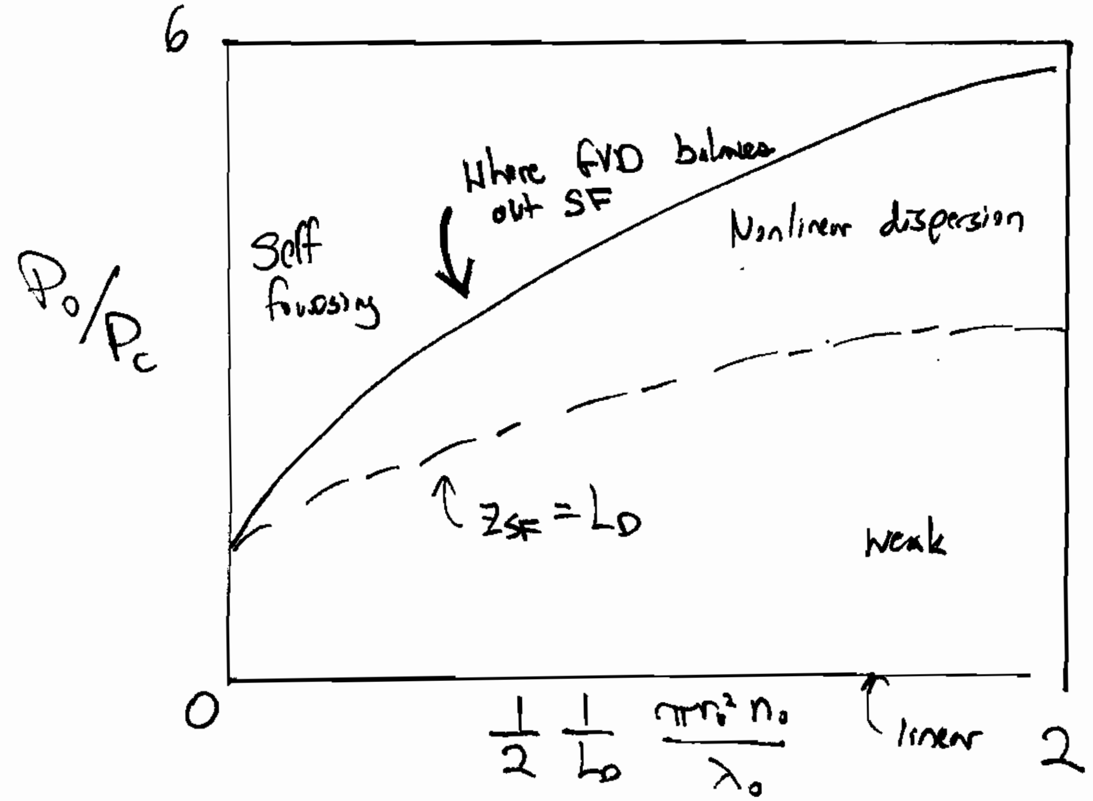
Need to include space-time coupling

Material dispersion ⇒ higher peak powers for self focusing compared to cw case

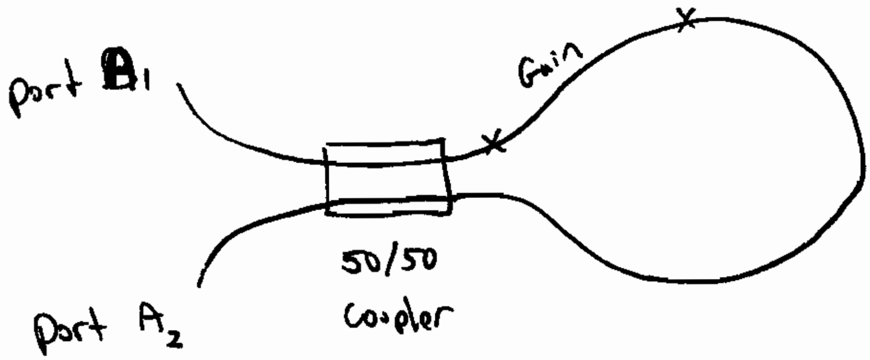
Need to consider higher order dispersion + nonlinearities

$$\frac{1}{2}\frac{1}{L_D}\left(\frac{\pi r_0^2 n_0}{\lambda_0}\right) \equiv \text{measure of dispersion relative to diffraction}$$

$$\frac{P_0}{P_{cr}} \equiv \text{Strength of pulse with respect to critical power}$$



Nonlinear Switching — Loop Mirrors

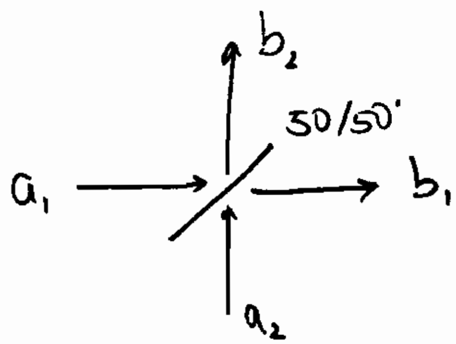


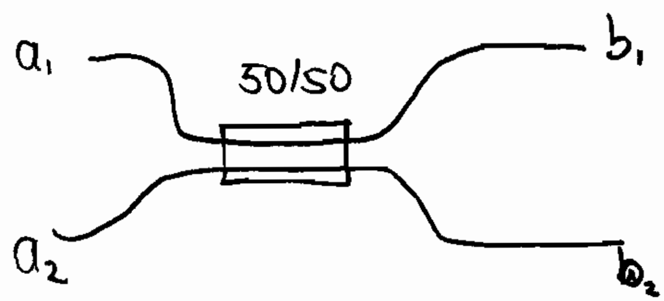
Loop mirror

Fast switch

Sagnac Interferometer

Difference between beam splitter + directional coupler





$\boxed{\text{beam splitter}}$

π phase shift between b_1 + b_2

$\boxed{\text{directional coupler}}$

(uses evanescent wave coupling)

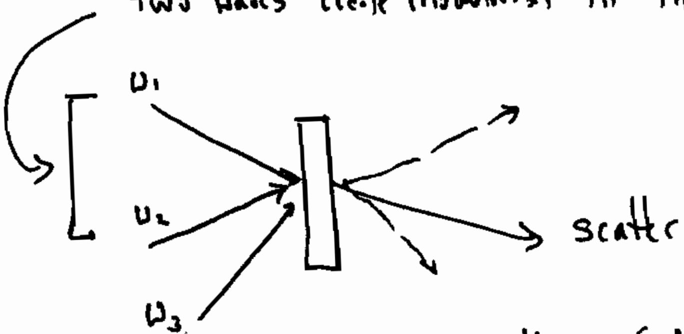
$\frac{\pi}{2}$ phase shift between b_1 + b_2 for 50/50.

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} t & e^{i\pi/2}\sqrt{1-t^2} \\ e^{i\pi/2}\sqrt{1-t^2} & t \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

3rd order effects : Nonlinear induced gratings self diffraction

Consider a noncolinear 3^{rd} order process

two waves create modulation in index

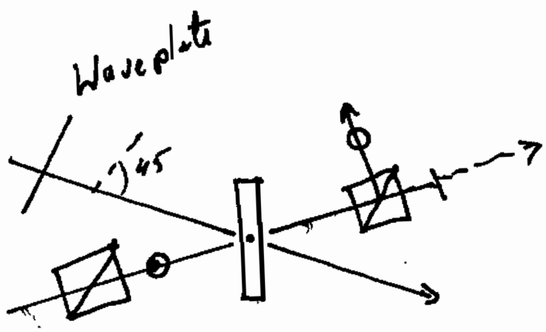


third wave scatters (diffracts) off volume grating

If $\omega_1 = \omega_2 = \omega_3 \Rightarrow$ self diffraction (used for $\chi^{(3)}$ FROG)

Polarization Gating

Also used for $\chi^{(3)}$ FROG



Rejects beam

Measure nonlinear change in polarization

Good SNR ratio

Two photon absorption ($\chi^{(3)}$ effect)

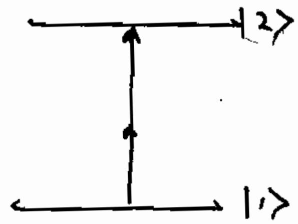
$$\alpha = \alpha_0 + \alpha_2 I$$

Where

$$\alpha_2 = \frac{3\omega_0}{4nc} \, \mathrm{Im}\left\{ \chi^{13}_{xxxx}(\omega_j - \omega, +\omega, -\omega) \right\}$$

Can write (single beam)

$$\frac{dI}{dz} = -\alpha_0 I - \alpha_2 I^2$$



Absorption of two photons

Can also be done using two beams

Saturable absorber (pg 611)

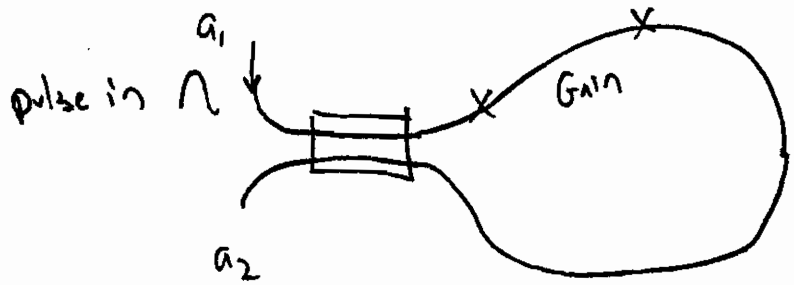
$$\boxed{\alpha(I) = \frac{\alpha_0}{1 + I/I_s}}$$

pulse shapes loss (Weiner)

Needs to be "fast"

SESAM Semiconductor saturable absorbing mirror

The nonlinear phase shift is asymmetric



$G \equiv$ Gain

i.e directional coupler will split the input pulse into two:

1) Clockwise Pulse: Pulse that goes clockwise get first amplified and then experiences a large nonlinear phase shift

2) Counter Clockwise Pulse: Pulse that goes counter clockwise goes thru fiber and get a small nonlinear phase then goes thru amplifier

Difference in phase shift between two pulses.

$$\Delta\phi \sim n_2 (G-1) I(t) L$$

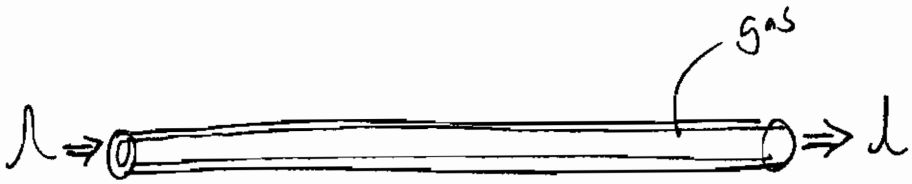
If the pulses are weak then $\Delta\phi = 0$ and the pulse will exit the loop out port a_1

If the pulses are strong then $\Delta\phi \simeq \pi$ and the pulse will exit the loop out port a_2.

Ultrashort pulse compression in Noble Gases.

Specific Noble gases (Ne, etc) exhibit a large n_2 with small dispersion. The SPM due to the gas in the presence of small GVD will cause spectral broadening / temporal compression.

$L_{NL} \ll L_D$



Short pulse
~1 mJ

Shorter pulse!

Compression from ~20 fs to < 5 fs

Using the interaction of SPM + GVD for pulse compression

~~Self Focusing~~

~~spatial analog to self~~

Lecture 14 Higher harmonic generation + attosecond pulse generation

- Optical Damage

  multiphoton absorption

  ionization

  Dependence on fluence + pulse duration

- HHG Requirements on pulses
  - CEP Definition

    ω_0

    single cycle intense pulses

  Nonlinear optics
  - Perturbation
  - Strong Field regime

  optical field ionization of atoms

  multiphoton
  tunnel
  above barrier

  3 step model of Corkum

  Consider effects
  absorption
  dephasing defocusing

- Phase effects

- Phase matching

- Attosecond pulse generation

  Train of pulses

  ~~Atto~~ Streak Camera characterization

  Single as pulse

  DOG

Intro

Why HHG?

Huge bandwidth

Cover X ray wavelengths

How related to attosecond pulses?

Wide bandwidth $\Rightarrow$ Support short pulse

What good are as pulses?

as timescales $\Rightarrow$ Frozen molecular motion

electronic timescale

$1 \text{ a.u.} = 2.4 \times 10^{-17} \text{ s}$

24. as

Femto chemistry to attophysics

Use ultrashort pulses in gases to generate HHG

Need
- nearly single cycle
- High intensity
- Phase stabilized

this is hard!

Single cycle pulses

$$\mathcal{E}(t) = A(t)\, e^{-i\omega_0 t + i\varphi_0} + c.c.$$

What does ω_0 mean?

$$T_0 = \frac{\lambda_0}{c} \qquad \omega_0 = \frac{\int_0^\infty \omega\, |E(\omega)|^2\, d\omega}{\int_0^\infty |E(\omega)|^2\, d\omega}$$

$\lambda_0 = 800\,\text{nm}$ $\quad T_0 = 2.67\,\text{fs}$

Expect concepts of carrier + envelope to ~~fail~~ as pulse gets to single cycle

Krause claims
- $\tau_p \geq T_0$ valid.
- Can use concept of carrier + phase for single cycle.

CEP matters if electrons move in the oscillation of one cycle!

Absolute phase
$$\varphi = \varphi_0 + \omega_0\left(\frac{1}{v_p} - \frac{1}{v_g}\right) z$$

$\Delta\varphi$: phase picked up per pulse

3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER
COMET

Nonlinear response of atoms in strong ~~laser~~ field

Show slide #1

Two regimes 1) Perturbative regime $< 10^{13}\ W/cm^2$
2) Strong field regime $> 10^{13}\ W/cm^2$

Need to consider electron ~~states~~ transitions
bound to bound states
bound to free states (ionization)

Perturbative regime : wavefunction within a_B

$$P_{NL} = \epsilon_0 \left[\chi^{(2)} E^{(2)} + \chi^{(3)} E^3 + \cdots \right]$$

response $\frac{1}{\Delta} < 1\,fs$

Bound-Bound

Conversence $\frac{\chi^{(k+1)} E^{k+1}}{\chi^k E^k} \simeq \frac{e E_0 a_B}{\hbar \Delta} \ll 1$ } bound-bound perturbative

Bound-Free

Keldysh $\frac{1}{\gamma} = \frac{e E_0}{\omega_0 \sqrt{2 m W_b}} = \frac{e E_0 a_B}{\hbar \omega_0} \ll 1$ } bound free perturbative

where $a_B \equiv \hbar / \sqrt{2 m W_b}$

Keldysh parameter

$$\gamma = \omega \frac{\sqrt{2 m W_0}}{e E_A}$$

m electron mass

ω ~~freq~~ laser freq.

W_0 (zero field ionization)

binding energy of most weakly bound e^-

$$\boxed{\gamma \equiv \text{ratio of laser field tunneling frequency to the laser frequency}}$$

$\gamma >> 1$ multiphoton ionization

$\gamma << 1$ tunneling ionization

$$\gamma = \sqrt{\frac{\cancel{W_\phi}\, W_\phi}{2U_p}}$$

W_ϕ ~~W_ϕ~~ ionization potential

CEP Sensitivity

Strong field ionization is phase dependent

Fig 46.

Ejection of photo electrons.

Fig 48 , Fig 49

$\phi_0 = 0$ cut off harmonics generated at single instant

$\phi_0 = \pi/2$ emerge at two instants $1/2$ cycle appart.

New Fronteors

Single atto second pulse Double optical gating (DOG)

- $\omega + 2\omega$ Break symmetry
- use polarization gating equal to one cycle to select one pulse

Longer wavelength sources

$U_p \sim \lambda^2 I$ increase cutoff frequency.

Strong Field regime ($\frac{1}{\gamma} < 1$, Wave function $> a_B$) ~~10^{13} W~~

$> 10\ W/cm^2$

most weakly bound e^- of energy $-W_b$ penetrates barrier at x_0 within fraction of laser oscillation cycle T_0

wiggling electron in external field

wiggle amplitude $\quad a_w = \dfrac{eE_a}{m\omega_0^2} \qquad x_0 = \dfrac{W_b}{eE_a}$

Pondermotive Energy

Cycle averaged KE of wiggling e^- $\qquad V_p = \dfrac{e^2 E_a}{4 m \omega_0^2}$

$$\frac{1}{\gamma^2} = \frac{2V_p}{W_b} = \frac{a_w}{2x_0}$$

e^- aquires large KE within fraction of T_0

Classical description $\Rightarrow$ Corkum
+ tunneling

$\Rightarrow$ wavefunction of e^- does not expand over T_0

~~3 Step Model ⇒ Slides~~

Cut off harmonic $\boxed{N_c \omega_0 = W_b + 3.17\, U_p}$ $\Big\{$ $U_p \sim \lambda^2 I$

Pulse propagation in gases for HHG (Fig 28)

Optical Field ionization of atoms (Fig 30)

multiphoton ionization
tunneling
above barrier

3 Step Model ⇒ Slides

Propagation effects
absorption
dephasing : diff in phase velocity of HHG + driving source
defocusing : free electrons cause defocusing

Optical Damage

Causes

- Linear absorption

$< 10^9 W/cm^2$

localized heating

CW + long pulse ($> 1 \mu s$)

- Avalanche breakdown

$10^9 W/cm - 10^{12} W/cm^2$

$10^9 W/cm^2 - 10^{12} W/cm^2$

$< 1 \mu s$

Dominant mechanism for pulsed lasers in this range.

Free electrons are accelerated by laser field

electrons impact ionize other atoms, producing more e^-

Fraction of energy to e^- causes local heating

- Multiphoton ionization or absorption

$10^{12} W/cm^2 - 10^{16} W/cm^2$

$10^{12} W/cm^2 - 10^{16} W/cm$ Dominates

$>10^{20} W/cm^2$ • Single field ionization

Dependence on pulse duration ⇒ Fluence dependence

10ps - 10ns : Damage Fluence $\sim \sqrt{T_p}$ Damage Intensity $\sim \frac{1}{\sqrt{T_p}}$

Damage depends on geometric mean of fluence + intensity

Fluence $\dfrac{\text{Energy}}{\text{area}}$ $\dfrac{N\hbar\omega}{\pi r^2}$

Intensity $\dfrac{\text{Power}}{\text{area}}$ $\dfrac{N\hbar\omega}{\pi r^2 \, T_p} = \dfrac{F}{T_p}$

$$F \sim \sqrt{T_p}$$

$$I T_p \sim \sqrt{T_p}$$

$$I \sim 1/\sqrt{T_p}$$

$$\text{Damage} \sim \sqrt{IF} \sim \sqrt{T^{-1/2}\, T^{1/2}} \sim 1$$

No time dependance

Quantum Mechanical Description of Nonlinear Optical Susceptibilities

Before we used a perturbative sol$^{\underline{n}}$ to a nonlinear oscillator to get the nonlinear response of a material

Lorentz Model

Solve $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = -eE(t)/m$, $P = \epsilon_0 \chi E$

$$\chi^{(1)}(\omega) = \frac{Ne^2/m}{\epsilon_0}\left(\frac{1}{\underbrace{\omega_0^2 - \omega^2 - 2i\gamma\omega}_{D(\omega)}}\right)$$

$$= \frac{Ne^2/m}{\epsilon_0}\,\frac{1}{D(\omega)}$$

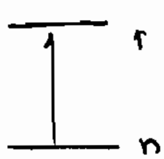
However a real material has many resonances

$$\chi^{(1)}(\omega) = \sum_j \frac{Ne^2/m}{\epsilon_0}\,\frac{f_j}{\omega_{0j}^2 - \omega^2 - 2i\gamma_j\omega}$$

oscillator strength

Our classical solution does not give information about f_j

A quantum mechanical approach will relate f_j to elements of dipole transition matrix μ_{nm}



$$f_{nr} = \frac{2m\,\omega_{nr}\,|\mu_{nr}|^2}{3\hbar e^2}$$

We can also get γ from the quantum mechanical treatment

Classical Derivation of Nonlinear susceptibilities

2nd order $\quad \ddot{x} + 2\gamma \dot{x} + \omega_0^2 x + a x^2 = -e E(t)/m$

$$E(t) = \left(E_1 e^{-i\omega_1 t} + E_2 e^{-i\omega_2 t} \right) + \text{c.c.}$$

Solve using perturbative expansion

$$x(t) = \lambda x^{(1)} + \lambda^2 x^{(2)} + \lambda^3 x^{(3)} + \cdots$$

Let eqs. $\quad \lambda \left[\ddot{x}^{(1)} + 2\gamma \dot{x}^{(1)} + \omega_0^2 x^{(1)} \right] = -eE(t)/m \ \lambda \qquad (1)$

$$\lambda^2 \left[\ddot{x}^{(2)} + 2\gamma \dot{x}^{(2)} + \omega_0^2 x^{(2)} + a \left(x^{(1)} \right)^2 \right] = (0)\,\lambda \qquad (2)$$

$$\lambda^3 \left[\ddot{x}^{(3)} + 2\gamma \dot{x}^{(3)} + \omega_0^2 x^{(3)} + a \left(x^{(1)} x^{(2)} \right) \right] = (0)\,\lambda \qquad (3)$$

Sol$\underline{\text{n}}$

$$x^{(1)}(\omega_j) = -\frac{e}{m} \frac{E_j}{\left(\omega_0^2 - \omega_j^2 - 2i\omega_j \gamma \right)} \qquad j = 1, 2$$

$$x^{(2)}(2\omega_2) = \frac{-a\,(e/m)^2 E^2}{D(2\omega_2)\, D^2(\omega_2)}$$

$$x^{(2)}(\omega_1 + \omega_2) =$$

$$x^{(2)}(\omega_1 - \omega_2) =$$

Find $\chi^{(2)} \qquad P^{(2)} = \epsilon_0 \chi^{(2)}(\quad)(E \cdots)$

Get $\quad \chi^{(2)}(2\omega_1; \omega_1, \omega_1) \quad \chi^{(2)}(\omega_1 + \omega_2; \omega_1, \omega_2) \quad \chi(\omega_1 - \omega_2; \omega_1, -\omega_2)$

$\chi^{(2)}(0; \omega, -\omega)$

Example

$$\chi^{(2)}(\omega_1+\omega_2;\ \omega_1,\omega_2) = \frac{Ne^2/m^2\ a}{\epsilon_0\ D(\omega_1+\omega_2)\,D(\omega_1)\,D(\omega_2)}$$

QM gives us

1) How $\chi^{(n)}$ depends on dipole transition moments + atomic energy levels

2) internal symmetries

3) Make numerical values of $\chi^{(n)}$

Works well for atomic vapors

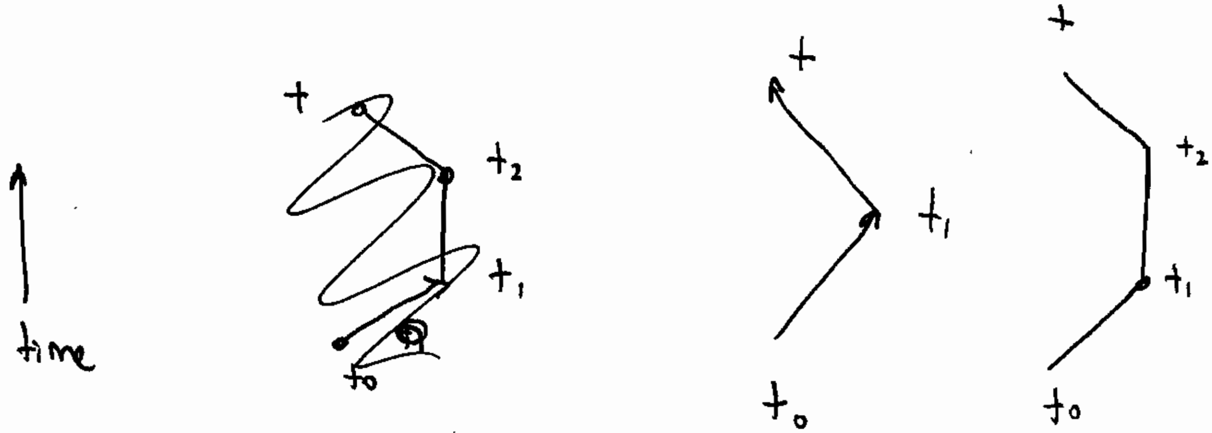
Use 1) time dependent perturbation theory
2) Density matrix

Time ordering of perturbation leads to different terms in the susceptibility

Account for terms using diagrams (Double sided

Timing ordering

$$|g\rangle \to \underbrace{|m\rangle \longrightarrow |n\rangle}_{\text{virtual transitions}} \to |f\rangle$$



$$\frac{i}{\hbar} \int_{t_0}^{t} \int_{t_0}^{t_1} U(t,t_0)\, U^\dagger(t',t_0)\, \mathcal{H}^{(0)} U(t_1,t_0)\, U(t_2,t_0) \times \mathcal{H}^{(1)} U(t_2',t)\, dt_2\, dt_2$$

Lecture 27 Quantum Mechanical Description of Nonlinear optical susceptibilities

So far, we have describe nonlinear optics in classical terms treating the medium as a collection of dipoles with a continuous spread of energies.

The question is, do we lose "Something" treating the system classically? Does a quantum description provide more or a better explaination? To answer these questions, we will need to develope a quantum treatment. More specifically, we will use a <u>semi-classical treatment</u>

- treat material quantum mechanically
- treat E/m field classically

We can do this since the number of photons are large. (High intensities)

Now we really have not discussed what a photon is, this will come later in our discussion of quantum optics

Density Matrix Formalism

A single quantum mechanical state can be described by the state vector

$$|\psi\rangle = \sum_n c_n |\psi_n\rangle$$

This is a <u>pure state</u>. If I have a collection of N quantum systems I cannot use a state vector to describe the total system. This is called a mixed state. Here, we have an ensemble of N systems, n_i are in state $|\psi_i\rangle$

The ensemble is described by an occupancy number n_i

A way to assembly the information on an ensemble is the density matrix

$$\boxed{\hat{\rho} = \sum_i p_i \, |\psi_i\rangle\langle\psi_i|}$$

$$p_i = \frac{n_i}{N}$$

probability to be picked Randomly

pure state : $p_i = 0$ except one state

Ensemble average of property Ω

$$\boxed{\langle\bar{\Omega}\rangle = \sum_i p_i \langle\psi_i|\Omega|\psi_i\rangle = \mathrm{Tr}(\hat{\rho}\Omega)}$$

Here there a two averaging 1) Quantum average $\langle\psi_i|\Omega|\psi_i\rangle$ for each system i

2) Classical average over different states $|i\rangle$

Remember orthonormality

$$1 = \sum_j |j\rangle\langle j|$$

Consider

$$\mathrm{Tr}(\Omega\rho) = \sum_j \langle j|\Omega\rho|j\rangle$$

$$= \cancel{\sum_i \sum_j \langle j|\Omega|i\rangle\langle i|\rho|j\rangle}$$

$$= \sum_j \sum_i \langle j|\Omega|i\rangle\langle i|j\rangle p_i \quad (\text{use definition of } \rho)$$

$$= \sum_i \sum_j \langle i|j\rangle\langle j|\Omega|i\rangle p_i$$

But $\quad 1 = \sum_j |j\rangle\langle j|$

so

$$Tr(\Omega\rho) = \sum_i \langle i|\Omega|i\rangle \rho_i$$

$$= \langle\bar{\Omega}\rangle \Rightarrow \text{Ensemble average of } \Omega$$

The density matrix contains all statistical information on the ensemble.

~~Define density matrix operator~~

$$\cancel{\hat{\rho} = |j\rangle\langle j|}$$

$$\cancel{Tr(\hat{\rho}) = \sum_n \langle i|j\rangle\langle}$$

For a pure state

$$\hat{\rho} = |\psi_j\rangle\langle\psi_j|$$

$$\cancel{Tr(\hat{\rho}) = \sum_j\sum_i \langle\psi_i|\psi_j\rangle\langle\psi_j|\psi_i\rangle \rho_i \neq \rho_i}$$

$$Tr(\hat{\rho}) = \sum_n \langle n|\hat{\rho}|n\rangle = \sum_i\sum_n \langle n|i\rangle \rho_i \langle i|n\rangle$$

$$= \sum_i\sum_n \rho_i \langle i|n\rangle\langle n|i\rangle = \sum_i \rho_i \langle i|i\rangle = 1$$

Also $\quad Tr(\hat{\rho}^2) = 1 \qquad$ pure state $\quad \rho^2 = \rho$

$$\mathrm{Tr}(\hat{\rho}\hat{\Omega}) = \sum_j \langle j | \rho \Omega | j \rangle$$

$$= \sum_j \sum_i \langle j | i \rangle \langle i | \Omega | j \rangle \rho_i$$

$$= \sum_j \sum_j \langle i | \hat{\Omega} | j \rangle \langle j | i \rangle \rho_i$$

$$= \sum_i \langle i | \hat{\Omega} | i \rangle \rho_i = \langle \Omega \rangle$$

Important Result

$$\boxed{\begin{array}{l} \mathrm{Tr}(\hat{\rho}^2) = 1 \\ \mathrm{Tr}(\hat{\rho}^2) < 1 \end{array}}$$

pure state / ensemble

mixed state

The density operator describes a <u>mixed state</u>

More properties of the density operator

$$\rho^\dagger = \rho \qquad \rho^2 = \rho \ (\text{pure ensemble})$$

$$\mathrm{Tr}\,\rho = 1 \qquad \mathrm{Tr}(\rho^2) \leq 1$$

We can express the density matrix in matrix space

$$\rho_{nm} = \sum_i p_i \, C_m^* C_n$$

where $\psi = \sum_n C_n \, |n\rangle$

↑ energy eigensolutions to Schrödinger Eq

Physical ~~At~~ Interpretation of density matrix

Diagonal Elements : probability of ~~state~~ system to be in eigenstate $n \Rightarrow \rho_{nn}$

Off diagonal elements : "Coherence" between levels n + m

$\rho_{nm} \neq 0$ if there is a coherent superposition of $|\psi_n\rangle$ + $|\psi_m\rangle$

Write Hamitonian

$$\hat{H} = \hat{H}_0 + \hat{V}(t)$$

Interaction $\rightarrow$ dipole $\qquad V(t) = -\hat{\mu} \cdot \bar{E}(t) \qquad \bar{\mu} = e\bar{r}$

classical E/m field

$$[\hat{H}, \hat{\rho}] = [\hat{H}_0, \hat{\rho}] + [\hat{V}(t), \hat{\rho}]$$

$\hat{H}_0$ satisfies time independent Schrodinger Eq

$$H_0 |\psi_n\rangle = E_n |\psi_n\rangle$$

or

$$H_{0,nm} = E_n \delta_{nm}$$

$|\psi_n\rangle$ eigensolutions to time-independent S.E.

So

$$[\hat{H}_0, \rho] = \hat{H}_0 \hat{\rho} - \hat{\rho} H_0 = \sum_{\nu} \left(H_{0,n\nu} \rho_{\nu m} - \rho_{n\nu} H_{0,\nu m} \right)$$

$$= \sum_{\nu} \left(E_n \delta_{n\nu} \rho_{\nu m} - \rho_{n\nu} \delta_{\nu m} E_m \right)$$

$$= E_n \rho_{nm} - E_m \rho_{nm} = (E_n - E_m) \rho_{nm}$$

Define $\qquad \omega_{nm} \equiv \dfrac{E_n - E_m}{\hbar}$

30

$$\boxed{\dot{\rho}_{nm} = -i\,\omega_{nm}\rho_{nm} - \frac{i}{\hbar}[\hat{V},\hat{\rho}]_{nm} - \frac{1}{T_2}(\rho_{nm} - \rho_{nm}^{eq})}$$

solution $\Rightarrow$ Expand $\rho(t) = \sum_n \rho^{(n)}(t)$

$$\rho^{(n)}(t) = \left(\frac{1}{i\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \left[V(t_1),\left[V(t_2),\cdots\left[V(t_n),\rho(t_0)\right]\cdots\right]\right]$$

(Dyson Series)

Note $t_0 \le t_n \le t_{n-1} \cdots \le t_1 \le t$ $\Leftarrow$ time ordering

$\rho^{(n)}$ contains $2n$ terms

$$[V(t_1),[V(t_2),\rho(t_0)]] = V(t_1)V(t_2)\rho(t_0) - V(t_1)\rho(t_0)V(t_2)$$
$$- V(t_2)\rho(t_0)V(t_1) + \rho(t_0)V(t_2)V(t_1)$$

Potential in interaction picture

$$V_I(t) = U^*(\mu\cdot E)\,U$$

$U \equiv$ unitary matrix $\exp(-i H_0 t/\hbar)$

Density matrix in interaction picture

$$\hat{\rho}_I = U^* \rho\, U$$

The off diagonal terms are important since they are proportional to an induced dipole moment.

Let's look at an example: Two level ~~system~~ atom

$$\overline{\qquad\qquad}\ |b\rangle$$

$$\overline{\qquad\qquad}\ |a\rangle$$

Describe specific state s

$$|\psi_s\rangle = C_a^s\,|a\rangle + C_b^s\,|b\rangle$$

Density matrix

$$\hat{\rho} \Rightarrow \begin{pmatrix} \rho_{aa} & \rho_{ab} \\ \rho_{ba} & \rho_{bb} \end{pmatrix} \qquad \rho_{nm} = \sum_s P_s\, C_m^{s*} C_n^s$$

Dipole moment operator

$$\hat{\mu} \Rightarrow \begin{pmatrix} 0 & \mu_{ab} \\ \mu_{ba} & 0 \end{pmatrix} \qquad \mu_{ij} = -e\,\langle i|\hat{z}|j\rangle$$

Find expectation value of dipole $\langle\hat{\mu}\rangle = Tr(\hat{\rho}\hat{\mu})$

$$\hat{\rho}\hat{\mu} \Rightarrow \begin{pmatrix} \rho_{ab}\mu_{ab} & \rho_{aa}\mu_{ab} \\ \rho_{bb}\mu_{ba} & \rho_{ba}\mu_{ab} \end{pmatrix}$$

Then

$$\boxed{\langle\bar{\mu}\rangle = Tr(\hat{\rho}\hat{\mu}) = \rho_{ab}\mu_{ba} + \rho_{ba}\mu_{ab}}$$

Example Time dependence for two level System

$$\dot{\rho}_{ba} = -i\,\omega_{ba}\,\rho_{ba} - \frac{1}{T_2}\,\rho_{ba}$$

equilibrium levels

$$\dot{\rho}_{bb} = \frac{1}{T_1}\left(\rho_{11} - \rho_{11}^{e}\right)$$

$$\rho_{ab}(t) = \rho_{ba}^{*}(t) \qquad \rho_{aa}(t) = 1 - \rho_{bb}(t)$$

Back to perturbation theory $\left(U(t)\,U^{*}(t') = U(t-t') \right)$

so in the interaction picture

$$V_I(t_1)\,\rho_I(t_0)\,V_I(t_2) = U^{*}(t_1)\left[-\mu\cdot E(t_1)\right]U(t_1)\,U^{*}(t_0)\,\rho^{(0)}(t_0)\,U(t_0)$$
$$U^{*}(t_2)\left[-\mu\cdot E(t_2)\right]U(t_2)$$
$$= U(t-t_1)\left(-\mu\cdot E\right)U(t_1-t_0)\,\rho^{0}(t_0)\,U(t_0-t_2)\left[-\mu\cdot E(t_2)\right]U(t_2-t)$$

for 2nd order

$$\left(\frac{1}{i\hbar}\right)\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\; U(t-t_1)\left[-\mu\cdot E(t_1)\right]U(t_1-t_0)\,\rho^{(0)}(t_0)$$
$$U(t_0-t_2)\left[-\mu\cdot E(t_2)\right]U(t_2-t)$$

For n^{th} order we have $\Rightarrow$ $2^n\,n!$ terms!

time ordering

Lecture 28 Nonlinear Optical Perturbation Theory

We wish to solve for the time dependence of $\hat{\rho}$. Use Liouville Eq.

$$\boxed{\dot{\rho}_{nm} = \frac{-i}{\hbar}[\hat{H}, \rho]_{nm}}$$ (interaction picture) or (Dirac picture) (1)

$$\boxed{\hat{H}(t) = \hat{V}(t) + \hat{H}_0}$$

However, we include a phenomenological damping term

$$\dot{\rho}_{nm} = \frac{-i}{\hbar}[\hat{H}, \hat{\rho}]_{nm} - \gamma_{nm}(\rho_{nm} - \rho_{nm}^{eq})$$

ρ_{nm} relates to ρ_{nm}^{eq} at rate γ_{nm}

Specifically

$$\rho_{nm}^{eq} = 0 \quad \text{for} \quad n \neq m$$

$$\gamma_{nm} = \gamma_{mn} \equiv 1/T_2 \Rightarrow \text{dephasing rate}$$

So

$$\boxed{\dot{\rho}_{nm} = \frac{-i}{\hbar}[\hat{H}, \hat{\rho}]_{nm} - \frac{1}{T_2}(\rho_{nm} - \rho_{nm}^{eq})} \quad (2)$$

If we ignore the dephasing, the differential eq (1) can be solved

Quantum "Pictures"	Heisenberg	Interaction	Schrödinger
State Ket	No change	Evolution by $V(t)$	Evolution by $H(t)$
Observable	Evolution by H	Evolution by H_0	No ~~work~~ change

Some notes on Quantum Pictures

	Heisenberg picture	Schrödinger Picture
...t	Stationary	Moving
...s.	Moving	Stationary
...nse ...t	moving oppositely	Stationary

Moving State
$$\begin{cases} c(t) = \langle a' | \left(U | a, t=0 \rangle \right) & \text{SP} \\ c(t) = \left(\langle a' | U \right) | a, t=0 \rangle & \text{HP} \end{cases}$$

Schrödinger Picture

State ket has time dependence

$$|\alpha, t_0; t\rangle = U(t, t_0) |\alpha, t_0\rangle$$

$$U(t, t_0) = \exp\left(\frac{-i H (t - t_0)}{\hbar} \right)$$

Heisenberg Picture

Observables have time dependence

$$\boxed{\frac{d\Omega^{(H)}}{dt} = \frac{-i}{\hbar} [H, \Omega^{(H)}]}$$

or

$$\boxed{\frac{d\Omega^{(H)}}{dt} = \frac{1}{i\hbar} [\Omega^{(H)}, H]}$$

Relationship between Schrödinger + Heisenberg Pictures

$$\boxed{\Omega^{(H)} = U^{\dagger} \Omega^{(S)} U}$$

Time dependences of Ensemble Systems

Expectation values are a function of time

$$\begin{pmatrix} \sum_s C_a^s(t)\, C_a^{s*}(t) & \sum_s C_a^s(t)\, C_b^{s*}(t) \\ \sum_s C_b^s(t)\, C_a^s(t) & \sum_s C_b^s(t)\, C_b^{s*}(t) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_s |C_a^s(t)|^2 & \sum_s C_a(t)\, C_b^*(t) \\ \sum_s C_b^s(t)\, C_a^s(t) & \sum_s |C_b^s(t)|^2 \end{pmatrix}$$

On diagonal terms $\Rightarrow$ positive + Real

off diagonal terms $\Rightarrow$ negative or complex

Coherences decay due to dephasing : Rates

Cancellation of emitted light.

Ensemble average of atomic wavefunctions add to zero over time

Two timescales

$T_1 \Rightarrow$ relaxation time ρ_{aa} or ρ_{bb} (level lifetime)

$T_2 \Rightarrow$ dephasing time ρ_{ab} or ρ_{ba} (coherence life time)
collision rate

Typically $T_2 << T_1$

Time scales

Medium	T_1 (s)	T_2	σ (cm^2)
Solids doped with resonant atomic systems	$10^{-3} - 10^{-6}$	$10^{-11} - 10^{-14}$	$\sim 10^{-20}$
dye molecules	$10^{-8} - 10^{-12}$	$10^{-13} - 10^{-14}$	$\sim 10^{-16}$
semiconductors	$10^{-4} - 10^{-12}$	$10^{-11} - 10^{-14}$	

$T_1 \Rightarrow$ lifetime, longitudinal relaxation time

$T_2 \Rightarrow$ dephasing time, transverse relaxation time

Deriving the Polarization using perturbation theory $\Rightarrow \chi^{(n)}$

$$\langle \bar{P} \rangle = \langle \bar{P}^{(1)} \rangle + \langle \bar{P}^{(2)} \rangle + \langle \bar{P}^{(3)} \rangle \ldots$$

Where $\langle \bar{P}^{(n)} \rangle = T_r \langle \rho^{(n)} \bar{P} \rangle$

where

$$\begin{cases} V = e\vec{r} \cdot \vec{E} \qquad \vec{P} = -Ne\vec{r} \qquad N \equiv \dfrac{\#\text{ dipoles}}{\text{volume}} \\ E = E_1(\omega_1) + E_2(\omega_2) + \ldots\ldots \end{cases}$$

50

$$\chi_{ij}^{(1)} = \frac{P_i^{(1)}(\omega)}{\epsilon_0 E_j(\omega)}$$

$$\chi_{ijkl}^{(3)} = \frac{P_i^{(3)}(\omega)}{\epsilon_0 E_j(\omega_1) E_k(\omega_2) E_l(\omega_3)}$$

$$\chi_{ijk}^{(2)} = \frac{P_i^{(2)}(\omega)}{\epsilon_0 E_j(\omega_1) E_k(\omega_2)}$$

Lets write out the first ~~two~~ 2nd order solutions for $\rho_{nm}^{(n)}$

$$\rho^{(1)}_{nm}(\omega_j) = \frac{[V(\omega_j)]_{nm}}{\hbar(\omega_j - \omega_{nm} + i\, 1/T_2)} \left(\rho^{(0)}_{mm} - \rho^{(0)}_{nn}\right)$$

$$\rho^{(2)}_{nm}(\omega_j + \omega_k) = \frac{[V(\omega_j), \rho^{(1)}(\omega_k)]_{nm} + [V(\omega_k), \rho^{(1)}(\omega_j)]_{nm}}{\hbar(\omega_j + \omega_k - \omega_{nm} + i\, 1/T_2)}$$

vow We use

$$\langle \bar{P}^{(n)} \rangle = Tr\left(\rho^{(n)} \vec{P}\right)$$

find the polarization using the $\rho^{(n)}$ from the Dyson series

For $P^{(1)} \Rightarrow$ Two term $\boxed{(2^n n!)} \Rightarrow$ Terms from Dyson Series

$P^{(2)} \Rightarrow$ Eight terms

$P^{(3)} \Rightarrow$ 48 terms

ien to find $\chi^{(n)}$ use the above expressions

$$\chi^{(2)} = \frac{P^{(2)}}{E(\,)E(\,)}$$

$$\chi^{(3)} = -\frac{Ne^3}{\hbar}\left[\sum_{s,n,n'} \left(\begin{matrix}\text{terms from} \\ \text{perturbation} \\ \text{theory}\end{matrix}\right)\right]$$

<u>Expand terms from the Dyson series:</u> $\underline{\underline{\text{one}}}$ term

In interaction picture

$$V_I(t) = U^\dagger(t)(-\bar{\mu}\cdot\bar{E})U(t) \qquad U(t) = \exp(-i\, t H_0/\hbar)$$

So $\quad V_I(t_1)\,\rho_I(t_0)\,V_I(t_2)$

$$= U^\dagger(t_1)\left[-\mu\cdot\bar{E}(t_1)\right]U(t_1)\,U^\dagger(t_0)\,\rho_I^{(0)}(t_0)\,U(t_0)$$

$\rho_I(t) = U^\dagger(t)\rho(t)U(t)$

$$U^\dagger(t_2)\left(-\mu\cdot E(t_2)\right)U(t_2)$$

So $\Rightarrow$

$$= \underbrace{\left[U(t-t_1)\left[-\mu\cdot E(t_1)\right]U(t_1-t_0)\right]}_{\text{left}}\rho^0(t_0)\underbrace{\left[U(t_0-t_2)\left(-\bar{\mu}\cdot E(t_2)\right)U(t_2-t)\right]}_{\text{Right}}$$

<u>Left side</u> (~~Ket~~ Bra evolution)

propagation from

$$t_0 \rightarrow t_1 \longrightarrow t$$

<u>Right side</u> (~~B~~ ket evolution)

propagation from

$$t_0 \rightarrow t_2 \rightarrow t$$

This is just $\underline{\underline{\text{one}}}$ term, there will be many more.

We need two sided diagram to handle both the ket + bra evolution

Double sided Feynman Diagrams (Yee + Gustafson) PRA 18 4 1979

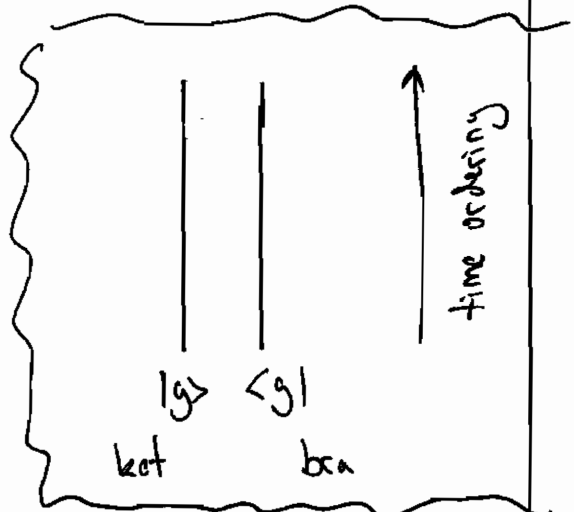
Used to keep track of terms in perturbation calculations
The density matrix involves products of two wave functions
so two diagrams are needed.

Diagrams give a simple picture of the corresponding physical process, allowing one two write down the corresponding mathematical expression.

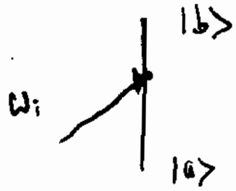
1) Start at $|g\rangle \rho_{gg}^{(0)} \langle g|$

2) Draw ket left, bra right

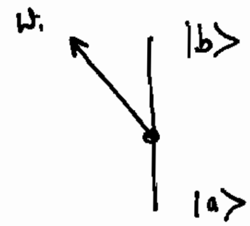
3. Consider a vertex from $|a\rangle \to |b\rangle$



ON Left (ket)



absorption



emission

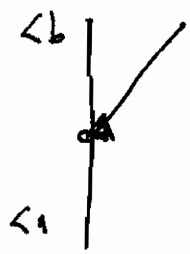
matrix elements

$\frac{1}{i\hbar}\langle b | \mathcal{H}_I^{+}(\omega_i) | a\rangle$ emission

$\frac{1}{i\hbar}\langle b | \mathcal{H}_I(\omega_i) | a\rangle$ absorption

ON RIGHT (bra) adjoint between bra + ket

emission



Absorption

$-\frac{1}{i\hbar}\langle b | \mathcal{H}_I^{+} | b\rangle$ emission $\quad -\frac{1}{i\hbar}\langle b | \mathcal{H} | b\rangle$ abs.

4. Propagation from j^{th} vertex to $(j+1)$ along $|\ell\rangle\langle k|$

$$\Pi_j = \pm \left[\sum_{m=1}^{j} \left(\pm \omega_m - \omega_{\ell k} + i \overbrace{\gamma_{\ell k}}^{1/(T_2)_{\ell k}} \right) \right]^{-1} \frac{1}{\hbar}$$

+ ket side
− bra side

Sign of term ω_m

	left (ket)	Right (bra)
Abs.	+	−
emission	−	+

Time ordering

n interactions
k on left (ket)
$n-k$ on right (bra)

possible $\dfrac{n!}{k!\,(n-k)!}$

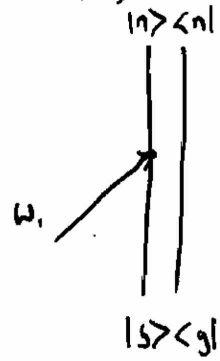
$$\chi^{(n)} = -\frac{N\rho^2}{\;}\left(\sum_{\text{terms}} - \cdots - \right)$$

Example : Linear optics absorption + emission

one photon, no virtual processes

$|g\rangle$ to $|n\rangle$

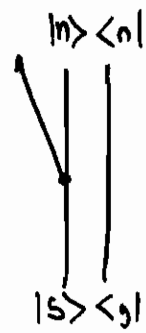
absorption



$$+\frac{\rho_{gg}\,\mu^{(1)}_{gn}\,\mu^{(1)}_{ng}}{\hbar\left(-\omega_{ng}+\omega_1+i\,{}^{1}\!/(T_2)_{ng}\right)}$$

complex Lorenzian

emission —



$$\frac{1}{\hbar}\left(\omega-\omega_{gn}+i\,{}^{1}\!/T_2\right)^{-1}$$

$$=\frac{1}{\hbar}\left(\omega+\omega_{ng}+i\,{}^{1}\!/T_2\right)^{-1}$$

$$-\frac{\rho_{nn}\,\mu^{(1)}_{ng}\,\mu^{(1)}_{gn}}{\hbar\left(\omega+\omega_{ng}+i\,{}^{1}\!/T\right)}$$

use

$\rho_{nn}=1-\rho_{gg}$

So

$$\chi^{(1)}_{ij}=\rho^{(0)}_{gg}\frac{N}{\hbar}\left[\sum_{gn}\frac{\mu^{(i)}_{ng}\,\mu^{(j)}_{gn}}{\left(\omega+\omega_{ng}+i\,{}^{1}\!/(T_2)_{ng}\right)}+\frac{\mu^{(j)}_{ng}\,\mu^{(i)}_{gn}}{\omega-\omega_{ng}+i\,{}^{1}\!/(T_2)_{ng}}\right]$$

To get classical result define ~~opera~~ oscillation strength + keep resonant term

$$f_{ng} \equiv \frac{2 m \omega_{ng} \, |\mu_{ng}|^2}{3 \hbar e^2} \qquad \rho_{gg}^{(0)} \simeq 1$$

$$\boxed{\chi_{ij}^{(1)} \cong \sum_{ng} f_{ng} \frac{Ne^2/m}{\left(\omega_{ng}^2 - \omega^2 - 2i\,\omega_{ng} \, ^1/_{(T_2)_{ng}}\right)}}$$

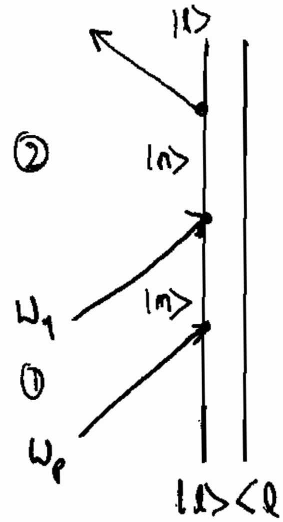
Boyd 3.5.25

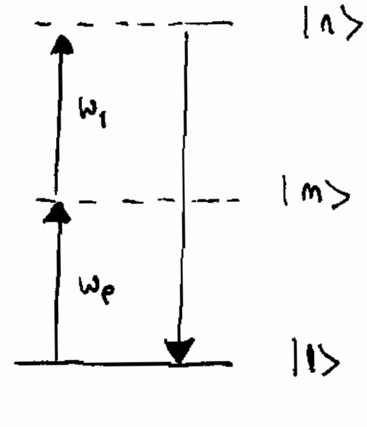
QM more than one resonant frequency

ω_{ng}

Strength $\Rightarrow$ oscillator

Second Harmonic generation (Follow Boyd pg 176) 8 Terms





(a_1)

Term

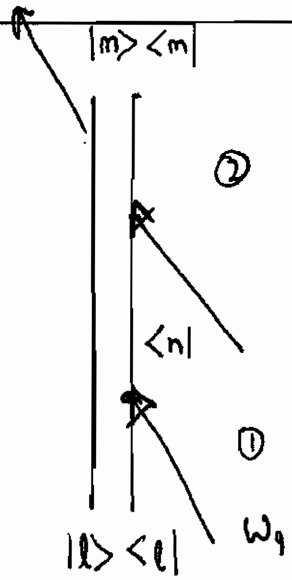
$$\frac{\mu_{ln}^{i}\,\mu_{nm}^{j}\,\mu_{ml}^{k}}{\left[(\omega_{nl}-\omega_p-\omega_q)-i\,\gamma_{nl}\right]\left[(\omega_{ml}-\omega_p)-i\,\gamma_{ml}\right]}$$

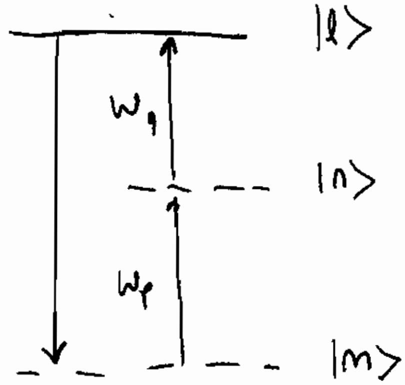
① $$\frac{\mu_{ml}}{\left[(\omega_{ml}-\omega_p)-i\,\gamma_{ml}\right]}$$

② $$\frac{\mu_{nm}}{\left[(\omega_{nl}-\omega_p-\omega_q)-i\,\gamma_{nl}\right]}$$

operate on ~~bra~~ ket side!

$$\rho_{ll}^{(0)} \to \rho_{ml}^{(1)} \to \rho_{nl}^{(2)}$$





Term

$$\frac{\mu^j_{ln}\,\mu^k_{nm}\,\mu^i_{ml}}{\underbrace{\left[(\omega_{ml}+\omega_p+\omega_q)+i\gamma_{ml}\right]}_{2}\;\underbrace{\left[\omega_{nl}+\omega_q+i\gamma_{nl}\right]}_{1}}$$

$$\rho^{(0)}_{ll} \rightarrow \rho^{(1)}_{ln} \rightarrow \rho^{(2)}_{lm}$$ } operate on ~~[illegible]~~ bra side

$$U(t-t_1)[-\mu\cdot E]\,U(t_1-t_0)\quad \rho^{0}(t_0)\quad U(t_0-t_2)\,[-\mu\cdot E(t_2)]\,U(t_2-t)$$

From Shen "The principles of nonlinear optics"

$)_k$. The calculation can be $+ \cdots$), which will have 48 literature[5] and is not however, is discussed in stants in the denominators ability can then be reduced rms in the expression for

$$\frac{\;_{nn'}(r_j)_{n'g}}{(\omega_2 - \omega_{ng})} .$$

ules per unit volume, the e for gases or molecular an distribution. For solids ucture, the eigenstates are stribution. The expression Since the band states form e resonant denominators with the photon wavevec- e form[3]

$$\frac{c', \mathbf{q}|r_k|v, \mathbf{q}\rangle}{\omega_{c'v}(\mathbf{q})]}$$
$$\frac{\mathbf{q}\rangle\langle c', \mathbf{q}|r_j|v, \mathbf{q}\rangle}{\;_v(\mathbf{q})]}$$
$$\frac{\mathbf{q}\rangle\langle c', \mathbf{q}|r_i|v, \mathbf{q}\rangle}{+ \omega_{cv}(\mathbf{q})]} \qquad (2.18)$$
$$\frac{\mathbf{q}\rangle\langle c', \mathbf{q}|r_i|v, \mathbf{q}\rangle}{+ \omega_{cv}(\mathbf{q})]}$$
$$\frac{\mathbf{q}\rangle\langle c', \mathbf{q}|r_k|v, \mathbf{q}\rangle}{+ \omega_{c'v}(\mathbf{q})]}$$
$$\left.\frac{\mathbf{q}\rangle\langle c', \mathbf{q}|r_j|v, \mathbf{q}\rangle}{- \omega_{cv}(\mathbf{q})]}\right\} f_v(\mathbf{q})$$

are the band indices, and $\rangle$. arising from the induced factor $L^{(n)}$ should then

appear as a multiplication factor in $\chi^{(n)}$. We discuss the local field correction in more detail in Section 2.4. For Bloch (band-state) electrons in solids with wavefunctions extended over many unit cells, the local field tends to get averaged out, and $L^{(n)}$ may approach 1.

2.3 DIAGRAMMATIC TECHNIQUE

Perturbation calculations can be facilitated with the help of diagrams. Feynman diagrams have been used in perturbation calculations on wavefunctions. Here, since the density matrices involve products of two wavefunctions, perturbation calculations require a kind of double-Feynman diagram. We introduce in this section a technique devised by Yee and Gustafson.[6] Only the steady-state response is considered here.

The important aspects of any diagrammatic technique are that the diagrams provide a simple picture to the corresponding physical process as well as allowing one to write down immediately the corresponding mathematical expression. It is essential to find the complete set of diagrams for a perturbation process of a given order. The scheme we adopt for calculating $\rho^{(n)}$ involves in each diagram a pair of Feynman diagrams with two lines of propagation, one for the $|\psi\rangle$ side of ρ and the other for the $\langle\psi|$ side. Figure 2.1 shows one of

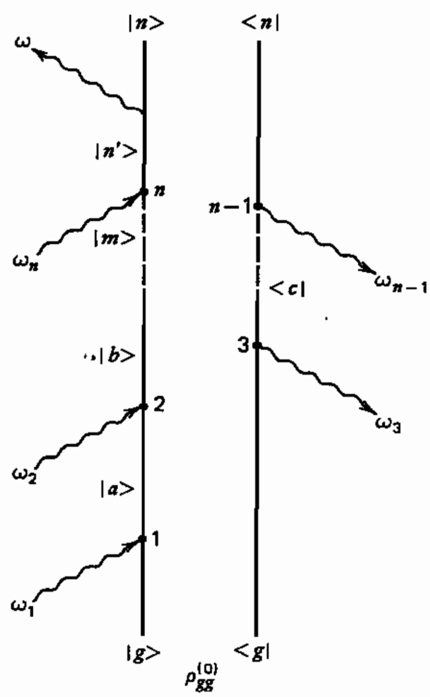


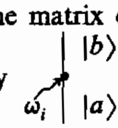
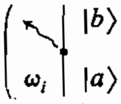
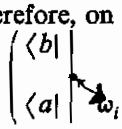
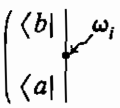
Fig. 2.1 A representative double-Feynman diagram describing one of the many terms in $\rho^{(n)}(\omega = \omega_1 + \omega_2 + \cdots + \omega_n)$.

the many diagrams describing the various terms in $\rho^{(n)}(\omega = \omega_1 + \omega_2 + \cdots + \omega_n)$. The system starts initially from $|g\rangle\langle g|$ with a population $\rho_{gg}^{(0)}$. The ket state propagates from $|g\rangle$ to $|n'\rangle$ through interaction with the radiation field at $\omega_1, \omega_2, \ldots, \omega_n$, and the bra state propagates from $\langle g|$ to $\langle n|$ through interaction with the field at $\omega_3, \ldots, \omega_{n-1}$. Then, the final interaction with the output field at ω puts the system in $|n\rangle\langle n|$. Through permutation of the interaction vertices and rearrangement of the positions of the vertices on the lines of propagation, the other diagrams for $\rho^{(n)}$ can also be drawn.

The microscopic expression for a given diagram can now be obtained using the following general rules describing the various multiplication factors:

1 The system starts with $|g\rangle\rho_{gg}^{(0)}\langle g|$.

2 The propagation of the ket state appears as multiplication factors on the left, and that of the bra state on the right.

3 A vertex bringing $|a\rangle$ to $|b\rangle$ through absorption at ω_i on the left (ket) side of the diagram is described by the matrix element $(1/i\hbar)\langle b|\mathscr{H}_{\text{int}}(\omega_i)|a\rangle$ with $\mathscr{H}_{\text{int}}(\omega_i) \propto e^{-i\omega_i t}$ (denoted by  in Fig. 2.1). If it is emission () instead of absorption, the vertex should be described by $(1/i\hbar)\langle b|\,\mathscr{H}_{\text{int}}^{\dagger}\,(\omega_i)|a\rangle$. Because of the adjoint nature between the bra and ket sides, an absorption process on the ket side appears as an emission process on the bra side, and vice versa.* Therefore, on the right (bra) side of the diagram, the vertices for emission () and absorption () are described by $-(1/i\hbar)\langle a|\mathscr{H}_{\text{int}}(\omega_i)|b\rangle$ and $-(1/i\hbar)\langle a|\,\mathscr{H}_{\text{int}}^{\dagger}\,(\omega_i)|b\rangle$, respectively.

4 Propagation from the jth vertex to the $(j+1)$th vertex along the $|l\rangle\langle k|$ double lines is described by the propagator $\prod_j = \pm[i(\Sigma_{i=1}^{j}\omega_i - \omega_{lk} + i\Gamma_{lk})]^{-1}$ The frequency ω_i is taken as positive if absorption of ω_i at the ith vertex occurs on the left or emission of ω_i on the right; it is taken as negative if absorption of ω_i occurs on the right or emission on the left.

5 The final state of the system is described by the product of the final ket and bra states, for example, $|n'\rangle\langle n|$ after the nth vertex in Fig. 2.1 for $\rho^{(n)}$.

6 The product of all factors describes the propagation from $|g\rangle\langle g|$ to $|n'\rangle\langle n|$ through a particular set of states in the diagram. Summation of these

*If the field is also quantized, $\mathscr{H}_{\text{int}}(\omega_i)$ operating on a ket state will annihilate a photon at ω_i, while if operating on a bra state it will create a photon.

$\rho^{(n)}(\omega = \omega_1 + \omega_2 + \cdots +$
population $\rho^{(0)}_{gg}$. The ket state
the radiation field at
$|$ to $\langle n|$ through interaction
tion with the output field at
of the interaction vertices
n the lines of propagation,

can now be obtained using
ultiplication factors:

ltiplication factors on the

at ω_i on the left (ket) side
ent $(1/i\hbar)\langle b|\mathscr{H}_{\text{int}}(\omega_i)|a\rangle$
Fig. 2.1). If it is emission

should be described by

ture between the bra and
appears as an emission
re, on the right (bra) side
) and absorption

$(\omega_i)|b\rangle$ and $-(1/i\hbar)\langle a|$

vertex along the $|l\rangle\langle k|$
$l_j = \pm[i(\sum_{i=1}^{j}\omega_i - \omega_{lk} +$
sorption of ω_i at the ith
he right; it is taken as
emission on the left.
duct of the final ket and
in Fig. 2.1 for $\rho^{(n)}$.
n from $|g\rangle\langle g|$ to $|n'\rangle\langle n|$
n. Summation of these

nihilate a photon at ω_i, while

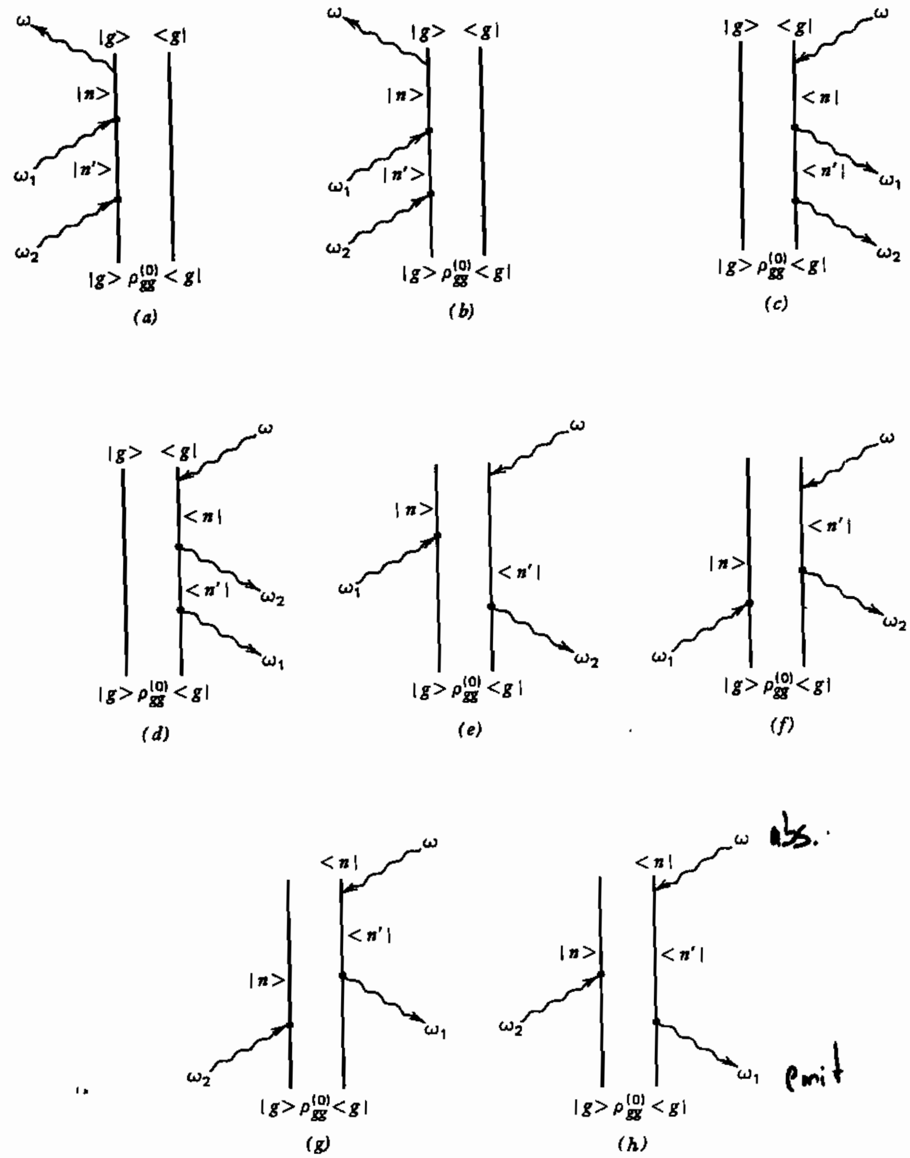


Fig. 2.2 The complete set of eight diagrams for the eight terms in $\rho^{(2)}(\omega = \omega_1 + \omega_2)$.

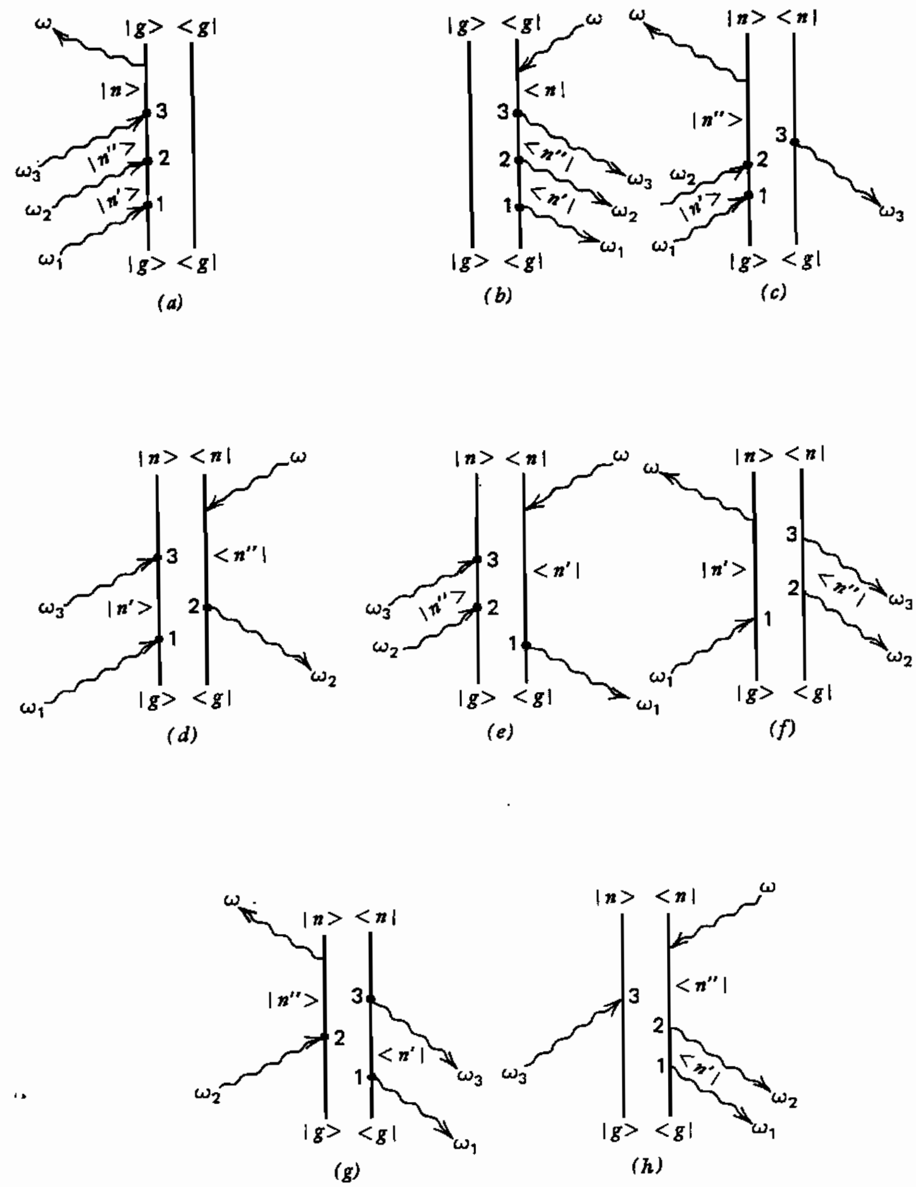


Fig. 2.3 The eight basic diagrams for $\rho^{(3)}(\omega = \omega_1 + \omega_2 + \omega_3)$.

Feynman diagrams for continuous wave case

Used to keep track of terms in perturbation ~~solutions~~ calculations. The density matrix involves products of two wavefunctions so Two diagrams are needed.

All diagrams give a simple picture of the corresponding physical process, allowing one to write down the corresponding mathematical expression.

1) Start system at $|g\rangle \rho_{gg}^{(0)} \langle g|$

2) Draw ket on left, bra at right

3. A vertex bringing $|a\rangle$ to $|b\rangle$

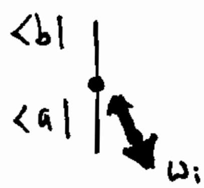
On left

absorption

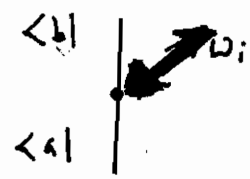
emission

$\Longrightarrow$ matrix elements $(1/i\hbar)\langle b|V|a\rangle$

On Right



emission



absorption

$\Longrightarrow$ matrix elements $-(1/i\hbar)\langle a|V|b\rangle$

4. Propagation from j^{th} vertex to $(j+1)$ along $|l\rangle\langle k|$ described by

$$\frac{1}{\hbar}\,\pm\left[\left(\sum_{i=1}^{j}\omega_i - \omega_{\ell k} + i\,{}^{1}\!/\!_{T_2}\right)\right]^{-1}$$

$+$ ket side absorption

$-$ bra side absorption

5.

Sign of term	left	right
	+ if absorption	− if absorption
	− if emission	+ if emission

(OR)

	left	Right
Abs.	+	−
Emission	−	+

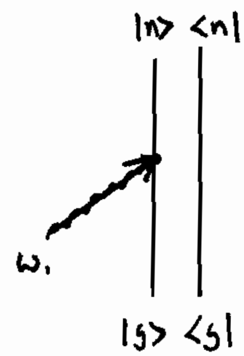
Term Sign $\pm[\quad]^{-1}$

∴ Final State of system given by final ket + bra

∴ Product of all factors yields the susceptibility

Example : Linear optics — Absorption

Involves only one photon
No virtual processes
$|g\rangle$ to $|n\rangle$



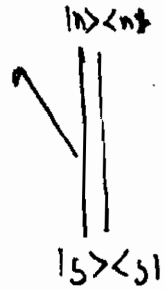
Absorption

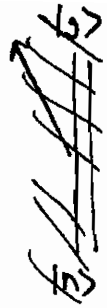
So $$\boxed{+\frac{\rho_{gg}\,\mu_{gn}^{(1)}\,\mu_{ng}^{(1)}}{\hbar\left(-\omega_{ng}+\omega_1+i\,1/(T_2)_{ng}\right)}}$$

To get classical result define oscillator strength

$$f_{ng} \equiv \frac{2m\omega_{ng}|\mu_{ng}|^2}{3\hbar e^2}$$

This is a complex Lorentzian function like we derived using our classical harmonic oscillator model.





$$\frac{1}{\hbar}\left(\omega - \omega_{gn} + i\, ^1/_{T_2}\right)^{-1}$$

$$\hbar\left(\omega + \omega_{ng} + i\, ^1/_{T_2}\right)^{-1}$$

Term $\Rightarrow \; - \dfrac{\rho_{nn}\, \mu_{ng}^{(i)}\, \mu_{gn}^{(i)}}{\hbar\left(\omega + \omega_{ng} + i\, ^1/_{T_2}\right)}$ But $\rho_{nn} = -\rho_{gg} + 1$

So

$$\chi_{ij}^{(1)} = \rho_{gg}^{(0)} \frac{N}{\hbar}\left[\sum_{gn} \frac{\mu_{ng}^{(i)}\, \mu_{gn}^{(j)}}{\left(\omega + \omega_{ng} + i\, ^1/_{T_{2ng}}\right)} + \frac{\mu_{ng}^{(j)}\, \mu_{gn}^{(i)}}{\left(\omega - \omega_{ng} + \, ^1/_{T_{2ng}}\right)}\right]$$

To get classical result, define the oscillator strength

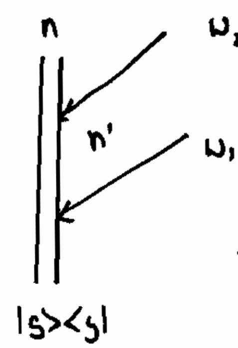
$$f_{ng} = \frac{2m\omega_{ng}\, |\mu_{ng}|^2}{3\hbar e^2} \qquad \rho_{gg}^{(0)} = 1$$

Dropping non resonant term

So

$$\chi_{ij}^{(1)} \simeq f_{na} \frac{Ne^2/m}{\left(\omega_{ng}^2 - \omega^2 - 2i\omega\, ^1/_{T_2}\right)}$$ ← Lorentzian

Absorption on bra side



$$-\left(\omega_2+\omega_1-\omega_{n'g}+1/T_2\right)^{-1}$$

$$-\left(\omega_1-\omega_{ng}+1/T_2\right)^{-1}$$

Emission on ket side

$$\left(\omega_1-\omega_{nn'}+1/T_2\right)^{-1}$$
$$=\left(\omega_1+\omega_{n'n}+1/T_2\right)^{-1}$$

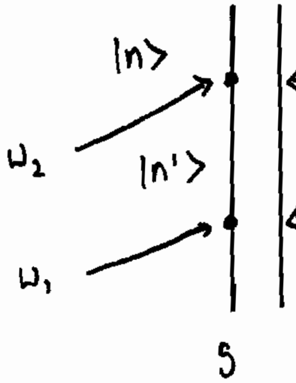
Emission on bra side

$$-\left(\omega_1-\omega_{n'n}+1/T_{2\,n'n}\right)^{-1}$$

$$=-\left(\omega_1+\omega_{nn'}+1/T_{2\,n'n}\right)^{-1}$$

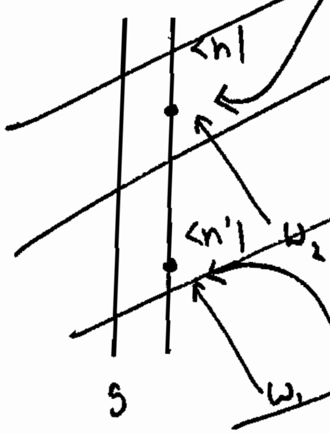
More Examples

Absorption



$(\omega_1 - \omega_{n'g} + i\ 1/T_{ng})^{-1}$

$(\omega_2 + \omega_1 - \omega_{nn'} + i\ 1/T_{ng})^{-1}$



$-(\omega_2 + \omega_1 - \omega_{n'n} + i\ 1/T_{nn'})^{-1}$

$-(+\omega_1 - \omega_{n'g} - i\ 1/T_{ng})^{-1}$ — here because on bra side due to complex conjugate

$= -(+\omega_1 + \omega_{gn'} - i\ 1/T_{ng})^{-1}$

$= +((\omega_1 + \omega_{gn'} - i\ (1/T_{ng})))^{-1}$

So

$$\boxed{\chi_{ij}^{(1)} = +\frac{Ne^2}{\hbar}\left(\frac{\rho_{gg}\,\mu_{gn}^{(i)}\,\mu_{ng}^{(j)}}{-\omega_{ng}+\omega_1+i\,(1/T_2)_{ng}}\right)}$$

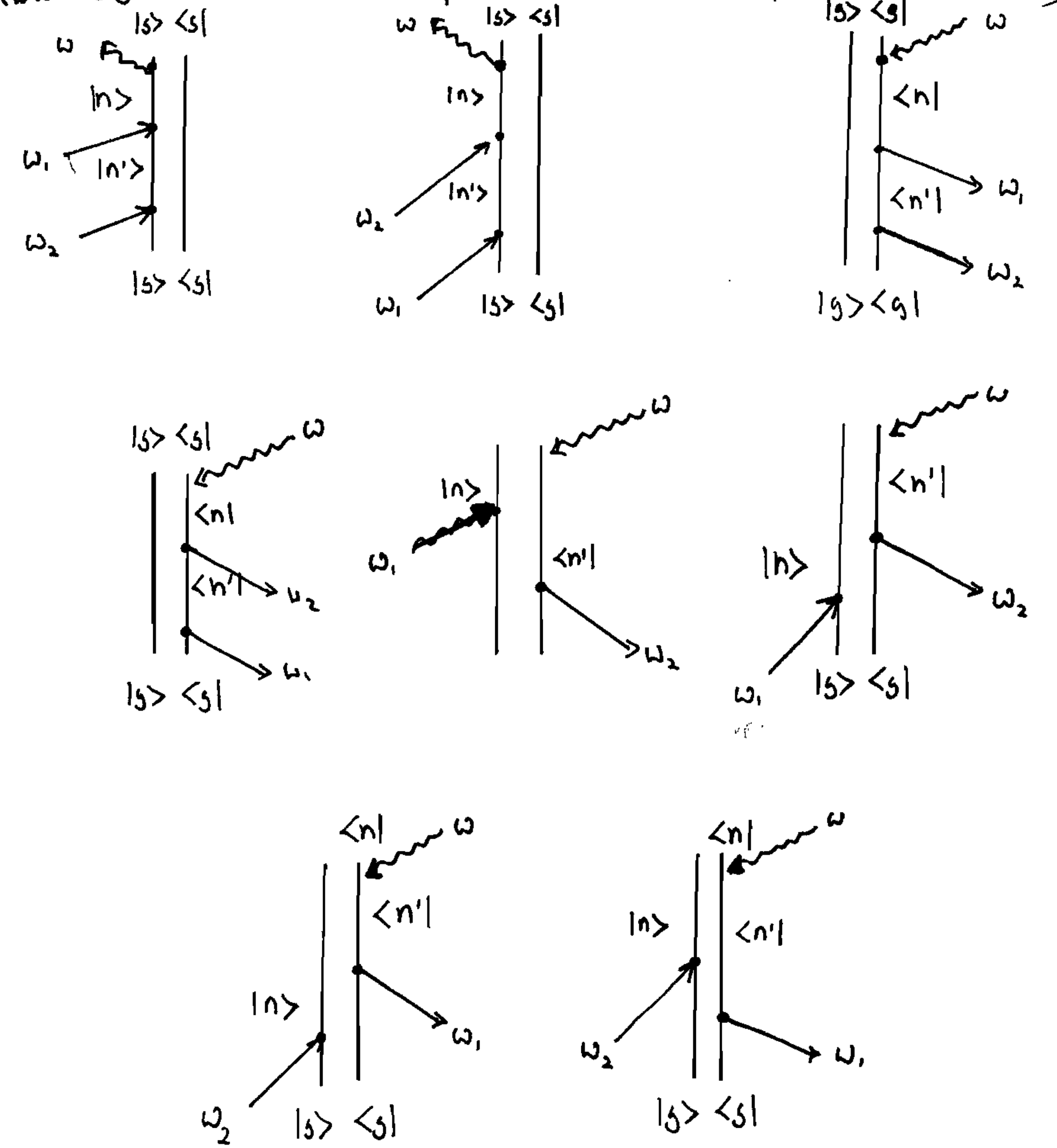
Relate to classical result by defining oscillator strength

$$f_{ng} = \frac{2m\omega_{ng}|\mu_{ng}|^2}{3\hbar e^2}$$

$\rho_{gg} \approx 1$

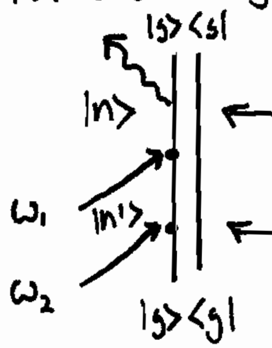
2nd order susceptibility

Draw Eight Diagrams for $\rho^{(2)}(\omega=\omega_1+\omega_2)$ (Three photon process)



Write down terms for $\chi^{(2)}$

We have a total of eight terms



$\leftarrow (\omega_1 + \omega_2 - \omega_{ng} + i\, 1/T_{ng})^{-1}$

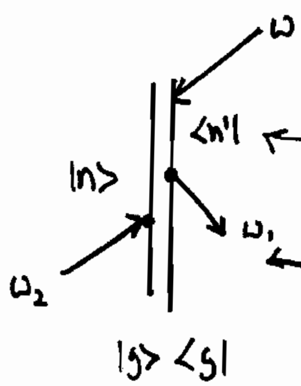
$\leftarrow (\omega_2 - \omega_{n'g} + i\, 1/T_{n'g})^{-1}$

(ω_1 + ω_2 are + since on right)

The term corresponding to this figure is

$$+ \frac{\mu^{(2)}_{sn}\, \mu^{(1)}_{nn'}\, \mu^{(2)}_{n'g}\, \rho^{(0)}_{ss}}{\hbar^2 (\omega_1 + \omega_2 - \omega_{ng} + i\, 1/T_{ng})(\omega_2 - \omega_{ng} + i\, 1/T_{n'g})}$$

Another figure



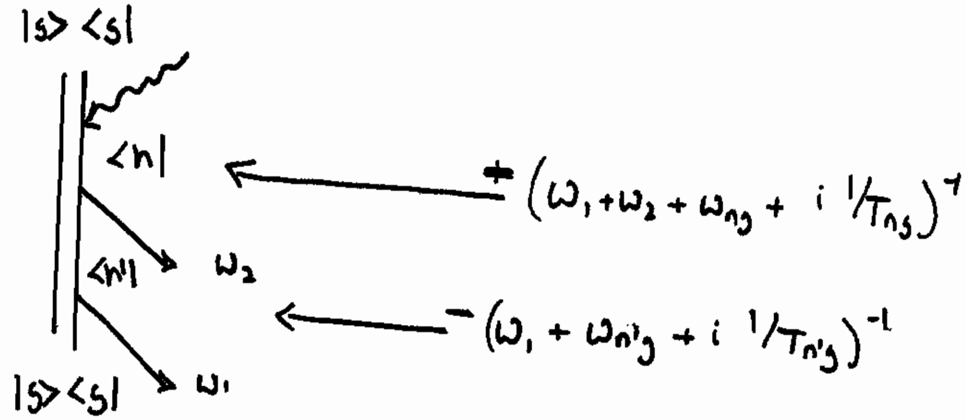
$\leftarrow -(\omega_1 + \omega_2 - \omega_{nn'} + i\, 1/T_{nn'})^{2}$

$\leftarrow (\omega_2 - \omega_{ng} + i\, 1/T_{ng})^{-1}$

$$- \frac{\mu^{(2)}_{ng}\, \mu^{(1)}_{n'n}\, \mu^{(0)}_{gn'}\, \rho^{(0)}_{ss}}{\hbar^2 (\omega_1 + \omega_2 - \omega_{nn'} + i\, 1/T_{nn'})(\omega_2 - \omega_{ng} + i\, 1/T_{ng})}$$

($\mu^{(2)}_{n'n} \Rightarrow$ 2 ⇒ field 2, n' ⇒ final state, n ⇒ initial state)

Another figure



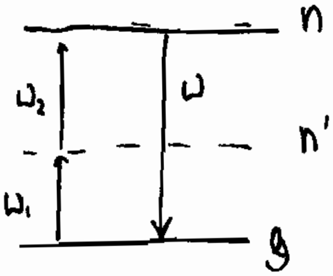
$$+\frac{\mu^{(1)}_{sn'}\,\mu^{(2)}_{n'n}\,\mu^{(0)}_{ng}\,\rho^{(0)}_{ss}}{\hbar^2\left(\omega_1+\omega_2+\omega_{ng}+i\,{}^{1}\!/T_{ng}\right)\left(\omega_1+\omega_{n'g}+i\,{}^{1}\!/T_{n'g}\right)}$$

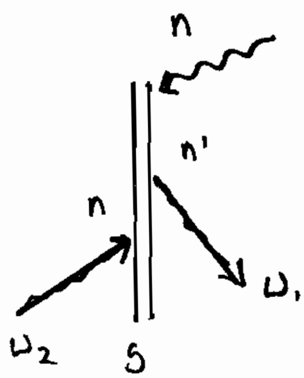
What are the other states $|n\rangle$ + $|n'\rangle$?

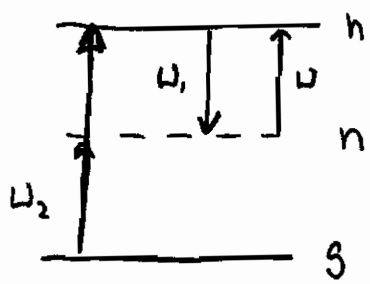
These are virtual transitions

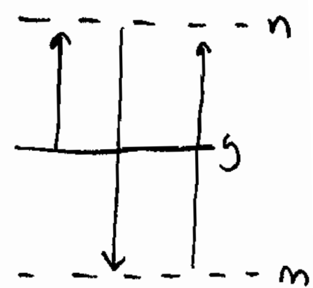
Transitions that occur + do not conserve energy.

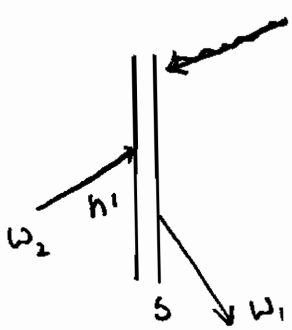
Physical Interpretation

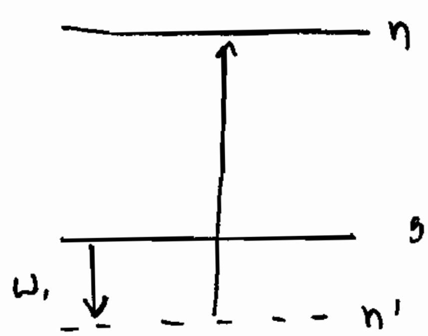












Intro to Quantum Optics: What is a photon?

Introduction into the quantum nature of light ⇒ Feynman nature of Science

Issues with "Wave-particle duality" ~~(Bad Term)~~

Nature of language / Need to be careful

" light exhibits wave + particle properties "

Two mutually inconsistant descriptions : particle + wave

Concepts from Newton + Huygens

Newton – corpuscular (<u>not</u> photons)
Huygens – 'waves' (different from modern)

(However, some will argue that Newton understood the dual nature of light (used particle + wave descriptions in Opticks))

Why is 'wave-particle duality' a potentially bad term?

Can be misrepresented/ misinterpreted as light 'acting' as a particle or wave depending on experiment.

light doesn't 'act', light just <u>is</u>!

Simple wave-particle description is flawed as we will see thru the next lectures

Issues with duality

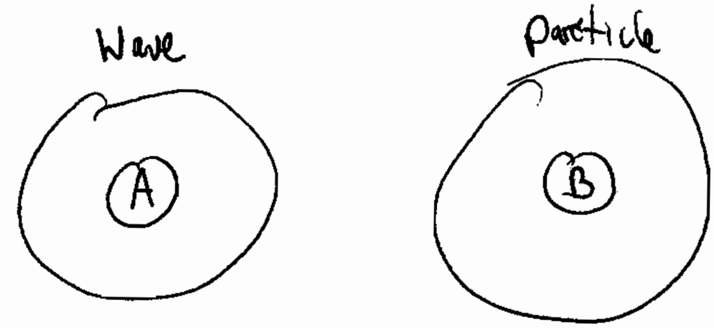
Particle / wave mutually exclusive concepts

localized — everywhere

Problem with duality

Perform Exp. A see particle nature

Venn Diagrams



Perform Exp. B see wave nature

Light ~~[illegible]~~ exhibits very different properties

Does it 'act' in a particular way?

NO!

Light doesn't 'act', it 'is' . . . .

Light isn't a particle or a wave, its light.

Experiments 'experience' either property but not both

⟹ Spork analogy

Spork?

Spork = spoon + fork

Spork is not a spoon, its not a fork

However, when we <u>use</u> a spork, we experience its 'spoon-like' nature <u>or</u> its 'fork-like' nature, but <u>never</u> both at the same time.

Problems with
Understanding Duality — Western vs. Eastern Philosophy

Subjects + Objects → Aristotle (Western)

One thing having two mutually exclusive properties → (Dao) Lao Tzu / Laozi (Eastern)

Aristotle → origins of scientific method.
Bacon, etc. ...

A better term... Principle of Complementarity

A experiment can reveal the particle nature of light or the wave nature of light, but never both

Quantum mechanics allows one to ~~measure~~ know one observable but not know a second, non commuting observable.

More about this later....

I cannot tell you about the quantum nature of light, must come to understand this on your own....

Thus, we will look closely at seminal experiments that tried to elucidate the quantum nature of light, starting with the photo electric effect.

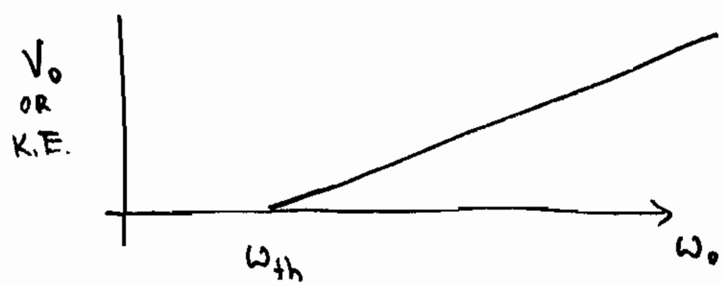
With these experiments in mind, we will develope a quantum theory of light.

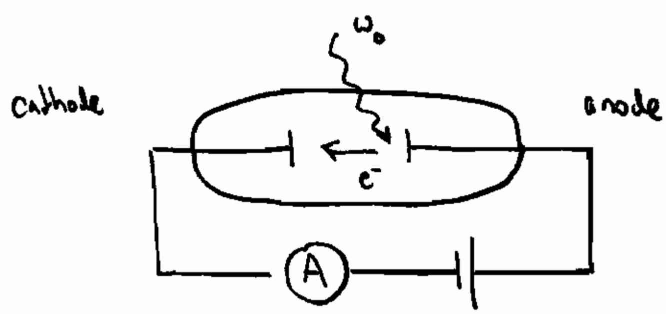
Lecture 29 What is a Photon? Part 1 : Photoelectric effect

Experiments of Lenard (1902) of light striking a metal

Observations

- Electrons are emitted a times very shortly after the onset of illumination ($\sim 10^{-9}$ s)
- Photocurrent rises linearly with light intensity
- Current to cathode decreases with increasing retarding potential, zero at stopping voltage V_0
- Stopping potential V_0 is linearly proportional to ω_0 and shows a threshold frequency ω_{th}





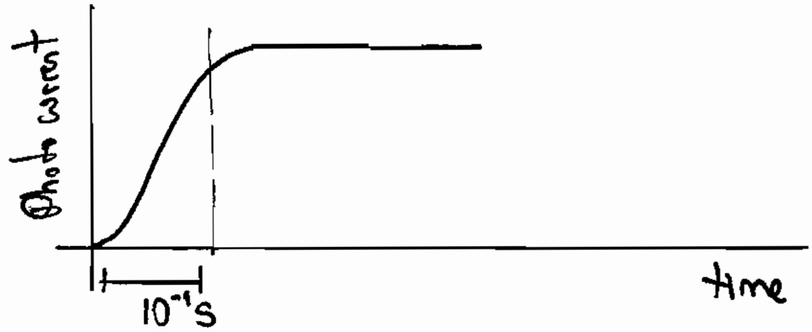
$V_0 \Rightarrow$ applied stopping voltage

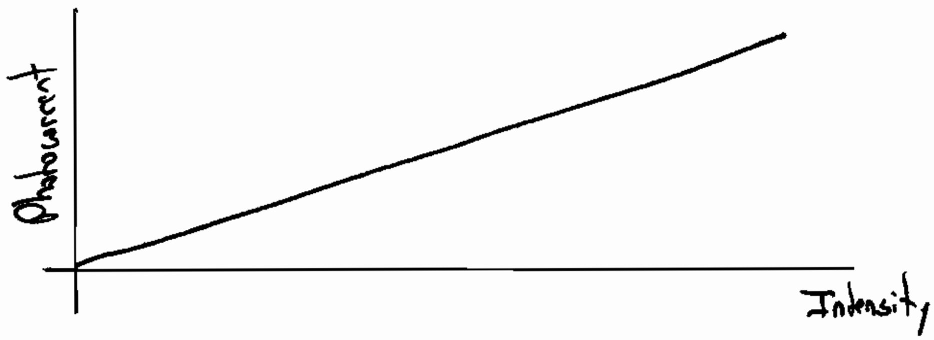
It was thought that.....

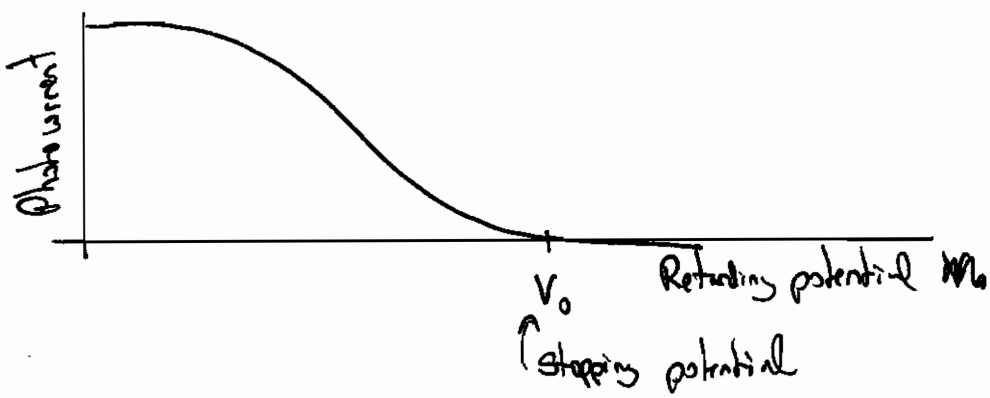
Classical E/m does not explain observed behavior

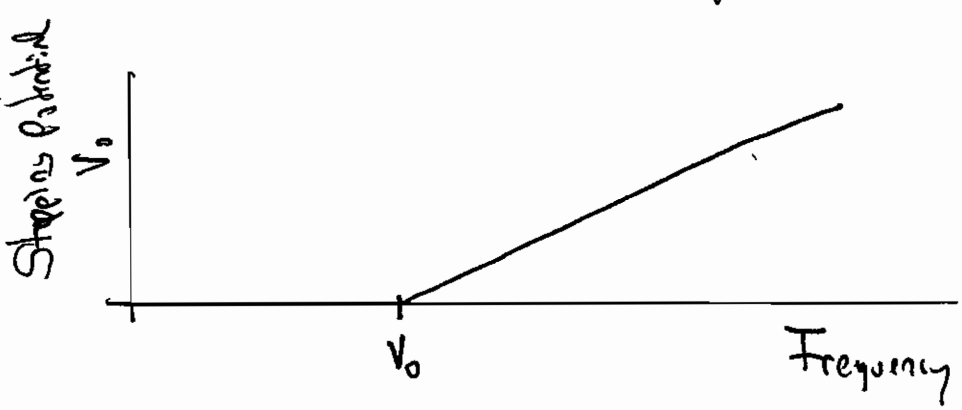
(or does it!!) (Son Tsu Hesel)

Results from Lenard









$$h\nu = KE + \Phi$$

$$KE = V_0 e$$

Einstein's Description (1906)

- Started with thermodynamical considerations
- Considered Blackbody radiation

"monochromatic radiation at low density behaves with respect to theory of heat as if it consisted of independent energy quanta of magnitude $h\nu$."

"if this is the case it is natural to investigate whether the laws of generation & transformation of light are such a kind as if light would consist of such energy quanta."

$$\hbar\omega_0 = KE + \phi_0$$

$$h\nu = \frac{1}{2}mv^2 + \phi_0$$

work function

The energy of a material oscillator with eigenvenfrequency ω_0 interacting with the radiation field can only take discrete values of $n\hbar\omega_0$

As einstein said

"our concept and the properties of the light electric field observed by Mr. Lenard, as far as I can see, are not in contradiction."

Einstein comment on nonlinear optics

"the # of energy quanta per unit volume being simultaneously converted is so large than an energy quantum of light generated can obtain its energy from several generating quanta.

1921 Nobel Prize "Services to Theoretical Physics, especially for his discovery of the law of photoelectric effect"

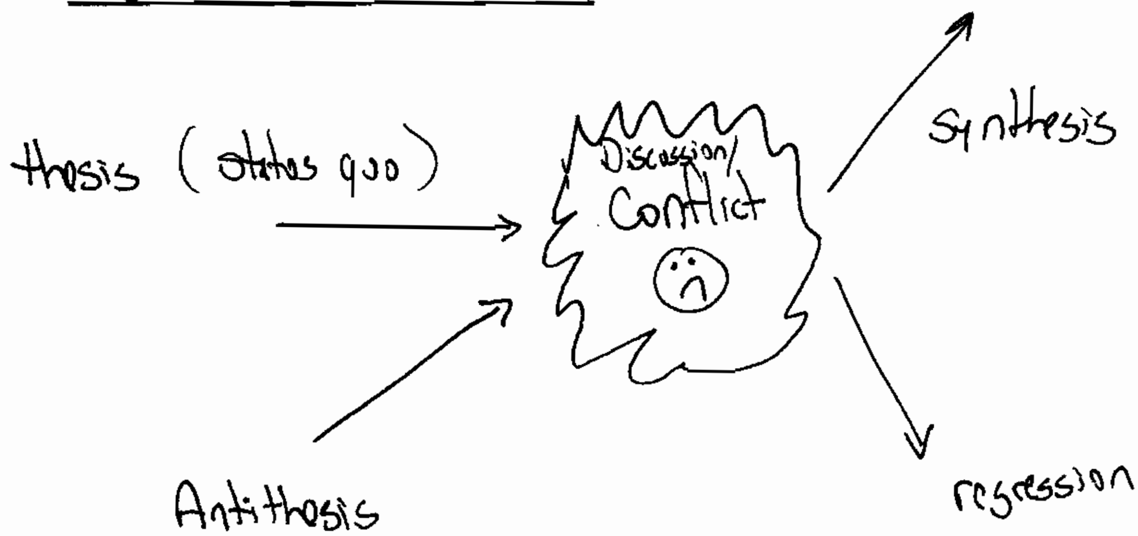
Photons (by name) 1926

<u>Gilbert Lewis</u>

" hypothetical new atom ... photon "

" hypothetical new entities as a particle of light ....
spends a minute fraction of its existance as a
carrier of radiant energy, the rest of the time as
an important structural element of the atom

Hegelian Dialectic



Discussion of Hanbury Brown + Twiss

Hanbury Brown + Twiss (Paul)

intensity Fluctuation : photon statistics

measuring diameters of fixed stars

stellar interometer

Shifting mirrors

$d\alpha_0 \simeq \lambda$ ($d\alpha_0 = 1.22\lambda$)

star's angular diameter

distance between mirrors (HBT eq 3)

Measure 0.02 arc seconds

HB+T $\Rightarrow$ Measured correlations of intensities at different positions

phase no longer appeared in measurement

photocurrents

Spatial coherence of starlight

$\Rightarrow$ transverse coherence length

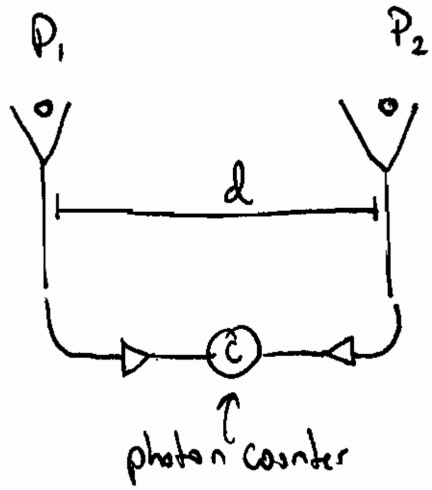
$\Delta I_1(t) \simeq \Delta I_2(t)$ $\quad d < l_{trans}$ $\quad l \equiv$ coherence length

$\overline{I_1(t)\, I_2(t)} = \overline{I}^2 + \overline{\Delta I_1^2}$

For $d > l_{trans}$ $\quad \overline{\Delta I_1(t)\, \Delta I_2(t)} \to 0$

so $\overline{I_1(t)\, I_2(t)} = \overline{I}^2$

$\Delta I(t) = I(t) - \overline{I}$



Experimental Idea

First attempted with radio waves

Then at optical frequencies.

<u>Intensity correlations</u> $\quad I_1 \equiv$ intensity at point 1, I_2 intensity at point 2

$$\Delta I_1(t) \simeq \Delta I_2(t) \qquad d < l_c$$

where d distance between ~~mirrors~~ ~~(sources)~~ detectors / point 1 (P_1) + point 2 (P_2)

l_c coherence length

$\bar{I}$ time averaged value of intensity

$$\Delta I(t) \equiv I(t) - \bar{I}$$

Correlator looks at time average product of $I_1(t) I_2(t)$

$$\overline{I_1(t) I_2(t)} = \bar{I}^2 + \overline{\Delta I_1^2 (t)}$$

with $\overline{\Delta I} = 0$

When $d > l_c$, the intensities at P_1 + P_2 fluctuate independently

$$\overline{\Delta I_1(t) \cdot \Delta I_2(t)} = 0 \qquad d > l_c$$

So

$$\overline{I_1(t) I_2(t)} = \bar{I}^2 \qquad d > l_c$$

Decrease in correlation with increasing d

Photon counting

respons prob. of detector $\sim$ instaneous intensity on surface

$n(t;T)$ counted within $t - T/2$ to $t + T/2$

$T \equiv$ integration time

Intensity correlations

$$n_1(t;T)\, n_2(t,T)$$

Count coinicidences : events at same time

$$\sim I_1(t) I_2(t)$$

$\Rightarrow$ HBT saw more coincidences for $d < l_{trans}$

For $d < l_c$

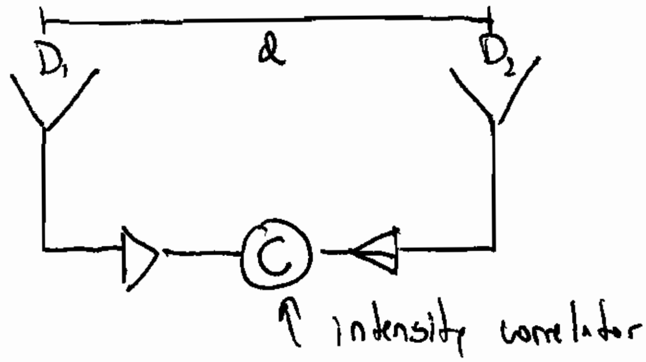
When a photon was obseved at P_1
the probability for detecting a photon at P_2
was larger than the ~~[illegible]~~ random case.

How does a photon "know" its surroundings.

$\Rightarrow$ Star I ~~[illegible]~~ demo
two atoms

Review HBT

Wanted to measure stellar diameters of stars.



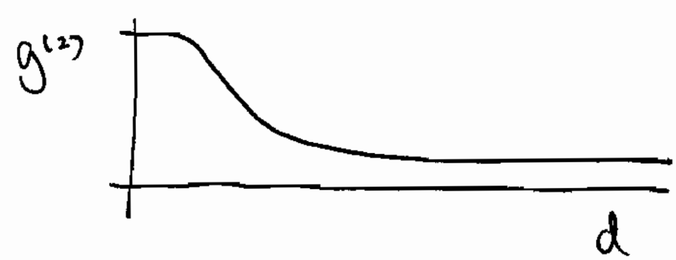
$d\alpha \simeq \lambda$

↑ distance ↑ star's angular diameter

Measure intensity correlations as a function of d

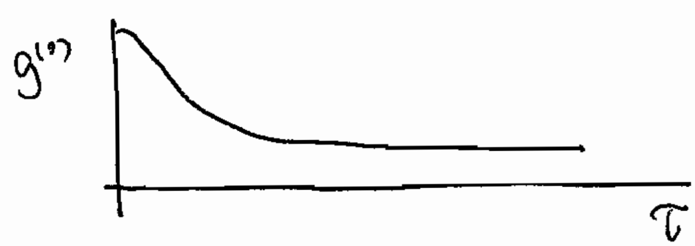
Spatial Coherence of star light

For $d = 0$ Saw strong correlation between detectors



$g^{(2)}(0) = 2$

Repeat measurement for time τ. Saw strong correlation at short times



Correlations $g^{(2)} = 2$ ~~The probability for detection of photon at D_1 was + D_2 was~~

$$\boxed{g^{(2)}(0) = 2}$$

If a photon is detected at D_1 there was a greater than random probability of detecting a photon at D_2.

HBT Effect $\Rightarrow$ Photon bunching

As we will see, classical physics explains this.

However a naive photon picture does not*. We need a better picture (developed by Glauber, Fano, etc.)

Let's look at a simplified version of HBT experiment.

* Purcell — Bosons
Glauber — Classical

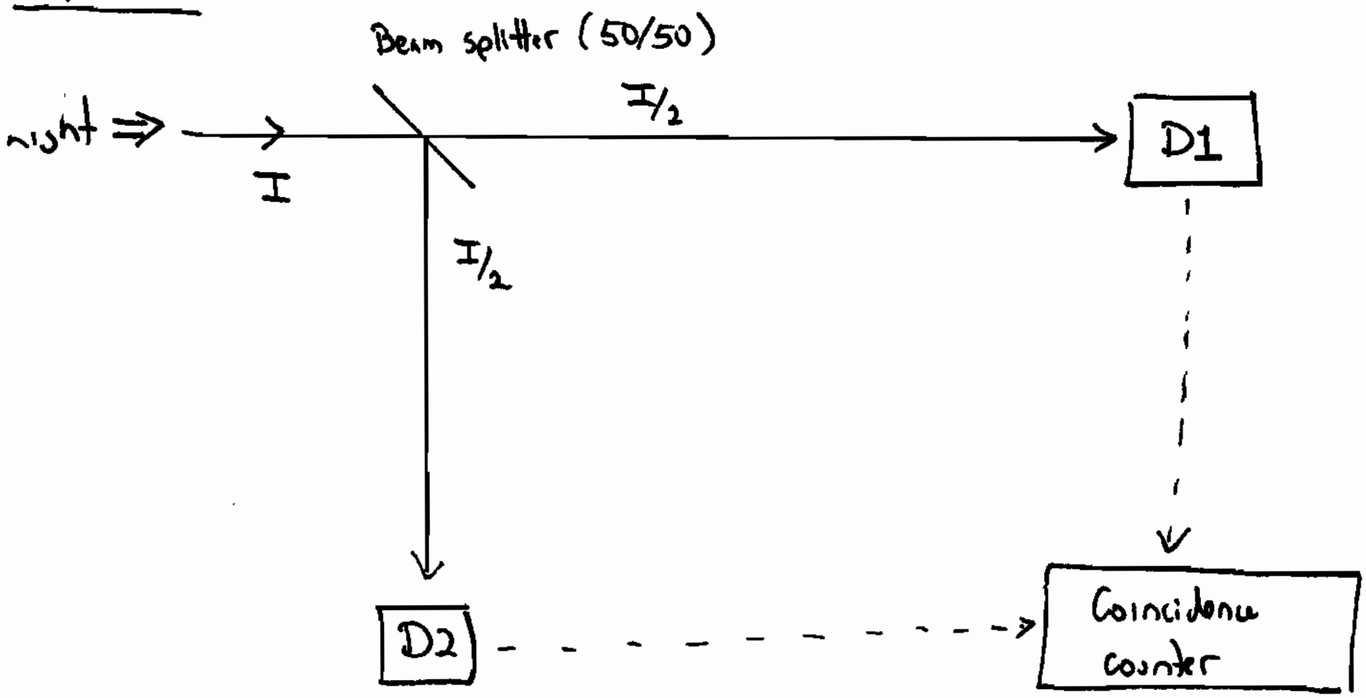
Hanbury-Brown + Twiss Experiment (1956)

How to design an experiment to detect single photons?!

- photon ⇒ particle at one place
- Experiment to determine the position (here or there) of a photon
- single photon, two detectors

  Do these detectors "click" at the same time?

Experiment



Do detectors D1 + D2 "click" at the same time. Within the particle picture they should <u>not</u>.

- Coincidence ~~[illegible]~~ detection

Records a count only if both detectors "click" at the same time

Experiment: Send light to beam splitter and measure the number of coincidences counts relative to the number of individual counts on the detector.

Anti correlation parameter

$$\boxed{A = \frac{P_c}{P_1 \; P_2}}$$

where

$P_c \equiv$ probability of coincidence

$P_1 \equiv$ measured prob. of detector 1 responding

$P_2 \equiv$ measured prob. of detector 2 responding

Outcomes

- Light as particles ~~photons (particles)~~ $A = 0$

- ~~Light as waves~~ ~~$A \neq 0$~~

more specifically

- Light as a wave
  If detectors click randomly + independent $A = 1$

(predicted by wave theory) $\left(P_{\text{both click}} = \left(P_{\substack{\text{one}\\\text{clicks}}}\right)\left(P_{\substack{\text{other}\\\text{clicks}}}\right) \right)$

Independent of intensity

$P_c = P_1 P_2$ so $A = 1$

- If two detectors clicking together more often than random "[illegible]" $A \geq 1$ "Clustering"

Write in terms of experimental results

Probability of detection

$$P_{1 \text{ or } 2} = \frac{N_{1 \text{ or } 2}}{\left(\frac{T}{\Delta t}\right)}$$

$\Delta t \equiv$ time resolution

$T \equiv$ experiment time

so

$$A = \frac{N_c}{N_1 N_2} \left(\frac{T}{\Delta t}\right)$$

Hanbury-Brown + Twiss

Quantum mechanics Prediction

for small intensities single photons are not split by the photodetector so $(A = 0)$ (anticorrelation)

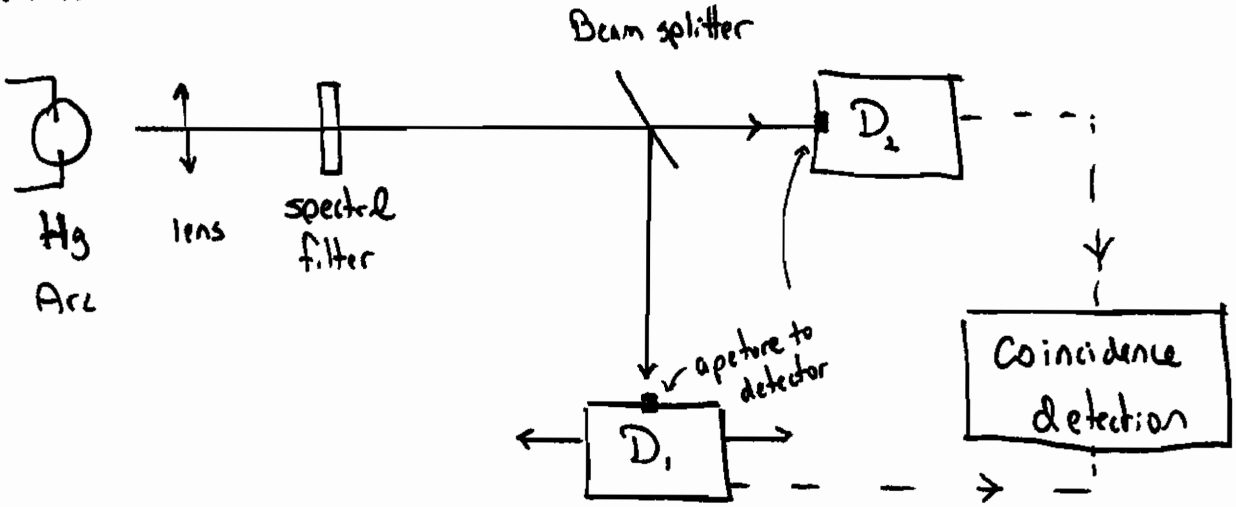
Wave theory Prediction

No matter the intensity of light at beam splitter it is halved in both directions, so random coincidences would be expected $(A = 1)$

$\Rightarrow$ They found the opposite result $A = 2$!!

When a click occured at one detector they found a higher than random probability that a click had simultaneously occurred at the other.

Experiment



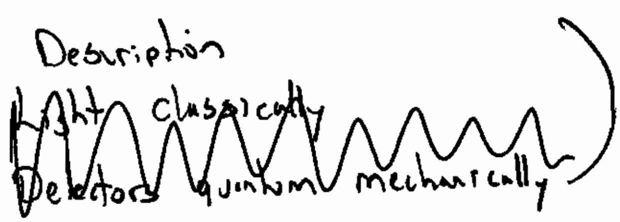
Detector 1 on slide
translation stage

This experiment failed to demonstrate the existance of photons and the indivisiblity of light. It showed that light travels thru space bunched up, you can divide the bunches in half but the bunches arrive at the same time

Origins of quantum optics $\Rightarrow$ understanding photon correlations

But is this classical or quantum description?

- Classical Description



Note
They describe things classically?

Probability for transition

$$P_1 = \alpha_1 I \Delta t$$

$$P_2 = \alpha_2 I \Delta t$$

$$P_c = \alpha_1 \alpha_2 I^2 (\Delta t)^2$$

$$A = \frac{\alpha_1 \alpha_2 I^2 (\Delta t)^2}{(\alpha_1 I \Delta t)(\alpha_2 I \Delta t)} = 1 \Rightarrow$$ Not their result
Result if used laser.

(Divide out detector response)

However if the light source produced a time varying intensity $\langle I \rangle$ produced by a collection of atoms:

$$P_1 = \alpha_1 \langle I \rangle \Delta t$$

$$P_2 = \alpha_2 \langle I \rangle \Delta t$$

$$P_c = \alpha_1 \alpha_2 \langle I^2 \rangle \Delta t$$

ave of intensity squared

Then $$A = \frac{\langle I^2 \rangle}{\langle I \rangle^2}$$

However $\langle I^2 \rangle \geq \langle I \rangle^2$ (Cauchy Swartz inequality)

Thus $A \geq 1$

What happens when you use a laser?!

$A = 1$ since the intensity fluctuations are small

$$\boxed{\langle I^2 \rangle = \langle I \rangle^2}$$

(For more details see Chapter 5 Gerry & Knight) from this experiment they noticed ...

for $d < l_{\text{tran coh}}$ when a ~~photon~~ click measured a D_1 the probability of detecting a "click" at D_2 was larger than the random case.

Does a photon "know" the outcome of the two detectors?!

Classical

<u>Coherence Functions</u>

2nd order (in Field) $\quad g^{(1)}(\vec{r}_1, \vec{r}_2\ t)$

$$\boxed{g^{(1)} = \frac{\langle E(r_1, t)\ E(\vec{r}_2, t)\rangle}{\langle E(\vec{r}_1, t)\, E^*(r_1, t)\, E(r_2, t)\, E^*(r_2, t)\rangle}}$$

Michelson Interferometer

Values of $g^{(1)}$ $\quad 0 < g^{(1)} < 1$

$g^{(1)} = 1$ complete coherence
$0 < g^{(1)} < 1$ partial coherence
$g^{(1)} = 0$ incoherent

4^{th} order (in Field) Intensity

$$\boxed{g^{(2)}(\vec{r}_1, \vec{r}_2, t) = \frac{\langle E^*(\vec{r}_1, t)\, E(r_1, t)\, E^*(r_2, t)\ E(r_2, t)\rangle}{\langle E(r_1, t)\ E^*(r_1, t)\rangle\langle E(r_2, t)\, E^*(r_2, t)\rangle} = \frac{\langle I(r_1, t)\ I(r_2, t)\rangle}{\langle I(r_1, t)\rangle\langle I(r_2, t)\rangle}}$$

Intensity correlator

HBT

values for $g^{(2)}$ $\quad 1 < g^{(2)} < \infty$

$$g^{(2)}(0) = \frac{\langle I^2(t)\rangle}{\langle I(t)\rangle^2}$$

Cauchy inequality $\langle I(t)^2\rangle \geq \langle I(t)\rangle^2$

COMET

3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER

Values of $g^{(1)}$ + $g^{(2)}$

— For laser light $\boxed{g^{(2)}(\tau) = 1}$ (Coherent State)

photons arrive randomly { Probability of Delayed coincidence is independent of time

$\boxed{\text{Poisson distribution}}$ $\boxed{P_n = e^{-\bar{n}} \frac{\bar{n}^n}{n!}}$

— Thermal light (single mode) (Chaotic light)

$\boxed{g^{(2)}(\tau) = 2}$ HBT Effect

start from density matrix approach of thermal fields (chpt. 2.5)

$$\boxed{P_n = \frac{\bar{n}^n}{(1+\bar{n})^{n+1}}}$$ $\boxed{\text{Bose - Einstein}}$

$$g^{(2)}(\tau) = 1 + \frac{\langle(\Delta \hat{n})^2\rangle - \langle \hat{n}\rangle}{\langle \hat{n}\rangle^2}$$

For Thermal light

$$g^{(2)} = 1 + |g^{(1)}|^2$$

Uncorrelated detection (shot noise) independent noise

Correlated detection (excess noise) beat notes of random waves

Back to Stellar Astronomy

HBT used intensity correlation ($g^{(2)}$) instead of Michelson ($g^{(1)}$) because path lengths from source must be nearly equal for using a Michelson. Also atmospheric issues causes phase fluctuations. An intensity correlator is ~~ins~~ insensitive to both problems.

3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER

COMET

Photon bunching (Paul) & Light sources

Coincidence rate $\tau < t_c$ is higher since

$$\overline{I^2(t)} \text{ is smaller than } \overline{I(t)\,I(t+\tau)} \quad \text{for} \quad \tau < t_c$$

For $\tau >> t_c$, coincidences are random

Need detector with $T < t_c$

$$t_c \sim \frac{1}{\Delta\nu}$$

Polarized Thermal Light

photons follow Bose-Einstein distribution

max at $n=0$

$$P_n = \frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}$$

Variance

$$\Delta n^2 = \langle n^2\rangle - \langle n\rangle^2 = \langle n\rangle^2 + \langle n\rangle$$

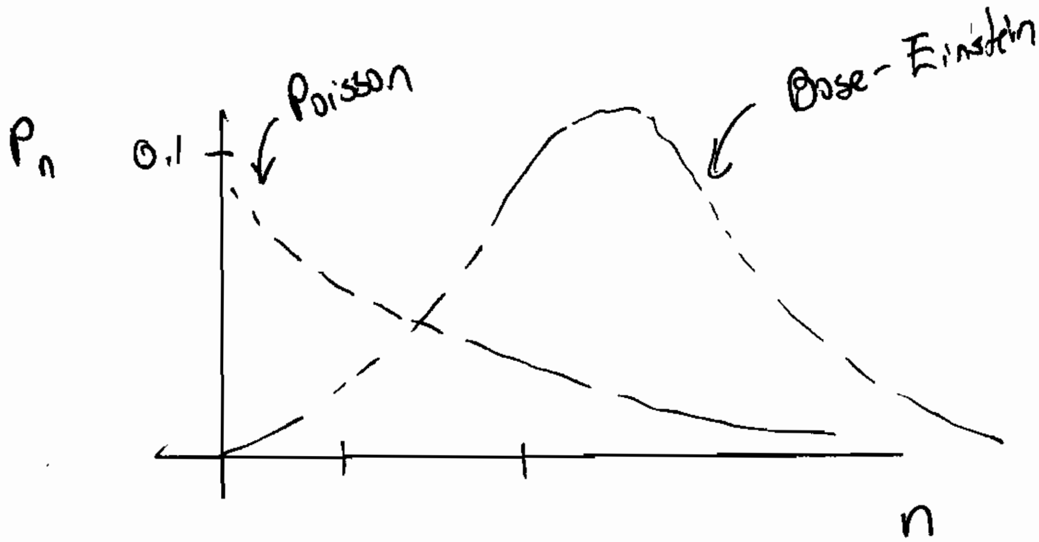
prob of not finding any photons is larger than the prob. of finding any given # of photons

photon # flucuates strongly

Monochromatic light: Poisson distribution

$$P_n = e^{-\bar{n}} \frac{\bar{n}^n}{n!}$$

Maximum near $\bar{n}$



Poisson ≡ completely random process

OK $\qquad T > t_c$ ✓

Onset of lasing Fig 8.8

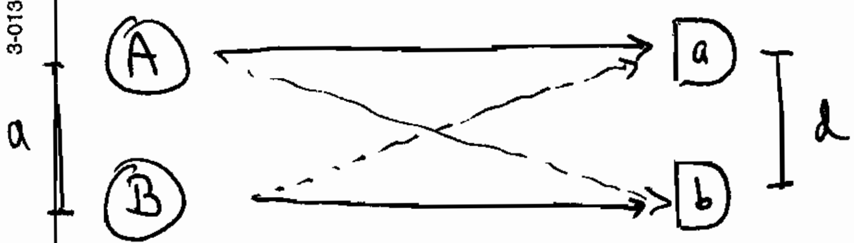
Scattered laser light ⇒ pseudothermal

Fano's Quantum Interpretation AJP 29 p. 539 (1961)

Looked at intensity correlations of light emitted by atoms

We can apply this to HBT experiment schematically

Sources Detectors



$R = \frac{1}{2}(a + d)$

Probabilities

Solid $\langle a|A\rangle\langle b|B\rangle$

Dotted $\langle a|B\rangle\langle b|A\rangle$

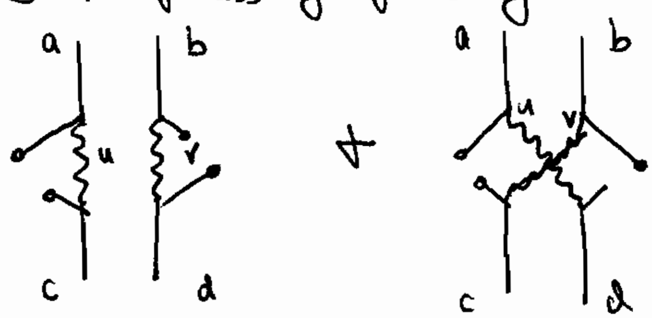
If photons are indistinguishable, amplitudes interfer constructively to get joint detection larger than individual events.

$$\langle a|A\rangle\langle b|B\rangle \text{ interfers with } \langle a|B\rangle\langle b|A\rangle$$

If $d > l_c$, interference 'washes out'

$$\frac{ad}{R\lambda} > 1 \qquad (\text{Eq } 34)$$

Paper actually looks at process of pairs of atoms



Feynman diagrams

Photoelectric effect revistited : Lamb + Scully (1969)
"The Photoelectric Effect without photons"

Jaynes +
Lamb + Scully develope a semi-classical theory for the photoelectric effect $\Rightarrow$ no more need for photons

Photo electric effect is not a proof of the existance of photons.

Summary

This theory along with the failing of experiments to detect photons raised a few questions. What is the nature of a photon? Are there really photons? Do they exist?! Or are they 'ifacts of the tools we used to investigate light?!

The problem with the photo electric effect + Hanbury-Brown Twiss Experiment was in the light source they used.

Anti correlations are expected if the source produces light in an eigenstate of the photon number operator.

For both experiments ~~all large number of photons used~~, using a quantum description, a large # of photons were used. If another experiment is designed which uses one photon (that is an eigenstate of a photon number operator) then we would expect an anticorrelation $A = 0$.

- Read: By Monday

1) Aspect el at Europhys Lett 1 (4) p 173 (1986)

2) Walther et al Phys Rev ~~MM~~ A vol 35, 6, 1987

Survey $\Rightarrow$ <u>out</u> <u>tonight</u>

Questions: (About Hanbury-Brown + Twiss Experiment)

1. Why Did Hanbury Brown + Twiss measure coincidences in "cathodes aligned" positions + <u>no</u> coincidences in "cathodes not aligned" positions?

2. Why did Brannen el al <u>not</u> measure any coincidences?

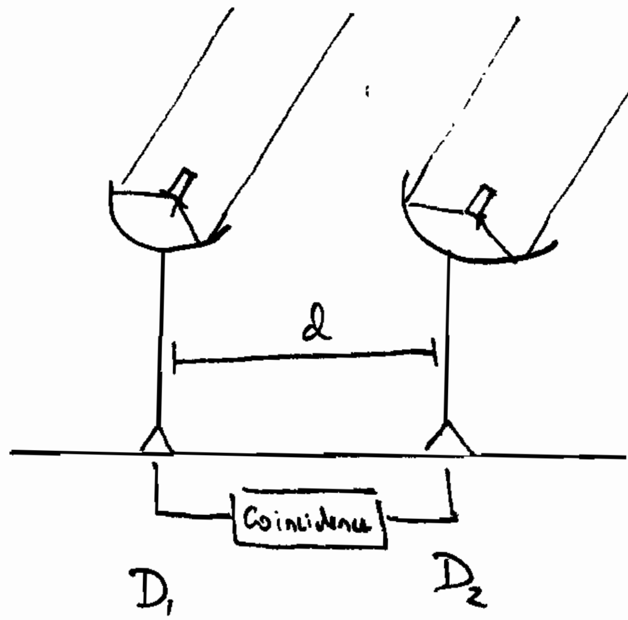
3. Given Brannen el at experiment, what is the one thing they need in order to observe coincidences.

   How would this "one thing" solve their problems?

   (Brannen + Ferguson Nature 1956)

Notes $\begin{cases} \tau_0 \sim \cancel{\text{XX}}\ 10^{-11}\text{s} \\ \text{resolving time } \ 10\,\text{ns} \end{cases}$ $\Rightarrow$ For Brannen et al so $T \simeq$ resolving time $\frac{\tau_0}{T} = \frac{10^{-11}}{10^{-8}} \simeq 10^{-3}$ $\Big\}$ Correlations 10^{-3}

Where did this experiment come from?: Radio astronomy



Measurement of spatial coherence of "light" from a star

- instantaneous phase of E changes slightly within coherence area.

- see similar time evolution between detectors if d is shorter than the transverse coherence length.

$$\Delta I_1 \simeq \Delta I_2 \qquad \text{for} \quad d < l_{coh}$$

$$\Delta I = I(t) - \bar{I}$$

Correlator gives $\overline{I_2(t)\, I_1(t)}$

$$\overline{I_1(t)\, I_2(t)} = \bar{I}^2 + \overline{\Delta I_1^2} \qquad d < l_{coh}$$

For $d > l_{coh}$

$$\overline{I_1(t) I_2(t)} = \bar{I}^2 \qquad \text{Decrease in intensity correlations.}$$

Instead of d we can use the star's angular diameter.

$d \simeq 188\text{m}$ $\qquad$ Δdiameter $\simeq$ 0.0005 arc sec.

Lecture 30 Aspect Experiments in 1986

Here, we will discuss two experiments performed by A. Aspect et al in Europhys. Lett. 1 (4) pp 173-179 (1986)

Both experiments used an atomic cascade as a light source. and a triggered detection scheme. The source provided single photons unlike the experiment of Hanbury-Brown + Twiss.

Two Experiments

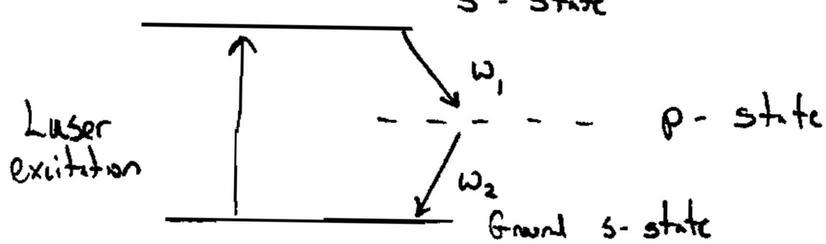
1. Test anticorrelation of the source (similar to Hanbury-Brown + Twiss)

2. Single photon interference experiments

The "single photon" Source.

Laser excitation of Ca atoms to a state that would decay by emitting two photons instead of one.

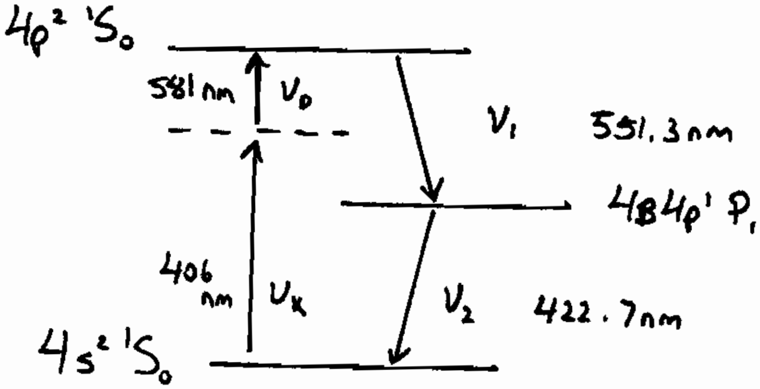
(Two photon absorption?)



How to detect these photon ω_2 from all other photons $\Rightarrow$ triggered detection

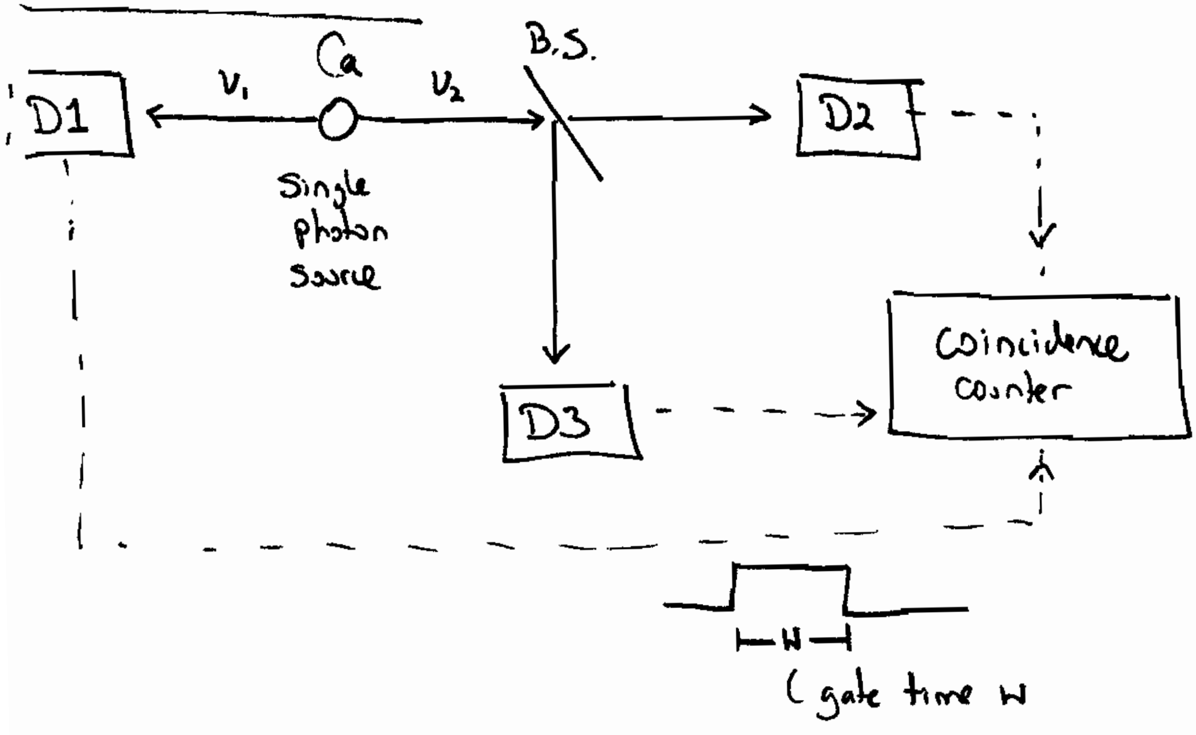
~~Experiment~~

Energy levels of Ca



Some numbers
- 240 coincidences/s
- 90 accidental coincidences/s
- 100 s → 150 true coincidences

"Detection Scheme



Photon ν_1 triggers gate of duration W

Look for coincidences during time W. Reject coincidences for times "outside" of W.

They measured $A = 0$! Single photons measured at last.

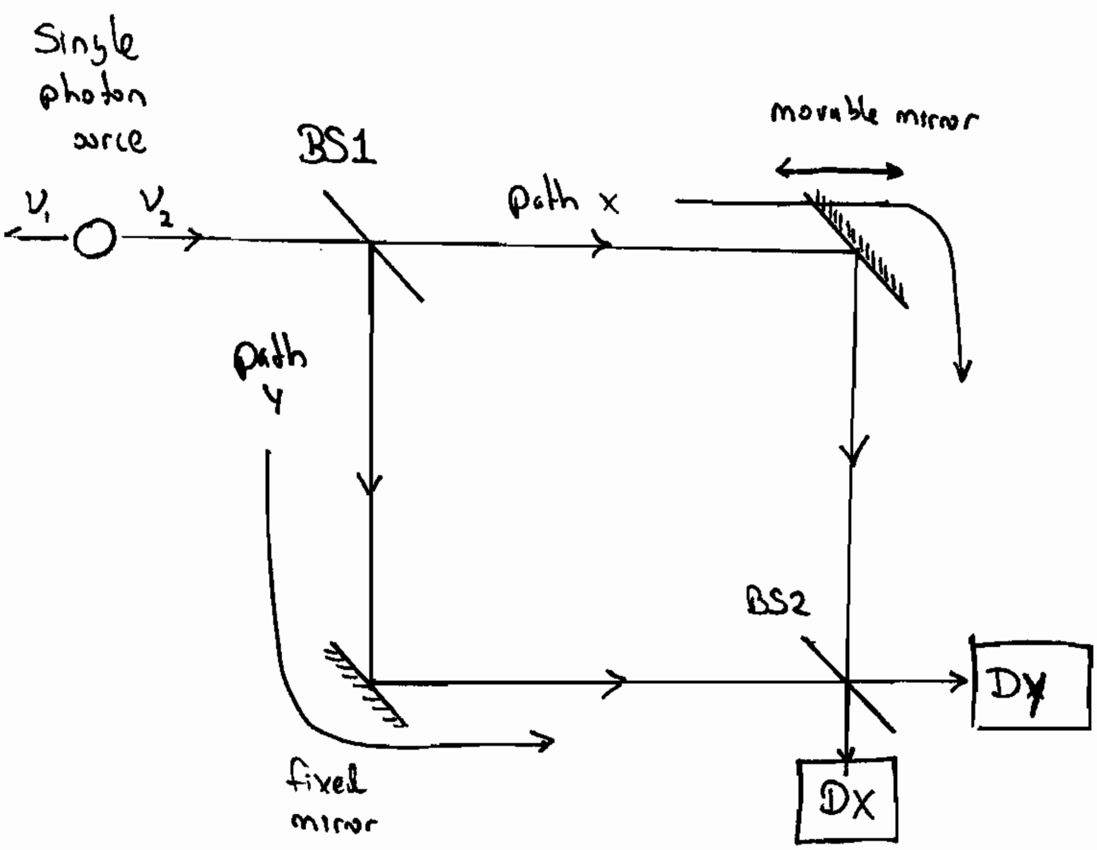
Clear evidence at last for the existance of photons.

What has been shown here?

Individual particles from their source were either reflected or transmitted, going one way or another, but never <u>both</u>

Experiment 2 : Single photon interference (huh?!)

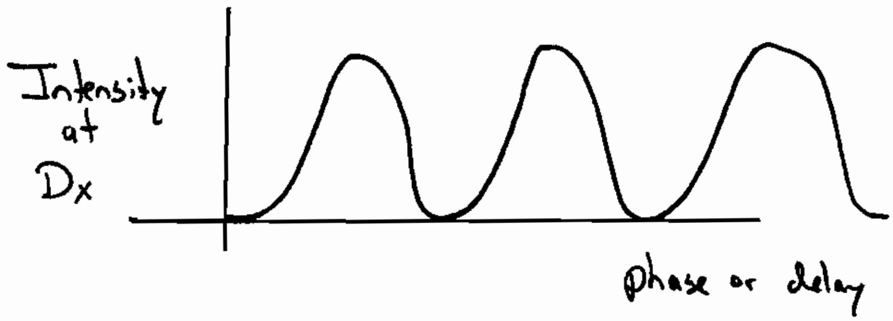
Allow single photons to enter a Mach-Zehnder Interferometer.



moving the mirror put a phase difference between the "arms" of the Mach Zehnder Interferometer.

What would happen if light is a wave?

One would see interference fringes as a function of changing the position of the movable mirror (just like for miniproject 1)

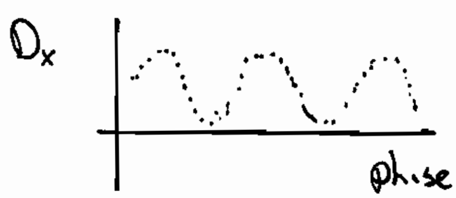


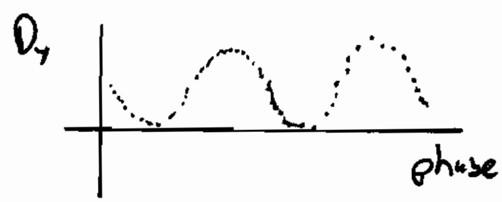
What would happen if light is particle?

From experiment #1 we know that the photon goes one way or the other at the BS. We would not expect any interference.

What happened?

They saw an interference pattern as they aquired counts!





Does the photon take both paths?!

Does the photon interfere with itself?!

- Wave particle duality   Can Wave-particle duality Explain this?

Does light "know" when to behave like a wave and when to behave like a particle?!

Explain HBT Classically

Chaotic light ⇒ thermal source

radiative
Doppler
collisional broadening } Kinetic theory of gases

Atom 1 $E_1(t) = E_0 \exp(i\omega_0 t + i\phi_1(t))$

From Loudon

phase changes per collision { Prob $p(\tau)\,d\tau = \frac{1}{\tau_0}\exp(-\tau/\tau_0)\,d\tau$

$$E(t) = \sum E_i(t)$$

$$= E_0 \exp(-i\omega_0 t)\, a(t) \exp(i\phi(t))$$

$$I(t) = \frac{1}{2}\epsilon_0 c E^2\, a^2(t)$$

HBT Measured 2^{nd} order correlations of chaotic light

For lorenztian of width γ

$$\langle (I(t_1) - \bar{I})(I(t_2) - \bar{I}) \rangle = \bar{I}^2 \exp(-2\gamma|t_1 - t_2|)$$

~~this happens~~ OR $\langle \bar{I}(t_1)\, I(t_2) \rangle = \bar{I}^2(\exp(-2\gamma|t_1-t_2|) + 1)$

So $g^{(2)} = \exp(-2\gamma|t_1 - t_2|) + 1$

Measuring correlated intensity Flucuations

vs. correlated intensities

$$g^{(2)} = \frac{\langle I(t_1)\, I(t_2) \rangle}{\langle I(t_1) \rangle \langle I(t_2) \rangle}$$

$$\langle I(t_1) \rangle = \langle I(t_2) \rangle = \bar{I}$$

$$g^{(2)} = \frac{\langle I(t_1)\, I(t_2) \rangle}{\bar{I}^2} \qquad \left\{ \begin{array}{l} \text{Cauchy} \\ \langle I(t)^2 \rangle \geq \langle I(t) \rangle^2 \end{array} \right.$$

For these light sources $p[I(t)] = \frac{1}{\bar{I}} \exp(-I(t)/\bar{I})$

$$\boxed{\langle I(t)^n \rangle = n!\ \bar{I}^n}$$

At $t=0$

$$g^{(2)} = \frac{2\bar{I}^2}{\bar{I}^2} = 2$$

$\tau_r \equiv$ response time

For $\tau_r < \tau_c$

$$\langle (\bar{I}(t_1) - I)(\bar{I}(t_2) - I) \rangle \simeq \bar{I}^2$$

For $\tau_r > \tau_c$

" " $= \bar{I}^2 / \gamma \tau_r$

HBT: Why $g^{(2)}=2$ Classically

Thermal source $\rightarrow$ chaotic light

Average long time compared to τ_c. Measure at two different times.

$$\langle E^*(t)\,E(t_2)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T E^*(t_1)\,E(t_2-t_1)\,dt_1$$

1st order correlation function of fields. Related to $g^{(1)}$

For Lorentzian $\quad \langle E^*(t_1)\,E(t_2)\rangle = \dfrac{2\bar{I}}{\epsilon_0 c}\exp(i\omega_0\tau - \gamma|\tau|)$

$$\tau = t_1 - t_2$$

$$\boxed{g^{(1)} = \exp(-\gamma|\tau|)}$$ Lorentzian $\gamma \equiv$ width

$$\boxed{g^{(1)} = \exp(-\tfrac{1}{2}\delta^2\tau^2)}$$ Gaussian $\quad \delta \equiv$ width

HBT (2nd order)

$$\langle (I(t_1)-\bar{I})(I(t_2)-\bar{I})\rangle = \langle I(t_1)\,I(t_2)\rangle - \bar{I}^2$$

Where $\langle I(t_1)\rangle = \langle I(t_2)\rangle = \bar{I}$

Put fields in

$$\langle I(t_1)\,I(t_2)\rangle = \left(\tfrac{1}{2}\epsilon_0 c\right)^2 \left|\langle E^*(t_1)\,E(t_2)\rangle\right|^2 + \bar{I}^2$$

For Lorenzation

$$\langle \bar{I}(t_1)\, \bar{I}(t_2) \rangle = \bar{I}^2 \left(\exp(-2\gamma|\tau|) + 1 \right)$$

- but always

$$\langle \bar{I}(t)^2 \rangle = 2\bar{I}^2$$

HBT $\langle (I(t_1) - \bar{I})(I(t_2) - \bar{I}) \rangle$

$$= \bar{I}^2 \exp(-2\gamma\, |t_1 - t_2|)$$

$$\boxed{g^{(2)} = \exp(-2\gamma|\tau|) + 1}$$

If $\gamma \to 0$ $\quad \exp(-\gamma) \to 1$

$$g^{(2)} \to 2$$

$g^{(2)}$ approaches laser for larger $\gamma|\tau|$

vs $g^{(1)}$ approaches laser for small $\gamma|\tau|$

HBT: Why $g^{(2)}=2$ Q.M.

Using Quantum Coherence Functions

$$g^{(2)} = \frac{\langle m_1 m_2 \rangle}{\bar{m}^2}$$

For Lorenztian $\langle m_1 m_2 \rangle = \bar{m}^2 \left(\exp(-2\gamma|\tau|) + 1 \right)$

HBT $\langle (m_1 - \bar{m})(m_2 - \bar{m}) \rangle = \langle m_1 m_2 \rangle - \bar{m}^2$

Chaotic light

$$E_1(t) = E_0 \exp(-i\omega_0 t + i\phi_1(t))$$

change per collision

$$E(t) = E_0 \exp(i\omega_0 t)\, a(t) \exp(i\phi(t))$$

$$I(t) = \frac{1}{2} \epsilon_0 c E_0^2 a^2(t)$$

Lecture 31 What is a photon? Part 3: Delayed Choice Experiment

Wave particle duality: Does this explain the Aspect Experiment?

Sometimes a wave / Sometimes a particle.

Photon "conspiracy theory"? to understand the wave/particle duality.

~~"senses" an experiment & should demonstrate its wave or particle properties!?~~

$\Rightarrow$ As the photon leaves a source, it "senses" the experiment and behaves accordingly, like a particle or wave.

Does this happen?! Test this: . . .

Delayed choice Experiment of John Wheeler

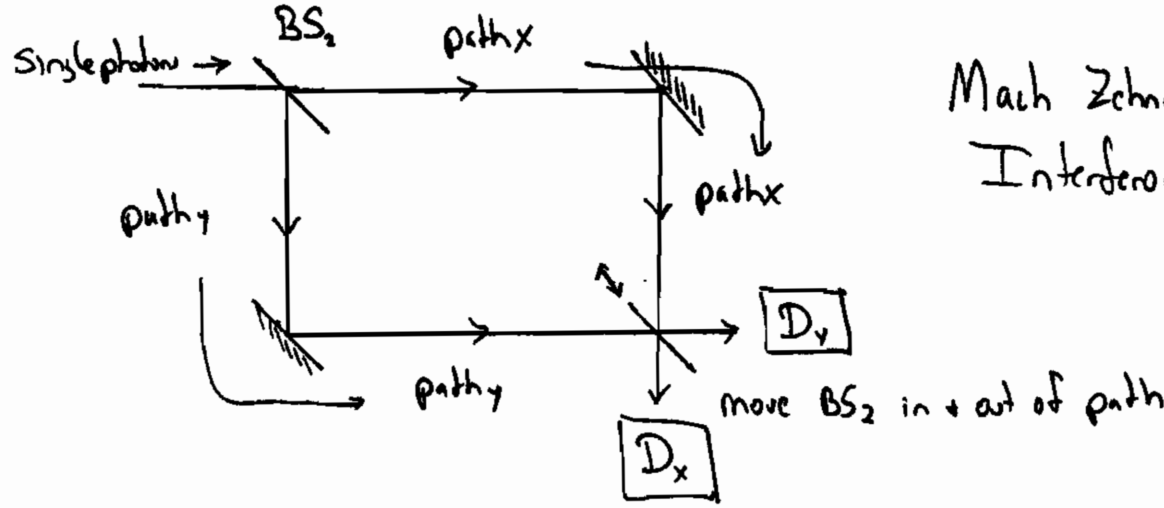
Ref. Wheeler "Mathematical Foundations of Quantum Mechanics" p. 9-48

- States that "conspiracy" theory is wrong. Also shows that wave-particle duality is more complicated
- Experiment of Delayed Choice
  - able to detect either particles (anti coincidences) or waves (interference)
  - Designed so that choice of which aspect to observe is delayed after the photon has "decided"

How to do this $\Rightarrow$ make BS2 in the MZ movable.

Delayed Choice Experiments PRA 35, 6 p 2532 (1987)

Delayed Choice Experiment



Mach Zehnder (MZ) Interferometer

What to expect?

- Single photon enters MZ
- Without BS_2 — Detectors D_x + D_y ascertain a specific path for the photon

  ⇒ Like Aspect Experiment #1 (anticorrelation exp.)
- With BS_2 — Expect to get interference. Lose all information on the path the photon takes

The idea is to insert BS_2 after the photon has entered the MZ interferometer

According to "conspiracy theory" the last minute insertion of BS_2 should "fool" light

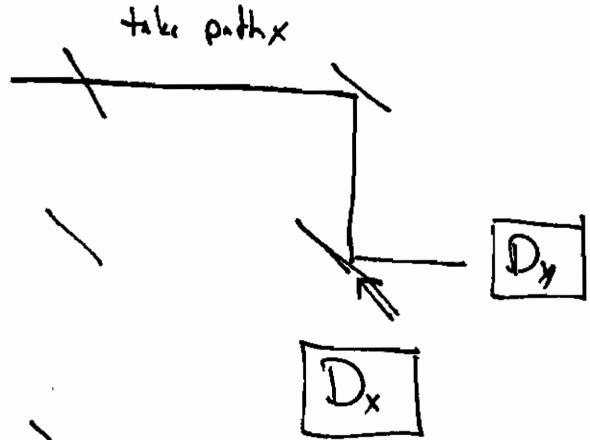
Example

1) No BS_2

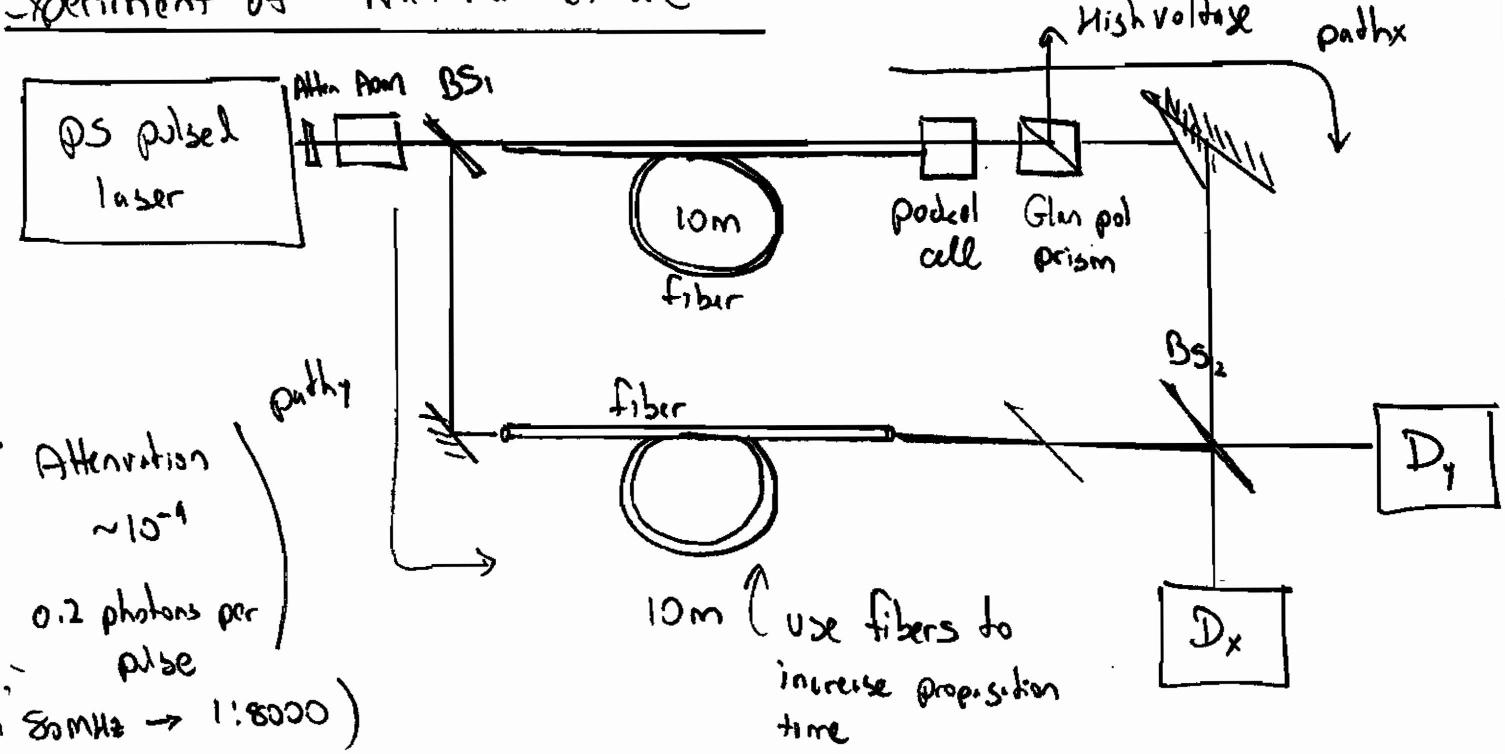
2) Light takes path x or y

3) Insert BS_2

4) Should observe interference?!!
(no according to conspiracy theory)



Experiment of Walther et al



Attenuation $\sim 10^{-9}$

0.2 photons per pulse

80 MHz → 1:8000

- Switching time 5ns
- Voltage High ⇒ Light reaching BS_2 came from path y
- Voltage Low ⇒ Light can take both paths ⇒ interference.

Used pulsed light source
path of interferometer arms
increased 30ns by 10m Fiber
Give time for Pockels cell to operate

Speed of light

Light travels... in vacuum

in 100 fs ⇒ 30 μm

in 500 fs ⇒ 150 μm Thickness of hair

in 1 ns ⇒ 30 cm Ruler length

In fiber

Light slower by 1.45

1 ns ⇒ 30 cm (1.45) = 43.50 cm

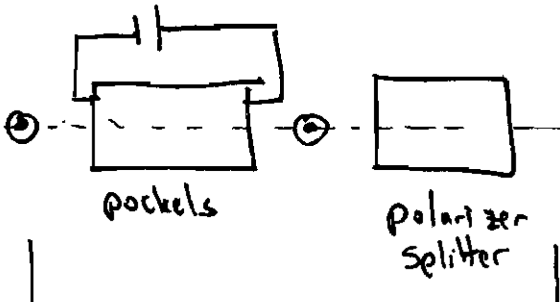
Pockels Cell

Applied electric field rotates input polarization by 90°

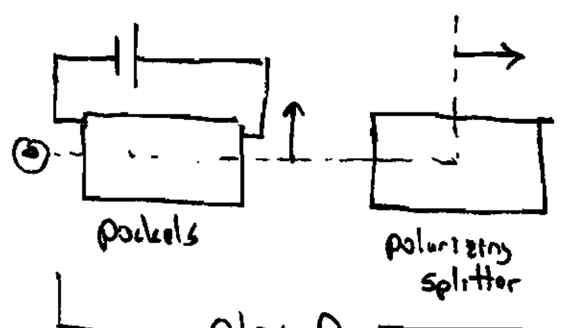
~~A polarizer splitter~~ Birefringent crystal with electro-optic effect

Rotate polarization ellipse by applying voltage

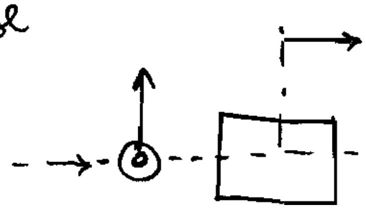
With combination of polarizer make polarization dependent switch



Open
no voltage



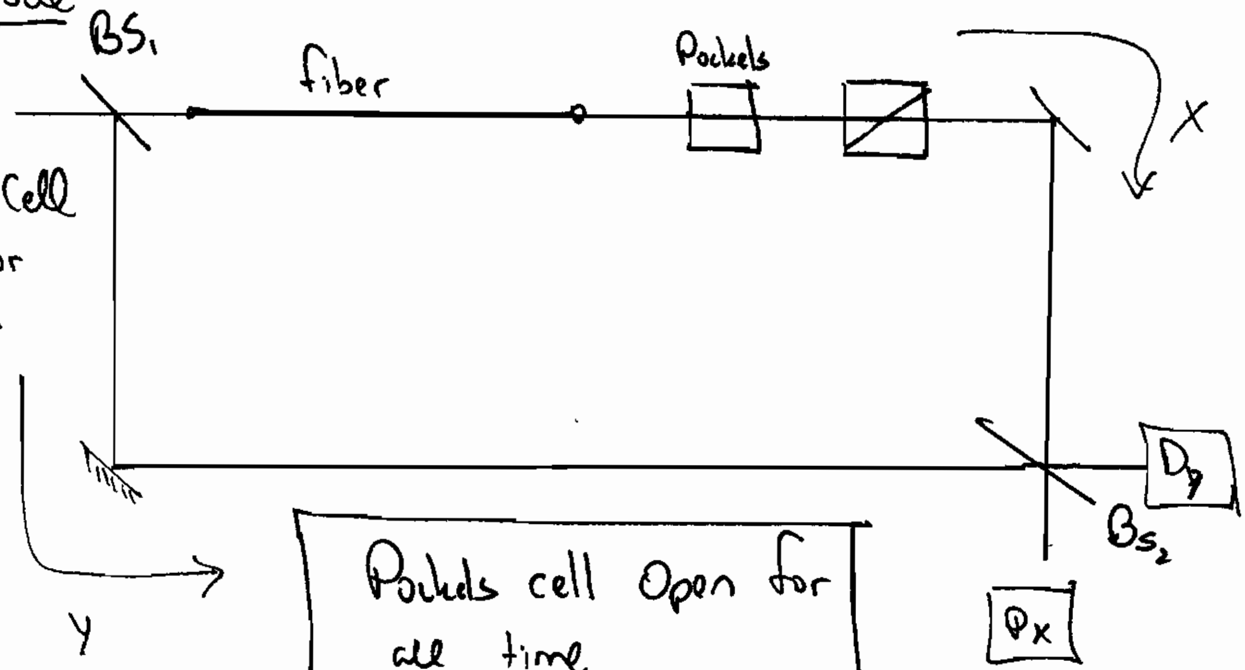
Closed
voltage

~~Polarizer~~  Glan Polarizing Beamsplitter

Review

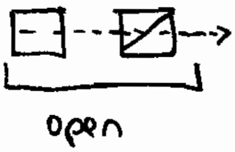
- Aspect et al
- Ask " which path did the photon take?"
- What does the photon interfere with to get the interference pattern?
- Does it interfer with itself?!

Normal Mode



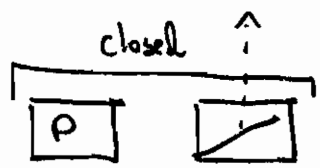
Pockels cell Open for all time

1.) Turn Pockel's Cell off (open) for all time t.



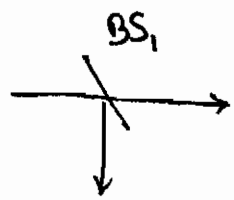
Delayed Choice Mode

1) t=0 Pockels Cell Closed ⟹



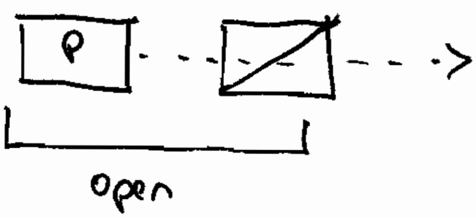
2) ~~t=5ns Open~~

t = 1ns Light passes BS1 ⟹



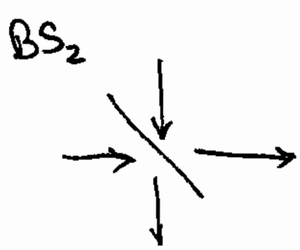
5ns

3) t = 6ns Pockel's Cell Open ⟹



4) t ≃ 30ns Light passes BS2

⟹



"Normal" mode — Pockel's Cell open (low voltage) when pulse reaches BS_2 and for the whole experiment

"Delayed Choice" mode — Pockels Cell Closed + opened 5 ns after pulse has passed BS_1. Pulse in fiber at this time

$N_\bullet \equiv$ Normal Mode
$N_+ \equiv$ Delayed Choice

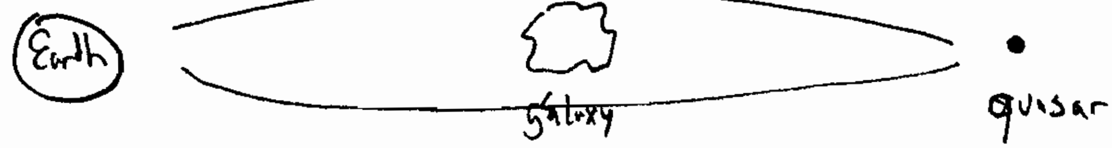
Results

- No matter when BS_2 was inserted an interference pattern was observed.
- If experiment began with BS_2 in place then removed interference was not observed.

Light is not fooled; when the apparatus is changed after light has made its "choice" the light still makes the correct choice.

"Double" Quasars (Wheeler)

Gravitational Lensing



Two paths

Two possible paths ⇒ interference can arise
Insert BS ⇒ see Interference
No BS — see two images

BS in place — photon has already traveled billions of yrs. via both paths

What possible difference does inserting the BS

Light has already traveled for billion of years.

What difference does inserting the BS make in the history of the ight?!

Do our actions at this present moment have consequenus that stretch back to the remote past?

"Smoky Dragon" (J. Wheeler)

simultaneously existing in everywhere in the interferometer.
which suddenly bends to bite the detector.

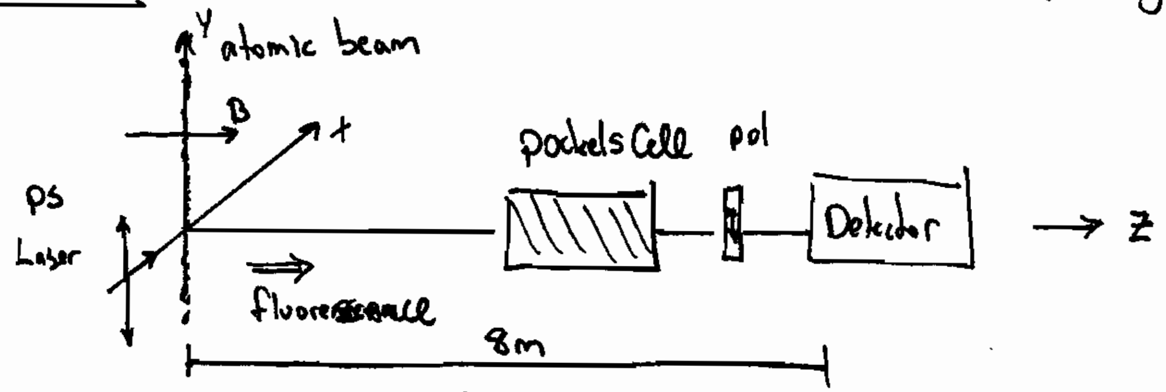
Quantum Beat Measurements (Walther et al)

- Atomic beam of Barium atoms
- Magnetic field applied $\perp$ to direction of atomic beam

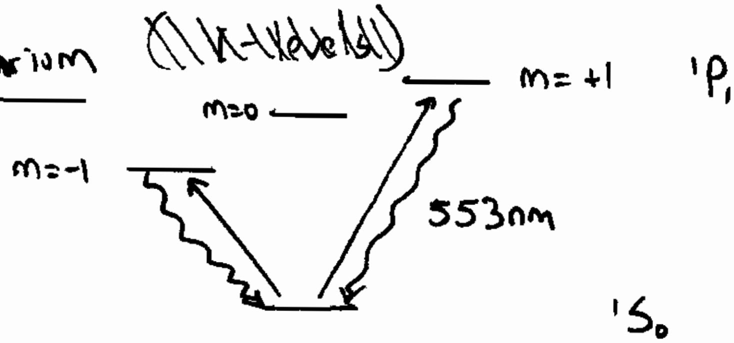
- Spectral measurement of Delayed Choice

<u>Experimental Setup</u>

Magnetic Field causes Zeeman splitting



<u>States for Barium</u>



Less than one photon per pulse

Interference between two paths

1) $|0\rangle \to |{+1}\rangle \to |0\rangle$
2) $|0\rangle \to |{-1}\rangle \to |0\rangle$ $\Big]\ \sigma^+$

Delayed Choice requires one path remains blocked untill the emitted photon arrives at detection system.

Use Pockel's Cell again

σ^+ transmitted

σ^- blocked

What does Pochels Cell Do here?

When the pockel's cell is on:

- The light from the $|0\rangle \rightarrow |+1\rangle \rightarrow |0\rangle$ path is changed to linearly polarized light which is transmitted by the polarizer

- The light from the $|0\rangle \rightarrow |-1\rangle \rightarrow |0\rangle$ path is changed to linearly polarized light and is blocked by the polarizer.

The pockels Cell effective blocks a path as in a similar manner as in the Mach Zehnder interferometer experiment.

See interference when both paths are present, see no interference if one path is blocked.

Results

- Normal Mode

Pockels Cell open ⇒ See interference
(No voltage)

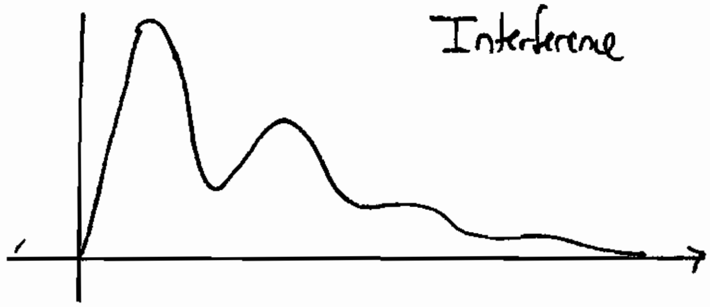
Apply voltage / Pockels Cell Closed ⇒ see exponential decrease

~~Delayed Choice~~

~~Apply voltage to pockels cell at different times.~~

Interference

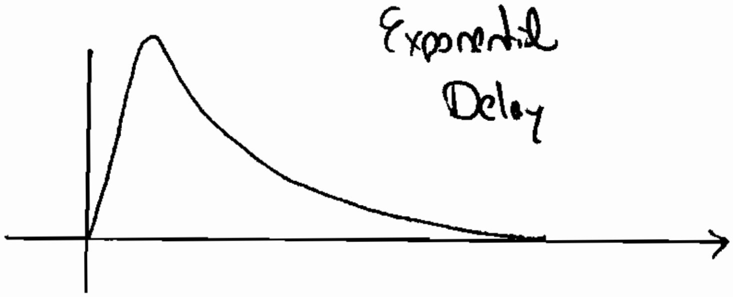
Type I



pockel's cell zero (open)

Exponential Delay

Type II



pockels cell voltage high (closed

Delayed Choice Experiment with Ba atoms

Turn on Pockels Cell

Turn off at different times 2ns 17ns 29ns

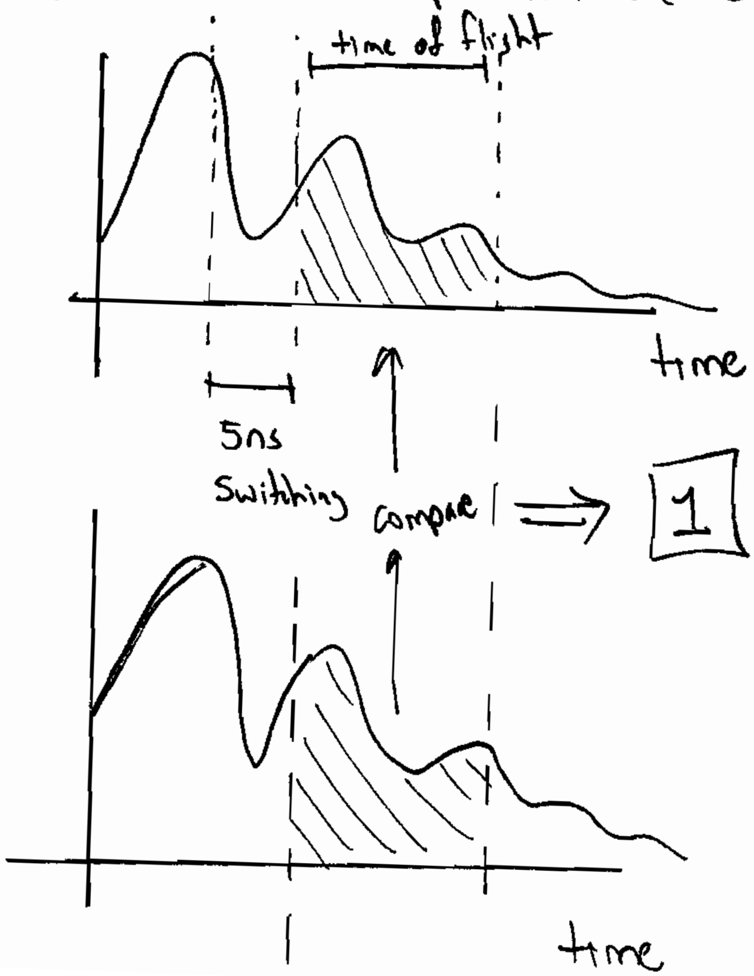
- See modulation of exponential decay after switching cell.

- They compare the normal operation to Delayed choice operation after swiching the pockels cell after 4ns

(Use Data from 10 - 30ns)

- They show that the data from 10 - 30ns for both the delayed choice and normal operation are basically the same.

Delayed Choice



Lecture 32 Quantization of a Single mode field

Complementarity and Quantum Beats

Principle of Complementarity

— Classical Physics ⇒ unity
  Quantum Mechanics ⇒ duality of two complementary pictures

particle ⟺ wave are complements

— Refers to what quantum mechanics allows us to know. ⇒ x OR p
  t or E

— What experiment allows us to know.

Knowledge of which path and the possibility of observing an interference pattern.

— All are mutually exclusive concepts

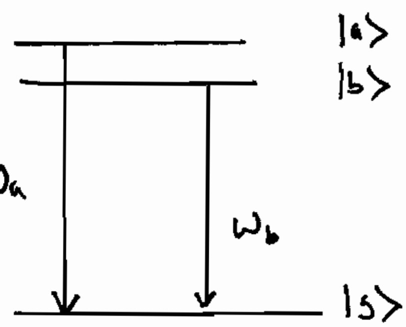
Complementarity + Information: Quantum Beats

Two Experiments

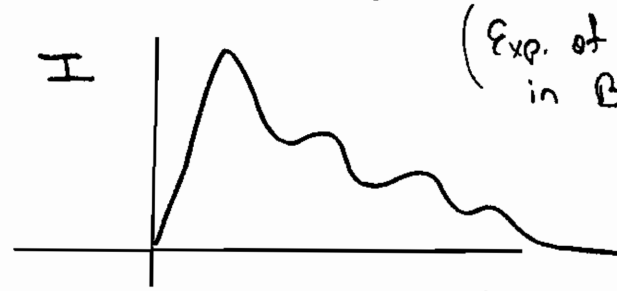
Fluoresence

(Exp. of Walther et al in Barium)

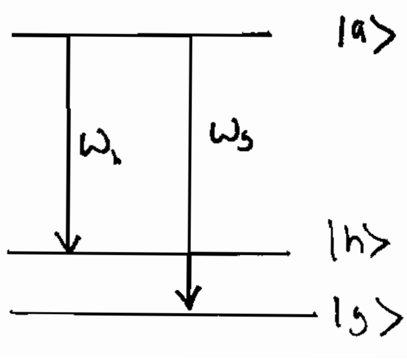
Type I



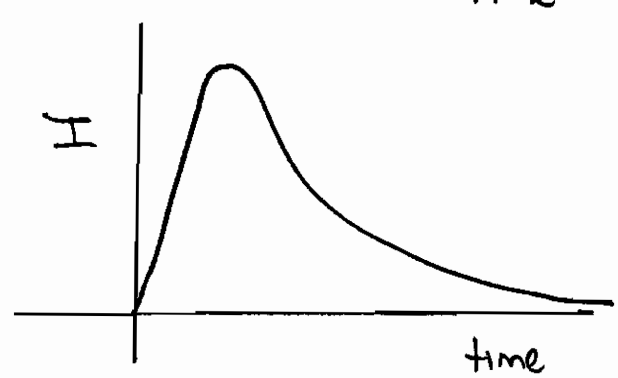
⇒



Type II



⇒



Why don't see beats for type II but see beats for Type I ?

complementarity:

If it is possible to distinguish by which history the atom has gone from initial to final state, then no beats will occur.

Do not need to perform experiment, sufficient that it is only possible!

Information (or lack of) leads to detection of beats.

Type I

Detector Cannot tell which "path" generated a photon $\left\{ \begin{array}{l} \text{Cannot tell} \\ |g\rangle \to |a\rangle \to |g\rangle \\ \text{OR} \\ |g\rangle \to |b\rangle \to |g\rangle \end{array} \right.$

- lack of information $\Rightarrow$ interference / beats
- paths are indistinguishable

Type I

Wavefunction

$$|\psi\rangle = C_1(t)\,|a\rangle\,|n_a\rangle + C_2\,|g\rangle\,|n_b\rangle$$

$$I \approx |\langle\psi|\psi\rangle|^2 = (|C_1|)^2 + (|C_2|)^2 + \underbrace{C_1^* C_2\, e^{i(\omega_a-\omega_b)t}\,\langle g|g\rangle}_{\text{interference term!}}$$

Type II

Paths here are not indistinguishable!

Do not expect interference.

1) Excite to $|a\rangle$

2) Decay to either $|g\rangle$ or $|h\rangle$. But one knows the final state

Can tell difference $\quad |g\rangle \to |a\rangle \to |g\rangle$

$\qquad\qquad\qquad\qquad |g\rangle \to |a\rangle \to |h\rangle$

Wavefunction

$$|\psi\rangle = C_1 \, |g\rangle \, |n_g\rangle + C_2 \, |h\rangle \, |n_h\rangle$$

$$|\langle\psi|\psi\rangle|^2 = |C_1|^2 + |C_2|^2 + \underbrace{C_1^* C_2 \, e^{-i(\omega_g-\omega_h)t} \langle g|h\rangle}$$

$= \text{zero!}$

since $\langle g|h\rangle = 0$

— orthogonal states

Amount of information $\Rightarrow$ Fringe Contrast / Sharpness

Go Back to the experiment of Walther et al. By turning the Pockel's Cell <u>on</u> (closed) one could tell the paths and thus the interference when away.

Now to the cool stuff . . . .

Single Mode States

E/M $\Rightarrow$ Harmonic oscillators

represent E + B as operators

Consider radiation field in 1D Cavity
perfecting conducting walls.

$$E_x(z,t) = \sqrt{\frac{2\omega^2}{V\epsilon_0}}\; q(t) \sin(kz)$$

$$B_y(z,t) = \frac{\mu_0 \epsilon_0}{k}\sqrt{\frac{2\omega^2}{V\epsilon_0}}\; \dot{q}(t) \cos(kz)$$

Total Energy $\quad \cancel{\frac{1}{2}\epsilon_0 E\cdot E + \frac{1}{2}\mu_0 B\cdot B}$ $\quad (D\cdot H)$

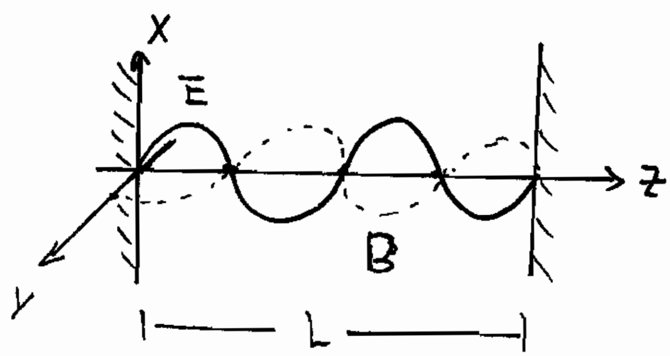
$$\frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) = W$$

$$H = \frac{1}{2}\int dV \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

$$H = \frac{1}{2}\left(p^2 + \omega^2 q^2\right) \Leftarrow$$ Hamiltonian For simple harmonic motion

Field Quantization

- Consider light in a cavity of perfectly conducting walls
- Get standing waves
- Electric field along $\hat{x}$, Magnetic Field along $\hat{y}$



$$E_x(z,t) = \left(\frac{2\omega^2}{V\varepsilon_0}\right)^{1/2} q(t)\ \sin(kz)$$

$$B_y(z,t) = \frac{\mu_0\varepsilon_0}{k}\left(\frac{2\omega^2}{V\varepsilon_0}\right)^{1/2} \dot{q}(t)\ \cos(kz)$$

$$\left\{\begin{array}{l} k = \frac{\omega n}{c} \quad n=1 \\ \omega_m = c\left(\frac{m\pi}{L}\right) \\ V \equiv \text{volume} \end{array}\right.$$

$q(t) \Rightarrow$ canonical position

$\dot{q}(t) = p(t) \Rightarrow$ Canonical momentum

Classical Energy or Hamilitonian

$$H = \frac{1}{2}\int dV \left(\varepsilon_0 E_x^2 + \frac{1}{\mu_0} B_y^2\right)$$

Rewrite in terms of Canonical terms

$$H = \frac{1}{2}\left(p^2 + \omega^2 q^2\right)$$

The system we have described in a harmonic oscillator. ~~To~~ ~~[illegible]~~ Thus we can use our quantum description of the harmonic oscillator

$$[\hat{q}, \hat{p}] = i\hbar$$

<u>Quantized fields</u>

$$\hat{E}_x(z,t) = \left(\frac{2\omega^2}{V\epsilon_0}\right)^{1/2} \hat{q}(t) \sin(kz)$$

$$\hat{B}_y(z,t) = \left(\frac{2\omega^2}{V\epsilon_0}\right)^{1/2} \hat{p}(t) \cos(kz)$$

$$\hat{H} = \frac{1}{2}\left(\hat{p}^2 + \omega^2 \hat{q}^2\right)$$

$\hat{p}$ + $\hat{q}$ are Hermitian (observables)

Introduce non Hermitian operators

$\hat{a}^+$ creation

$\hat{a}$ annihilation

$$\boxed{\begin{aligned} a^+ &= \frac{1}{\sqrt{2\hbar\omega}}\left(\omega\hat{q} - i\hat{p}\right) \\ a &= \frac{1}{\sqrt{2\hbar\omega}}\left(\omega\hat{q} + i\hat{p}\right) \end{aligned}}$$

so

$$\hat{E}_x = \mathcal{E}_0\left(\hat{a} + \hat{a}^+\right)\sin kz \qquad \mathcal{E}_0 \equiv \sqrt{\hbar\omega/\epsilon_0 V}$$

$$\hat{B}_y = B_0 \frac{1}{i}\left(a - a^+\right)\cos kz \qquad B_0 \equiv \frac{\mu_0}{k}\sqrt{\frac{\epsilon_0\hbar\omega^3}{V}}$$

Also $[\hat{a}, \hat{a}^+] = 1$

$$\boxed{H = \hbar\omega\left(\hat{a}^+\hat{a} + 1/2\right)}$$

Time Dependence of $\hat{a}$ + $\hat{a}^+$

Heisenberg picture $\Rightarrow$ Liouville Eq (as we did before)

$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar}[H, \hat{a}] = -i\omega\hat{a}$$

Solution $\boxed{\hat{a}(t) = \hat{a}(0)e^{-i\omega t}}$

For $\hat{a}^+$ $\boxed{\hat{a}^+(t) = \hat{a}^+(0)e^{i\omega t}}$

Show $\hat{H}\hat{a} = \hat{a}\hat{H} - \hat{a}\hbar\omega$

$\hat{H}\hat{a}^+ = \hat{a}^+\hat{H} + a^+\hbar\omega$

Number Operator $\hat{n}$

$\hat{n} \equiv a^+a \Rightarrow$ eigenstate $|n\rangle$ with energy E_n

Thus $\hat{H}|n\rangle = \hbar\omega(\hat{a}^+\hat{a} + \frac{1}{2})|n\rangle = E_n|n\rangle$

write $\hbar\omega\left(\hat{a}^+a + \frac{1}{2}\right)(a^+|n\rangle) = (E_n + \hbar\omega)(a^+|n\rangle)$

$\hat{H}(a^+|n\rangle) = (E_n + \hbar\omega)(a^+|n\rangle)$

eigenstate $\hat{a}^+|n\rangle$ has eigenvalue $E_n + \hbar\omega$

$\hat{a}^+ \Rightarrow$ creates quantum of energy $\hbar\omega$

also $\hat{H}(a|n\rangle) = (E_n - \hbar\omega)(a|n\rangle)$

So $\boxed{E_n = \hbar\omega(n + \frac{1}{2})}$

Show $E_n + \hbar\omega$ is eigenvalue of $a^\dagger|n\rangle$

$$\hat{H}|n\rangle = \hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right)|n\rangle = E_n|n\rangle$$

multilpy $a^\dagger$ on Left
$$\left\{\begin{array}{l} \hbar\omega\left(a^\dagger a + 1/2\right)|n\rangle = E_n|n\rangle \\ \hbar\omega\left(a^\dagger a^\dagger a + \tfrac{1}{2}a^\dagger\right)|n\rangle = E_n a^\dagger|n\rangle \end{array}\right.$$

$aa^\dagger - a^\dagger a = 1$ So $a^\dagger a = aa^\dagger - 1$

$$\hbar\omega\left(a^\dagger a a^\dagger - a^\dagger + 1/2\, a^\dagger\right)|n\rangle = E_n\left(a^\dagger|n\rangle\right)$$

$$\hbar\omega\left(a^\dagger a + 1/2\right)\left(a^\dagger|n\rangle\right) = \left(E_n + \hbar\omega\right)\left(a^\dagger|n\rangle\right)$$

So $$H\left(\hat{a}^\dagger|n\rangle\right) = \underset{\substack{\uparrow \\ \text{eigenvalue}}}{\left(E_n + \hbar\omega\right)}\left(a^\dagger|n\rangle\right)$$

Similarly

$$H\left(\hat{a}|n\rangle\right) = \left(E_n - \hbar\omega\right)\left(a|n\rangle\right)$$

Show $\hat{H}\hat{a}^{+} = \hat{a}^{+}\hat{H} + a^{+}\hbar\omega$ use $[a\ a^{+}] = 1$

$$H a^{+} = \hbar\omega\left(a^{+}a\,a^{+} + \tfrac{1}{2}a^{+}\right)$$

$$\left(a a^{+} = 1 + a^{+}a\right)$$

$$= \hbar\omega\left(a^{+}a^{+}a + a^{+} + \tfrac{1}{2}a^{+}\right)$$

$$= a^{+}\left(\hbar\omega\left(a^{+}a + \tfrac{1}{2}\right)\right) + \hbar\omega a^{+}$$

$$= a^{+}H + \hbar\omega a^{+}$$

Then

$$H\left(\hat{a}^{+}|n\rangle\right) =$$

$$= \left(a^{+}H + a^{+}\hbar\omega\right)|n\rangle$$

$$= a^{+}H|n\rangle + \hbar\omega\, a^{+}|n\rangle$$

$$= \left(E_n + \hbar\omega\right)\left(a^{+}|n\rangle\right) \checkmark$$

Zero point energy $\quad n=0$

$$\hat{H}|0\rangle = \hbar\omega\left(\hat{a}^{+}\hat{a} + 1/2\right)|0\rangle = \frac{1}{2}\hbar\omega|0\rangle$$

$$\frac{1}{2}\hbar\omega \Rightarrow \text{zero energy}$$

$$\hat{a}|0\rangle = 0 \qquad \hat{a}^{+}|0\rangle = |1\rangle$$

Normalizing Number States

$$\hat{n}|n\rangle = n|n\rangle \qquad \langle n|n\rangle = 1$$

$$\hat{a}|n\rangle = c_n|n-1\rangle$$

$$\Rightarrow c_n = \sqrt{n}$$

$$\hat{a}^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$\boxed{\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle \qquad \hat{a}^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle}$$

$$\boxed{\boxed{|n\rangle = \frac{(\hat{a}^{+})^{n}}{\sqrt{n!}}|0\rangle}}$$

Non vanishing elements

$$\boxed{\begin{aligned}\langle n-1|\hat{a}|n\rangle &= \sqrt{n}\\ \langle n+1|a^{+}|n\rangle &= \sqrt{1+n}\end{aligned}}$$

Lecture 33 Multimode Fields

$|n\rangle$ has a well defined energy but not of field since

$$\langle n|\hat{E}_x|n\rangle = 0 \qquad (\text{expectation of } \sin())$$

But the energy density which is proportional to E^2 is not zero

$$\underbrace{\langle n|E_x^2|n\rangle}_{\text{energy density}} = 2\mathcal{E}_0^2\sin^2(kz)\,(n+\tfrac{1}{2})$$

Variance of $\hat{E}$

$$\langle(\Delta\hat{E}_x)^2\rangle = \langle\hat{E}_x^2\rangle - \langle\hat{E}_x\rangle^2 =$$

so
$$\boxed{\Delta E_x = \sqrt{2}\,\mathcal{E}_0\,\sin(kz)\sqrt{n+1/2}}$$

Which is valid for even $n=0$! $\Rightarrow$ Vacuum fluctuations

Commutation between $\hat{n}$ + $\hat{E}$

$$[\hat{n},\hat{E}_x] = \mathcal{E}_0\sin(kz)(\hat{a}^+ - \hat{a})$$

or

$$\Delta n\,\Delta E_x \geq \frac{1}{2}\mathcal{E}_0\,|\sin(kz)|\,|\langle\hat{a}^+ - \hat{a}\rangle|$$

If field is accurately known, # of photons is uncertain

$\Rightarrow$ Amplitude + phase in QM cannot be both well defined.

Classical phase space

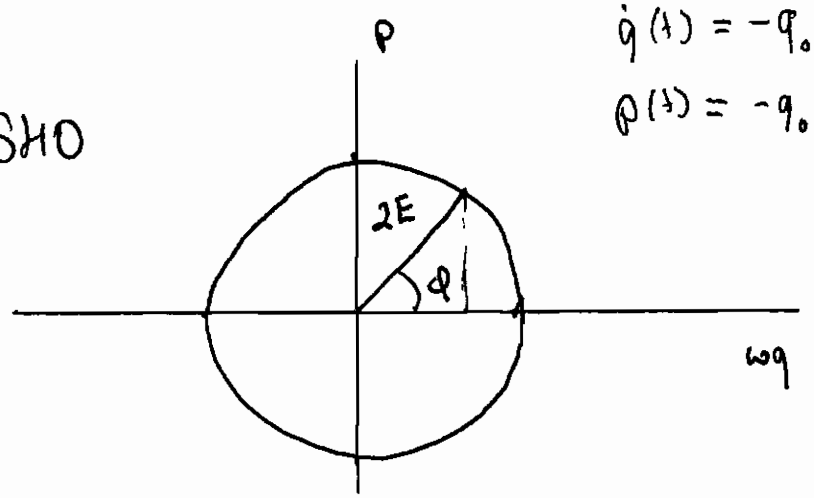
$$2E = 2H = p^2 + \omega_0^2 q^2$$

$$q(t) = q_0 \cos \omega_0 t$$

$$\dot q(t) = -q_0 \omega_0 \sin \omega_0 t$$

$$p(t) = -q_0 \omega_0 \sin \omega t$$

SHO



$$\tan\varphi = \frac{p}{\omega_0 q} = \frac{-q_0\omega_0 \sin\omega t}{\omega_0 q \cos\omega t} = -\tan \omega t = \tan(-\omega_0 t)$$

$$\boxed{\phi(t) = -\omega_0 t} \qquad \phi(t) \sim \omega_0 t$$

Classical

$$L = T - U \qquad \mathcal{L} = \mathcal{L}(q, \dot q)$$

$$= \tfrac{1}{2} m v^2 - \tfrac{1}{2} k x^2$$

$$= \tfrac{1}{2} m \dot q^2 - \tfrac{1}{2} k q^2$$

$$H(p,q) = p\dot q - \mathcal{L}(q, \dot q) \qquad \Big\{ \ \dot q = p/m$$

$$= p\frac{p}{m} - \frac{1}{2} m \frac{p^2}{m^2} + \frac{1}{2} k q^2$$

$$= p^2/2m + \frac{1}{2} k q^2 = \frac{p^2}{2m} + \frac{1}{2}\omega_0^2 m q^2$$

For number states

$$\langle X_1 \rangle = \langle n | \hat{X}_1 | n \rangle = \langle n | \hat{X}_2 | n \rangle = 0$$

$$\langle X_1^2 \rangle = \langle n | \hat{X}_1^2 | n \rangle = \langle n | \hat{X}_2^2 | n \rangle = \frac{1}{4}(2n+1)$$

Thus the varience

$$\langle (\Delta X_1)^2 \rangle = \langle X_1^2 \rangle - \langle X_1 \rangle^2$$

$$= \frac{1}{4}(2n+1)$$

For vacuum $n=0$

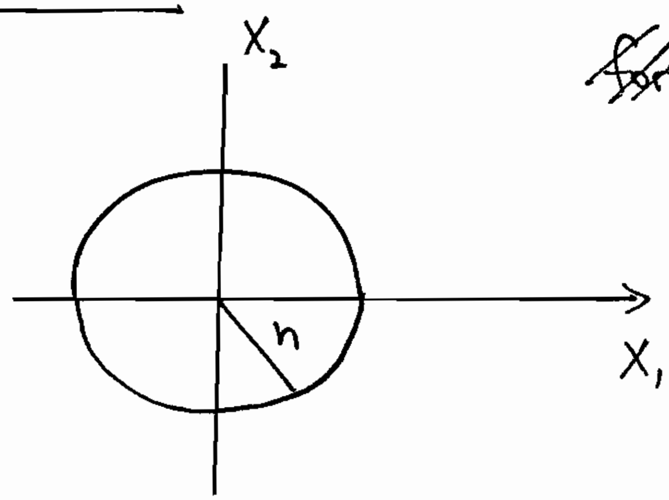
$$\boxed{\langle (\Delta X_1)^2 \rangle_{vac} = \frac{1}{4}}$$

Vacuum squeezing

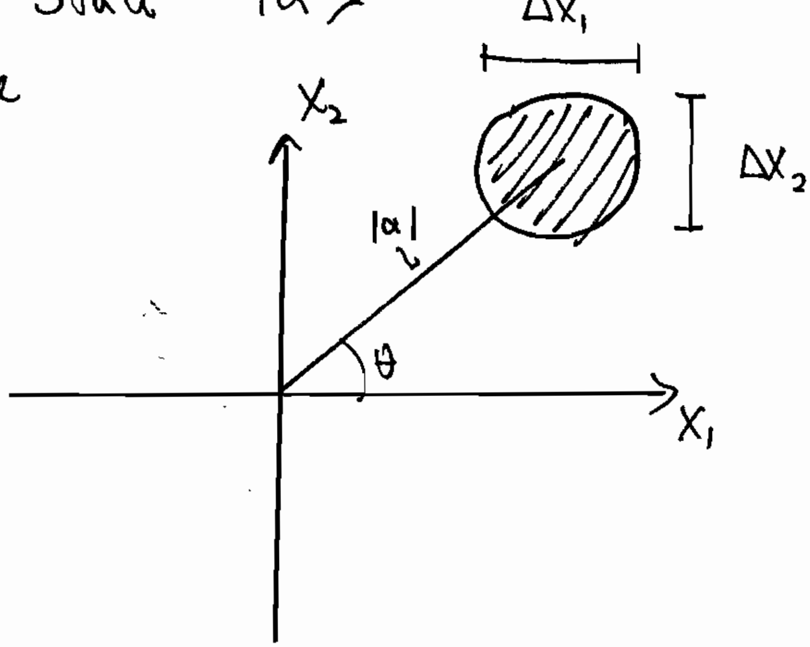
Phase space pictures

~~for these~~

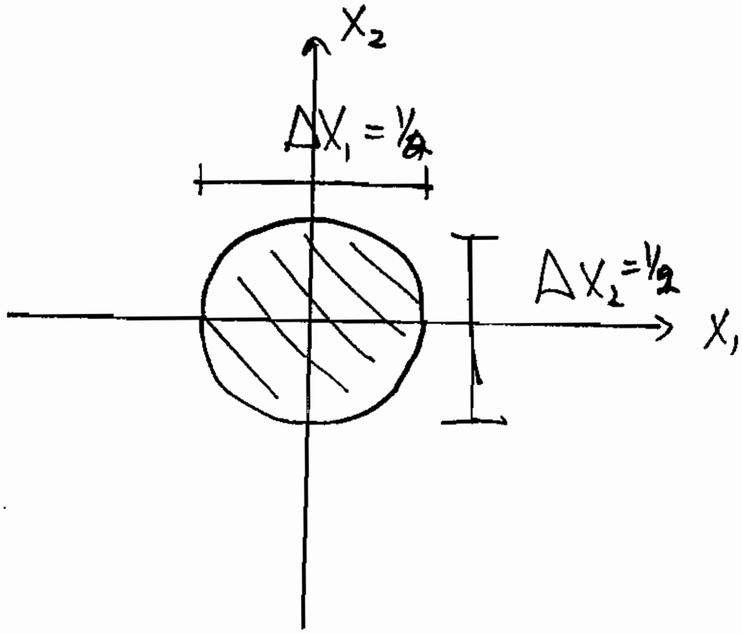
For $|n\rangle$
(number state)
a circle



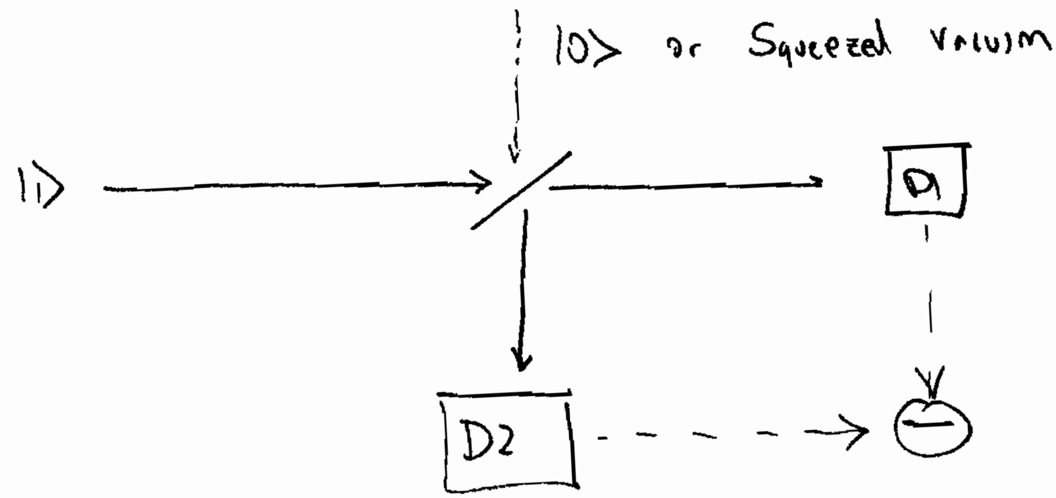
For a coherent state $|\alpha\rangle$
a filled circle



For vacuum



Balanced Detection with Squeezed Light



In analogy with $\Delta t \Delta E$ we get number/phase

$$\Delta n \Delta \phi \geq 1$$

Quadrature Operators

Write out time dependence

$$\hat{E}_x = \mathcal{E}_0 \left(\hat{a} e^{-i\omega t} + a^\dagger e^{i\omega t} \right) \sin(kz)$$

Define quadrature operators

$$\hat{X}_1 = \frac{1}{2}\left(\hat{a} + \hat{a}^\dagger\right) \qquad \text{"In-phase"}$$

$$\hat{X}_2 = \frac{1}{2i}\left(\hat{a} - \hat{a}^\dagger\right) \qquad \text{"In-quadrature"} \quad (90^\circ \text{ out of phase})$$

So $[\hat{X}_1, \hat{X}_2] = i/2 \Rightarrow \langle (\Delta X_1)^2 \rangle \langle (\Delta X_2)^2 \rangle \geq 1/16$

And $\langle n | X_1^2 | n \rangle = \frac{1}{4}(2n+1)$

in-phase in-quadrature

$$\boxed{\hat{E}_x = 2\,\mathcal{E}_0 \sin(kz)\left(\hat{X}_1 \cos\omega t + \hat{X}_2 \sin\omega t\right)}$$

~~Vacuum minimizes the uncertainty product~~

$$\cancel{\langle \hat{X}_1^2 \rangle_{vac} = 1/4 = \langle \hat{X}_2^2 \rangle}$$

~~Squeezed Vacuum~~ $\Rightarrow$ ~~Mini project~~ ~~Charmin~~

~~Number states~~

$$\cancel{\langle n | X_1^2 | n \rangle = 1/4\,(2n+1)}$$

$$\cancel{\langle n | X_2^2 | n \rangle}$$

~~So~~

$$\cancel{\langle X_1^2 \rangle = \frac{1}{4}\, n}$$

Relationship between $\cancel{X_1}$ $\hat{X}_1$ + $\hat{X}_2$ + $\hat{q}$ + $\hat{p}$

$$\hat{X}_1 = \frac{1}{2}\left(\hat{a} + \hat{a}^{+}\right)$$

$$= \cancel{\frac{1}{2}\sqrt{2\hbar\omega}} = \frac{1}{2}\frac{1}{\sqrt{2\hbar\omega}}\left[\omega\hat{q} + i\hat{p} + \omega\hat{q} - i\hat{p}\right]$$

$$\boxed{\hat{X}_1 = \omega\hat{q}\ \frac{1}{2}\frac{1}{\sqrt{2\hbar\omega}}}$$

$$\hat{X}_2 = \frac{1}{2i}\left(\hat{a} - \hat{a}^{+}\right)$$

$$= \frac{1}{2}\frac{1}{\sqrt{2\hbar\omega}}\frac{1}{i}\left(\omega\hat{q} + ip - \omega\hat{q} + i\hat{p}\right)$$

$$\boxed{\hat{X}_2 = +\frac{1}{2}\frac{1}{\sqrt{2\hbar\omega}}\hat{p}}$$

$$[X_1, X_2] = i/2$$

Review

Number states

$$\hat{H}|n\rangle = E_n|n\rangle = \hbar\omega\left(n + \tfrac{1}{2}\right)$$

$$\hat{a}^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$\hat{n} = \hat{a}^{+}\hat{a}$$

$$\langle n|\hat{n}|n\rangle = \langle n|\hat{a}^{+}a|n\rangle = \langle n|a^{+}\sqrt{n}|n-1\rangle$$

$$= \langle n|n-1+1\rangle\sqrt{n}\sqrt{n}$$

$$= n\langle n|n\rangle$$

Quadrature operators $\hat{X}_1$ + $\hat{X}_2$

$$[X_1, X_2] = i/2$$

Implies $\langle(\Delta X_1)^2\rangle < 1/4$ or $\langle(\Delta X_2)^2\rangle < 1/4$

$\boxed{\langle(\Delta X_1)^2\rangle\langle(\Delta X_2)^2\rangle \geq 1/16 \quad \text{For a general state}}$

$\boxed{\Delta X_1\,\Delta X_2 \geq 1/4}$

For number states

$$\langle n|X_2^2|n\rangle = \frac{1}{4}(2n+1) = \cancel{M}$$

$$\langle n|X_2|n\rangle = 0$$

So

$$\langle (\Delta X_1)^2 \rangle = \langle X_1^2 \rangle - \langle X_1 \rangle^2$$

$$= \frac{1}{4}(2n+1)$$

For vacuum state $|0\rangle$ + coherent state

$$\Delta X_1 \, \Delta X_2 \geq 1/4$$

$$\langle (\Delta X_1)^2 \rangle_{vac} = \frac{1}{4} = \langle X_1^2 \rangle$$

Chapter 7

We can create a squeezed vacuum state such that $|\psi_s\rangle$

$$\langle (\Delta X_1)^2 \rangle < \frac{1}{4} \quad \text{OR}$$

$$\langle (\Delta X_2)^2 \rangle < \frac{1}{4}$$

Review : Generalized uncertainty relationship

$$\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle \geq \frac{1}{4}\left|\langle[\hat{A},\hat{B}]\rangle\right|^2$$

where $\langle(\Delta\hat{A})^2\rangle$ is the varience

$$\langle(\Delta\hat{A})^2\rangle = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2$$

ΔA is called the root mean square deviation from mean

(Notice no bracket)
for specific basis

$$\Delta A \Delta B \geq \frac{1}{2}\left|\langle[A,B]\rangle\right|^2$$

Multimode Fields

Procedure

1) Write $E + B$ in terms of $\vec{A}$ (Coulomb gauge)

   Using Maxwell's eqs.

2) Consider E/m Fields in cubic cavity of length L

3) Integrate of all possible modes in 3D

4) Write $\vec{A}(r,t) = \sum_{k,s} \hat{e}_{ks}\left(A_{ks}(t)\, e^{i\vec{k}\cdot\vec{r}} + A^{+}_{ks}(t)\, e^{-i\vec{k}\cdot\vec{r}}\right)$

   Sum of plane waves

5) Write $E + B$ in terms of $\vec{A}(r,t)$ above, use Coulomb gauge

6) Find total energy using $H = \int_V \left(\frac{1}{2}\epsilon_0 E\cdot E + \frac{1}{\mu_0} B\cdot B\right) dV$

   H in terms of A

7) Write $A_{\vec{k}s}$ in terms of $q_{ks} + p_{ks}$ $\Rightarrow$ Get SHM Hamiltonian

8) Use operator form of $\hat{q}_{ks} + \hat{p}_{ks}$

9) Define $\hat{a}_{\vec{k}s} + \hat{a}^{+}_{ks}$ as before

10) Write $\hat{H}$

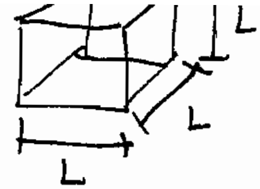
11) Write $\hat{A}_{\vec{k}s}$ in terms of $\hat{a}_{ks}$

& write $\hat{A}(\vec{r},t)$

12) Write $\vec{E}(\vec{r},t)$ & $\hat{\vec{B}}(\vec{r},t)$

in terms of $\hat{a}_{\vec{k}s}$, $\hat{a}^{\dagger}_{ks}$

Multimode Fields



Generalization of single mode result

Cubical Cavity with periodic boundary conditions

Express field in terms of the vector potential $\vec{A}$

$$\vec{E} = -\partial_t \vec{A} \quad + \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

$$\nabla^2 \bar{A} - \frac{1}{c}\partial_t^2 \bar{A} = 0$$

Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$

The boundary conditions impose

$$k_x = \frac{2\pi}{L} m_x \qquad k_y = \frac{2\pi}{L} m_y \qquad k_z = \frac{2\pi}{L} m_z$$

Total # of modes in k space

$$\Delta m = 2\left(\frac{L}{2\pi}\right)^3 \Delta k_x \Delta k_y \Delta k_z$$

$$dm = 2\frac{V}{8\pi^3} dk_x\, dk_y\, dk_z = 2\frac{V}{8\pi^3} k^2 dk \sin\theta\, d\theta\, d\varphi$$

(Go to next page)

The vector potential can be express a superposition of plane waves

$$\boxed{\vec{A}(\bar{r},t) = \sum_{\bar{k},s} \hat{e}_{\bar{k}s}\left(A_{ks}(t)\, e^{i\bar{k}\cdot\bar{r}}\ A_{\bar{k}s}(t)\, e^{-i\bar{k}\cdot\bar{r}}\right)}$$

Sum over $\vec{k}$ → sum over m

Sum over s → two polarizations orthogonal

to continuous integral $\sum_{\bar{k}} \to \frac{V}{\pi}\int k^2 dV$

Get time dependence of A_{ks} from Wave Eq

$$A_{ks}(t) = A_{ks} e^{-i\omega_k t}$$

$$dm = 2 \frac{V}{8\pi^3} \, dk_x \, dk_y \, dk_z$$

Integrate over solid angle $d\Omega = \sin\theta \, d\theta \, d\varphi$

$$\left.\begin{array}{l} \text{\# of modes} \\ \text{in all directions} \\ \text{from range} \\ \quad k \text{ to } dk \end{array}\right\} = V \frac{k^2}{\pi^2} \, dk$$

$$= V \rho_k \, dk$$

$\rho_k \, dk \equiv$ mode density

\# of modes per unit volume

~~[illegible]~~ = Using $k = \omega_k / c$

we have
$$dm = 2 \frac{V}{8\pi^3} \frac{\omega^2_k}{c^3} \, d\omega_k \, d\Omega$$

Integrate over $d\Omega$

$$\left.\begin{array}{l} \text{\# modes in all} \\ \text{directions with} \\ \quad \omega_k \text{ to } \omega_k + d\omega \end{array}\right\} \Longrightarrow V \frac{\omega_k^2}{\pi^2 c^2} \, d\omega_k = V \rho(\omega_k) \, d\omega_k$$

Write $\vec{A}$

$$\vec{A}(r,t) = \sum_{\bar{k},s} \left(A_{\bar{k}s}(t)\, e^{i\bar{k}\cdot\bar{r}} + A^*_{\bar{k}s}(t)\, e^{-i\bar{k}\cdot\bar{r}} \right)$$

— $\bar{k}$ sum over $n_x \; n_y \; n_z$

— s polarization

$$\hat{e}_{\bar{k}s} \cdot \hat{e}_{ks'} = \delta_{ss'}$$

and $\bar{k} \cdot \hat{e}_{\bar{k}s} = 0$

transversality / coulomb gauge

$$\hat{e}_{k1} \times \hat{e}_{k2} = \frac{\bar{k}}{|\bar{k}|}$$

Replace sum with $\sum_k \; \frac{V}{\pi^2} \int k^2\, dk$

Get Fields in terms of $A_{\bar{k}s}$

Write Electric + Magnetic fields

$$E(\bar{r},t) = i \sum_{\bar{k},s} \omega_k \hat{e}_{ks} \left[A_{\bar{k}s} e^{i(\bar{k}\cdot\bar{r} - \omega_k t)} - A^*_{\bar{k}s} e^{-i(\bar{k}\cdot\bar{r} - \omega t)} \right]$$

$$B(\bar{r},t) = \frac{i}{c} \sum_{\bar{k},s} \omega_k \left(\frac{\bar{k}}{|\bar{k}|} \times \hat{e}_{ks} \right) \left[\quad " \quad " \quad \right]$$

Write Hamitonian by integrating over volume Start from $H = \frac{1}{2}\iint_V \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) dV$

$$\boxed{H = 2\epsilon_0 V \sum_{\bar{k},s} \omega_k^2 A_{ks} A^*_{ks}}$$

Introduce canonical variables $\bar{q}_{\bar{k}s}$ + $\bar{p}_{\bar{k}s}$

$$\boxed{A_{\bar{k}s} = \frac{1}{2\omega_k \sqrt{\epsilon_0 V}} \left(\omega_k \bar{q}_{ks} + i \bar{p}_{ks} \right)}$$

Vector potential in terms of $\hat{q}$ + $\hat{p}$

Then $\boxed{H = \frac{1}{2} \sum_{\bar{k}s} \left(p_{ks}^2 + \omega_k^2 q_k^2 \right)}$ Classical field

operators quantize $\hat{p}_{\bar{k}s}$ + $\hat{q}_{\bar{k}s}$

$$\boxed{[\hat{q}_{\bar{k}s}, \hat{p}_{\bar{k}'s'}] = i\hbar \delta_{\bar{k}\bar{k}'} \delta_{ss'}}$$

$$\boxed{\hat{H} = \sum_{\bar{k}s} \hbar\omega_k \left(\hat{a}^\dagger_{\bar{k}s} \hat{a}_{\bar{k}s} + \frac{1}{2} \right) \qquad \hat{n}_{\bar{k}s} = \hat{a}^\dagger_{ks} \hat{a}_{\bar{k}s}}$$

same as before

$$\boxed{\begin{aligned} \hat{a}_{ks} &= \frac{1}{\sqrt{2\hbar\omega_k}}\left(\omega_k \hat{q}_{ks} + i p_{ks}\right) \\ \hat{a}^\dagger_{ks} &= \frac{1}{\sqrt{2\hbar\omega_k}}\left(\omega_k \hat{q}_{ks} - i \hat{p}_{ks}\right) \end{aligned}}$$

lead to

$$\hat{A}_{ks} = \sqrt{\frac{\hbar}{2\omega_k \epsilon_0 V}}\ \hat{a}_{ks}$$

For j^{th} mode and all modes

$$\hat{H} = \sum_j \hbar\omega_j \left(\hat{n}_j + 1/2\right)$$

Field Hamiltonian for all modes

where $\hat{n}_j = \hat{n}_{\vec{k}_j s_j}$

$\hat{n}_j \,|\, n_1\, n_2 \cdots n_j \cdots\rangle$

$\hat{n}_j |\{n_j\}\rangle$

$= n_j \,|\, n_1\, n_2 \cdots n_j \cdots\rangle$

Multimode photon number state

$$|n_1\, n_2 \cdots\rangle = |n_1\rangle|n_2\rangle|n_3\rangle \cdots$$

$\uparrow$ mode j

$$= |\{n_j\}\rangle$$

So

$$\hat{H}|\{n_j\}\rangle = E\,|\{n_j\}\rangle$$

$\uparrow$ all modes

using Loudon's notation

Multimode are orthogonal

$$\langle n_1\, n_2 \cdots n_j \cdots | n_1'\, n_2' \cdots\rangle = \delta_{n_1 n_1'}\delta_{n_2 n_2'} \cdots$$

operation by $\hat{a}_j$

$$a_j |n_1\, n_2 \cdots n_j \cdots\rangle = \sqrt{n_j}\, |n_1\, n_2 \cdots n_j - 1 \cdots\rangle$$

multimode vacuum $\quad |\{0\}\rangle = |0_1, 0_2 \cdots\rangle$

$$\boxed{|\{n_j\}\rangle = \prod_j \frac{(\hat{a}_j^{\dagger})^{n_j}}{\sqrt{n_j}}\, |\{0\}\rangle}$$

Back to the fields

complex amplitude

Now $$\boxed{\hat{A}_{\bar{k}s} = \sqrt{\frac{\hbar}{2\omega_k \varepsilon_0 V}}\ \hat{a}_{ks}}\ \cancel{\left[\hat{a}_{ks}\, e^{i(\bar{k}\cdot\bar{r}-\omega_k t)} + \hat{a}^{\dagger}_{ks}\, e^{-i(\bar{k}\cdot\bar{r}-\omega_k t)}\right]}$$

nd

$$\boxed{\hat{E}(\bar{r},t) = i \sum_{\bar{k}s} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\ \hat{e}_{\bar{k}s} \left(\hat{a}_{ks}\, e^{i(\bar{k}\cdot\bar{r}-\omega_k t)} - \hat{a}^{\dagger}_{\bar{k}s}\, e^{-i(\bar{k}\cdot\bar{r}-\omega_k t)} \right)}$$

$$\hat{a}_{ks}(t) = \hat{a}_{ks}(0)\, e^{-i\omega_k t}$$

$$\hat{E}(\bar{r},t) = i \sum_{k,s} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\ \hat{e}_{ks} \left[\hat{a}_{\bar{k}s}(t)\, e^{i\bar{k}\cdot\bar{r}} - a_{ks}(t)\, e^{-i\bar{k}\cdot\bar{r}} \right]$$

$$= \boxed{\hat{E}^{(+)}(\bar{r},t) + \hat{E}^{(-)}(\bar{r},t)}$$

$$\hat{E}^{(-)} = \left(\hat{E}^{+}\right)^{\dagger}$$

Collective annihilation

Collective creation

Thermal Fields (HBT)

Black body Source

In analogy to statistical mechanics

Classical

$$P_n = \frac{\exp(-E_n/k_BT)}{\sum_n \exp(-E_n/k_BT)} = \frac{\exp(-E_n/k_BT)}{Z}$$

Quantum

$$\hat{\rho}_{Th} = \frac{\exp(-\hat{H}/k_BT)}{Tr(\exp(-\hat{H}/k_BT))} \qquad \hat{H} = (a^\dagger a + 1/2)\hbar\omega$$

$$E_n = (n + 1/2)\hbar\omega$$

where

$$Tr(\exp(-\hat{H}/k_BT)) = \sum_{n=0}^{\infty} \langle n| \exp(-\hat{H}/k_BT) |n\rangle$$

$$= \sum_n \exp(-E_n/k_BT)$$

$$= Z \quad \text{partition Function}$$

$$\boxed{Z = \exp(-\hbar\omega/2k_BT) \underbrace{\sum_n \exp(-n\hbar\omega/k_BT)}_{}}$$

↳ sum for $\exp(\hbar\omega/k_BT) < 1$

$$\frac{1}{1 - \exp(-\hbar\omega/k_BT)}$$

3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER

COMET

$$\boxed{Z = \frac{\exp(-\hbar\omega/2k_BT)}{1 - \exp(-\hbar\omega/k_BT)}}$$

$$P_n = \langle n | \hat{\rho}_{Th} | n \rangle = \frac{1}{Z} \exp(-E_n/k_BT)$$

get classical result

Find average number of photons $\bar{n}$

$$\bar{n} = \langle \hat{n} \rangle = T_r(\hat{n}\hat{\rho}_{th}) = \sum \langle n | \hat{n} \rho_{Th} | n \rangle$$

$$= \sum_n n P_n$$

$$\frac{e^{-x}}{(1-e^{-x})^2}$$

$$= \exp(-\hbar\omega/2k_BT) \frac{1}{Z} \sum_n n \exp(-\hbar\omega n/k_BT)$$

Bose

$$\boxed{\bar{n} = \frac{1}{\exp(\hbar\omega/k_BT) - 1}} \quad \begin{cases} \bar{n} \simeq \dfrac{k_BT}{\hbar\omega} & k_BT \gg \hbar\omega \\ \bar{n} \simeq \dfrac{\hbar\omega}{k_BT} & k_BT \ll \hbar\omega \end{cases}$$

At room temp average # of photons is very small

At $T = 6000K$ $\quad \bar{n} \simeq 10^{-2}$ Light bulb.

~~Prob for finding photons~~

$$\cancel{\hat{\rho}_{Th} = \frac{1}{1+\bar{n}} \sum_n \left(\frac{\bar{n}}{1+\bar{n}}\right)^n |n\rangle\langle n|} \quad \cancel{\text{So } P_n = \frac{\bar{n}^n}{}}$$

Using

$$\hat{\rho}_{th} = \frac{1}{1+\bar{n}} \sum_n \left(\frac{\bar{n}}{1+\bar{n}} \right)^n |n\rangle\langle n| \qquad (2.138)$$

And

$$\hat{\rho}_{th} = \sum_n P_n \, |n\rangle\langle n| \qquad \text{(see next page)}$$

$$\boxed{P_n = \frac{\bar{n}^n}{(1+\bar{n})^{n+1}}} \rightarrow \text{photon distribution from HBT}$$

Fluctuations in average photon #

~~$\langle \ldots \rangle = \bar{n}$~~

$$\langle (\Delta n)^2 \rangle = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2$$

$$\langle \hat{n}^2 \rangle = \mathrm{Tr}(\hat{n}^2 \rho_{th}) = \bar{n} + 2\bar{n}^2$$

$$\text{So} \quad \langle (\Delta n)^2 \rangle = \bar{n} + \bar{n}^2$$

~~$\bar{n} = k_B T / h$~~

~~$\Delta n = (\bar{n} + \bar{n}^2)^{1/2}$~~

$$\boxed{\text{Fluctuations of } \hat{n} \text{ are larger than average } \bar{n}}$$

~~At ~~room~~ temp $k_B T \sim 25\,\text{meV}$~~

~~optical $\sim \hbar\omega \sim 1\,\text{eV}$~~

~~$\bar{n} \sim$~~

$$\boxed{\Delta n = \sqrt{\bar{n} + \bar{n}^2}} \qquad \boxed{\Delta n > \bar{n}}$$

$$\cancel{\hat{\rho}_{Th} = \sum_{n'=0}^{\infty} \sum_{n=0}^{\infty} |n'\rangle\langle n'| \rho}$$

$$\hat{\rho}_{th} = \sum_{n'=0}^{\infty} \sum_{n=0} \overbrace{|n'\rangle\langle n'|}^{\hat{1}} \rho_{Th} \overbrace{|n\rangle\langle n|}^{1}$$

$$\hat{\rho}_{Th} = \frac{1}{Z} \sum_{n} \exp(-E_n/k_BT) \, |n\rangle\langle n|$$

$$\hat{\rho}_{Th} = 1 \sum P_n |n\rangle\langle n|$$

But

$$\exp(-\hbar\omega/k_BT) = \frac{\bar{n}}{1+\bar{n}} \qquad E_n = (1/2 + n)\hbar\omega$$

$$\cancel{Z = \frac{\bar{n}}{1+\bar{n}} \quad \frac{1}{1 - \frac{\bar{n}}{\bar{n}+1}} =}$$

So

$$\exp(-\hbar\omega n / k_BT) = \left(\frac{\bar{n}}{1+\bar{n}}\right)^n$$

$$Z = \cancel{1+\bar{n}}$$

COMET 3-0235 — 50 SHEETS — 5 SQUARES 3-0236 — 100 SHEETS — 5 SQUARES 3-0237 — 200 SHEETS — 5 SQUARES 3-0137 — 200 SHEETS — FILLER

At room temp

$k_B T \sim 25\,\text{meV}$

$\hbar\omega \sim 2\,\text{eV}$

$\bar{n}$ Big or small #s at room temp.

$$\boxed{\bar{n} \sim 1.8 \times 10^{-35}}$$

$$\frac{\hbar\omega}{k_B T} \simeq 80 = \frac{2}{0.025} \simeq 2 * 40$$

$$\bar{n} = \frac{1}{\exp(80) - 1} \simeq \exp(-80) \simeq 10^{-35}$$

$$\boxed{\Delta\bar{n} = 10^{-18}}$$

$$\boxed{\frac{\Delta\bar{n}}{\bar{n}} \simeq 10^{17} \simeq \frac{1}{\sqrt{\bar{n}}}}$$

Flucuations dominate the Source

$\boxed{T = 3000}$

$$\left\{ \begin{array}{l} hc = 1240\ \text{eV}\cdot\text{nm} \\ k_B = 8.62\times10^{-5}\ \text{eV}/K \end{array} \right.$$

$\hbar\omega = 2\,\text{eV}$

$k_B T = 0.25\ \text{eV}$

$$\frac{\hbar\omega}{k_B T} = 7.73$$

$\bar{n} \simeq 0.00437$ photons at 2eV

$\Delta\bar{n} \simeq 0.0209$

$$\boxed{\frac{\Delta\bar{n}}{\bar{n}} \simeq 47}$$

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~~To get Plank's Law~~

$$\langle \bar{E} \rangle = \hbar \omega \bar{n}$$ Average energy

To get Plank's Law

multiply average energy of photons by the density of modes in ω in a unit volume

$$\bar{U}(\omega) = \overbrace{\hbar \omega \bar{n}}^{\text{Ave Energy}} \ \overbrace{\frac{\omega^2}{\pi^2 c^3}}^{}$$ Eq. (2.75)

$$\boxed{\bar{U}(\omega) = \frac{\hbar \omega^3}{\pi^2 c^3} \frac{1}{\exp(\hbar\omega/k_B T) - 1}}$$

Black body spectrum

Lecture 34 Coherent States

Review $\langle n | E_x | n \rangle = 0$

Brief discussion on the quantum phase

Dirac $\quad \hat{a} = e^{i\hat{\phi}} \sqrt{\hat{n}} \qquad \hat{a}^{+} = \sqrt{\hat{n}}\, e^{-i\hat{\phi}}$

$$[\hat{n}, \hat{\phi}] = i \qquad \Delta n \, \Delta \phi \geq 1/2$$

Problems with this definition:

1) $\hat{\phi}$ is not Hermitian

Eg. If $\hat{\phi}$ is Hermitian then $e^{i\hat{\phi}}$ is unitary

But $(e^{i\hat{\phi}})^{+} e^{i\phi} = 1$ <u>but</u> $e^{i\hat{\phi}} (e^{i\hat{\phi}})^{+} \neq 1$

2) Is $\hat{\phi}$ an angle operator?

Its <u>not</u> periodic $\quad -\infty < \phi < \infty \qquad \psi(\phi) \neq \psi(\phi + 2\pi)$

And $\Delta \phi > 2\pi$

Another approach Susskind - Glogower operators (SG)

operators analogous to exponential phase factor $e^{i\phi}$

$$\hat{E} \rightarrow e^{i\phi} \qquad \hat{E} = \sum_{n=0}^{\infty} |n\rangle\langle n+1| = (\hat{a}\hat{a}^{+})^{-1/2} \hat{a}$$

$$\hat{E}^{+} \rightarrow e^{-i\phi} \qquad \hat{E}^{+} = \sum_{n=0}^{\infty} |n+1\rangle\langle n| = \hat{a}^{+} (\hat{a}\hat{a}^{+})^{-1/2}$$

not field

Vacuum spoils the unitarity of $\hat{E}$

$$\hat{E}\hat{E}^{+} = 1 \qquad \hat{E}^{+}\hat{E} = 1 - |0\rangle\langle 0|$$

Unitary for large $n \Rightarrow$ approx unitary !!

3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER
COMET

Quantum phase (Loudon) (SG operators)

$$\boxed{\begin{aligned} \exp(i\hat{\phi}) &= (\hat{n}+1)^{-1/2}\,\hat{a} \\ \exp(-i\hat{\phi}) &= \hat{a}^{\dagger}\,(\hat{n}+1)^{-1/2} \end{aligned}}$$

Use instead of $\hat{E}$ + $\hat{E}^{\dagger}$

$$\exp(i\hat{\phi})\exp(-i\hat{\phi}) = 1$$

but

$$\exp(-i\hat{\phi})\exp(i\hat{\phi}) \neq 1$$

For number states

Not unitary

$$\exp(-i\hat{\phi})\exp(i\hat{\phi}) = 1 - |0\rangle\langle 0|$$

$\exp(i\hat{\phi})$ + $\exp(-i\hat{\phi})$ are not Hermitian

Write Hermitian operators $\hat{C}$ + $\hat{S}$

$$\hat{C} = \cos\hat{\phi} = \frac{1}{2}\left(\exp(i\hat{\phi}) + \exp(-i\hat{\phi})\right)$$

$$\hat{S} = \sin\hat{\phi} = \frac{1}{2i}\left(\exp(i\hat{\phi}) - \exp(-i\hat{\phi})\right)$$

$$[\cos\hat{\phi}, \sin\hat{\phi}] = \left(\hat{a}^{\dagger}(\hat{n}+1)^{-1}\hat{a} - 1\right)/2i$$

For $|n\rangle \Rightarrow [\cos\hat{\phi}, \sin\hat{\phi}] = \frac{1}{2}i\ |0\rangle\langle 0|$

Commutes for all states except vacuum

$$\cos^2\hat{\phi} + \sin^2\hat{\phi} = 1 - \frac{1}{2}|0\rangle\langle 0|$$

Relationship between phase + number of photons

$$[\hat{n}, \exp(i\hat{\phi})] = -\exp(+i\hat{\phi})$$

$$[\hat{n}, \exp(-i\hat{\phi})] = \exp(-i\hat{\phi})$$

$$[\hat{n}, \cos\hat{\phi}] = -i \sin\hat{\phi}$$

$$[\hat{n}, \sin\hat{\phi}] = i\cos\hat{\phi}$$

So

$$\boxed{\begin{aligned} \Delta n \, \Delta(\sin\hat{\phi}) &\geq \tfrac{1}{2} \left| \langle \cos\hat{\phi} \rangle \right| \\ \Delta n \, \Delta(\cos\hat{\phi}) &\geq \tfrac{1}{2} \left| \langle \sin\hat{\phi} \rangle \right| \end{aligned}} \quad \Big\} \; \Delta \text{ - rms deviation}$$

For number states $\Delta n = 0$ (Get $0 = 0$)

$$\langle n | \sin\hat{\phi} | n \rangle = \langle n | \cos\hat{\phi} | n \rangle = 0$$

$$\langle n | \sin^2\hat{\phi} | n \rangle = \langle n | \cos^2\hat{\phi} | n \rangle = \begin{cases} 1/2 & n \geq 1 \\ 1/4 & n = 0 \end{cases}$$

So for $n \geq 1$ $\quad \Delta(\sin\hat{\phi}) = \dfrac{1}{\sqrt{2}} = \Delta(\cos\hat{\phi})$

~~corresponds~~ means that to a phase angle equally likely to have <u>any</u> value from 0 to 2π.

11/11/10

Review of number states $|n\rangle$ $\qquad \hat{n}|n\rangle = n|n\rangle$

~~Field/number~~ Amplitude / # of photons

$\langle n|\hat{E}_x|n\rangle = 0$

$\langle n|E_x^2|n\rangle = 2\mathcal{E}_0^2 \sin^2 kz \,(n+1/2)$

$\boxed{\text{Number of photons is well defined}}$

So $\langle(\Delta E)^2\rangle = \langle E_x^2\rangle - \langle E_x\rangle^2 = 2\mathcal{E}_0 \sin^2 kz\,(n+1/2)$

$\Delta E = \sqrt{2}\,\mathcal{E}_0 \sin^2 kz \sqrt{n+1/2} \equiv \Delta E_n = \sqrt{\dfrac{2\hbar\omega}{\epsilon_0 V}}\sqrt{n+1/2}$

Phase

~~$\Delta n\, \Delta(\sin\hat\phi) \geq 1/2\, |\langle\cos\hat\phi\rangle|$ (in general)~~

~~But For # states $\Delta n = 0$ + $\langle\cos\hat\phi\rangle = 0$~~

Phase

$\langle n|\sin\hat\phi|n\rangle = \langle n|\cos\hat\phi|n\rangle = 0$

$\langle n|\sin^2\hat\phi|n\rangle = \langle n|\cos^2\phi|n\rangle = \begin{cases} 1/2 & n\geq 1 \\ 1/4 & n=0 \end{cases}$

$[\cos\hat\phi, \sin\hat\phi] = \frac{1}{2} i\, |0\rangle\langle 0|$

So For $n \geq 1$ $\quad \Delta(\sin\hat\phi) = 1/\sqrt{2}$

number state is equally likely to have <u>any</u> phase from 0 to 2π.
phase is <u>not</u> well defined for $|n\rangle$

Commutation between n + ϕ

$\Delta n\ \Delta(\sin\hat\phi) \geq 1/2\, |\langle\cos\hat\phi\rangle|$ In general

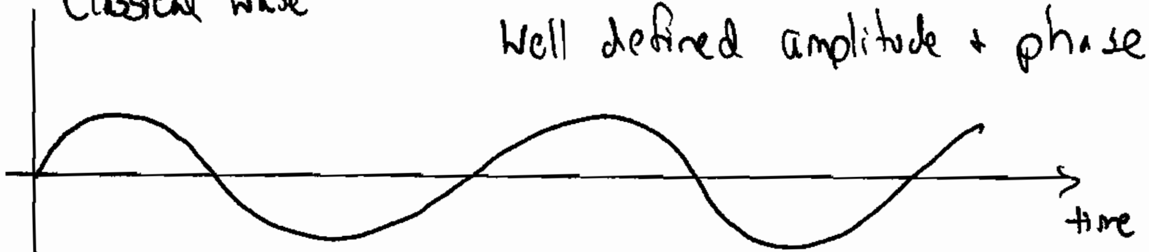
For $|n\rangle$: $\Delta n \to 0$, $\langle\cos\hat\phi\rangle \to 0$

but $\Delta(\sin\hat\phi) = 1/\sqrt{2}$ (Not ∞)

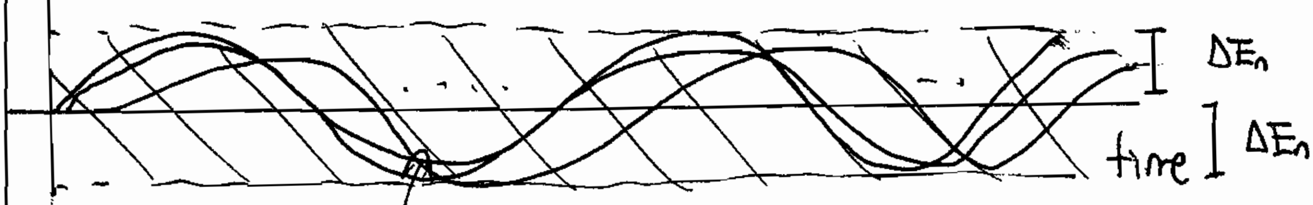
Can we draw a picture of the electric field of a number state?

Classical wave

Well defined amplitude + phase



Number state $|n\rangle$



made up of sine waves of same amplitude but phase from 0 to 2π

At any point $\langle n|\hat{E}_x|n\rangle = 0$

but

$$\langle n|\hat{E}_x^2|n\rangle \neq 0$$

$$\Delta E = \Delta E_n$$

$[\cos\hat{\phi}, \sin\hat{\phi}] = \frac{1}{2} i \, |0\rangle\langle 0|$

Problem
$\cos\hat{\phi}$ + $\sin\hat{\phi}$
do not commute
Only one element of commutation matrix does not vanish

Eigenstates of $\hat{\phi}$

$$e^{i\hat{\phi}} |\phi\rangle = e^{i\phi} |\phi\rangle$$

where $$|\phi\rangle = \sum_{n=0}^{\infty} e^{in\phi} |n\rangle$$

(Loudon $$|\phi\rangle = \lim_{s\to\infty} \frac{1}{\sqrt{1+s}} \sum_{n=0}^{s} \exp(in\phi) |n\rangle$$)

$|\phi\rangle$ not normalizable or orthogonal

$$\langle \phi | \phi' \rangle \neq \delta(\phi - \phi')$$

(Loudon def^{n} is normal + orthogonal)

$$\langle \phi | \cos\hat{\phi} | \phi \rangle = \cos\phi \quad \text{in } \lim_{s\to\infty}$$

$$\langle \phi | \sin\hat{\phi} | \phi \rangle = \sin\phi \quad \text{in } \lim_{s\to\infty}$$

$\cos\hat{\phi}$ is not a strict eigenstate of $|\phi\rangle$

$|\phi\rangle$ is <u>not</u> a strict eigenstate of $e^{i\hat{\phi}}$

because of the commutation result $[\cos\hat{\phi}, \sin\hat{\phi}]$

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Physical properties of single mode number states $|n\rangle$

n photons excited in cavity $\quad \Delta n = 0$

$$\langle n|\cos\hat{\phi}|n\rangle = \langle n|\sin\hat{\phi}|n\rangle = 0$$

$$\langle n|\cos^2\hat{\phi}|n\rangle = \langle n|\sin^2\hat{\phi}|n\rangle = \begin{cases} 1/2 & n \neq 0 \\ 1/4 & n = 0 \end{cases}$$

$$n \geq 1 \Rightarrow \Delta(\cos\hat{\phi}) = \Delta(\sin\hat{\phi}) = 1/\sqrt{2}$$

(phase is equally likely from 0 to 2π)

consistent with uncertainty relationships

Shows that E/m with $|n\rangle$ has definite amplitude but ϕ is random from 0 to 2π

$$(\Delta n)\,\Delta(\cos\phi) \geq \tfrac{1}{2}|\langle\sin\phi\rangle|$$

Consider $\hat{E}_x$ (electric field)

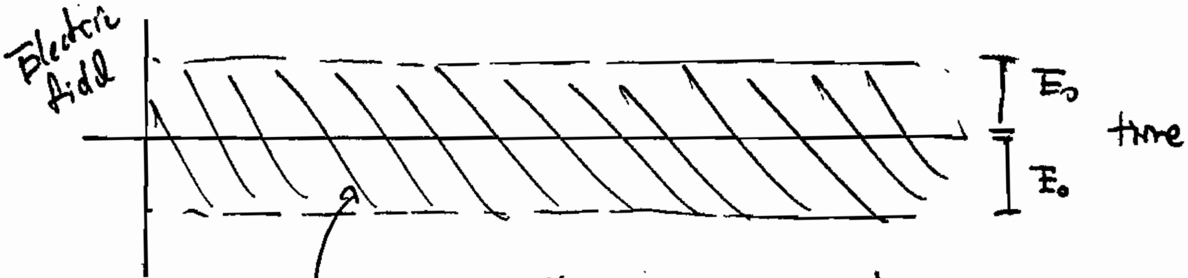
$$\langle n|\hat{E}_x|n\rangle = 0$$

$$\langle n|\hat{E}_x^2|n\rangle = \left(\hbar\omega/\epsilon_0 V\right)\left(n + 1/2\right)$$

$$\Delta E = \sqrt{\hbar\omega/\epsilon_0 V}\;\sqrt{n + 1/2}$$

well defined amplitude $E_0 = \sqrt{\dfrac{2\hbar\omega}{\epsilon_0 V}}\;\sqrt{n + 1/2}$

but not a well defined phase



sum of multiple sine waves of freq ω with phases 0 to 2π

$$\Delta E_x = E_0$$

<u>Physical properties of single mode phase states</u> $|\phi\rangle$

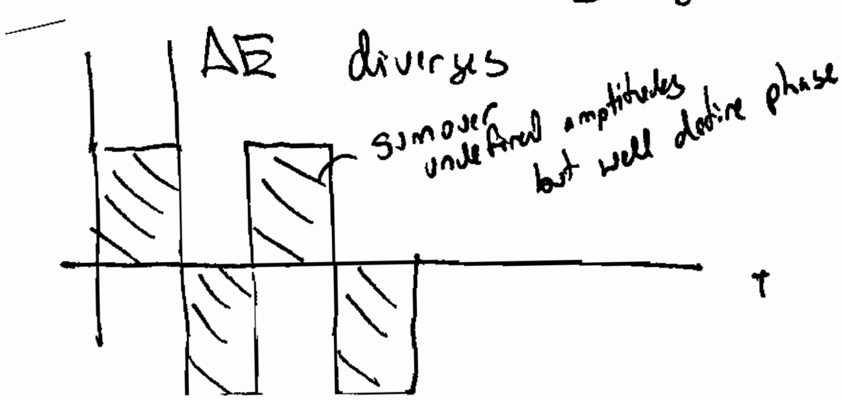
$$\Delta(\cos\phi) = \Delta(\sin\phi) = 0 \qquad \lim_{s\to\infty}$$

$$\langle\phi|\hat{n}|\phi\rangle = \lim_{s\to\infty} \frac{1}{2}s = \infty$$

$$\langle\phi|\hat{n}^2|\phi\rangle = \lim_{s\to\infty} \frac{1}{6}s(2s+1) = \infty$$

$$\langle\phi|\hat{E}_x|\phi\rangle = -2\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\sin(\;)\lim_{s\to\infty}\frac{1}{s+1}\sum_{n=0}^{s}\sqrt{n+1}$$

Diverges!



Eigenstate of $\hat{E}$

$$\hat{E}|\phi\rangle = e^{i\phi}|\phi\rangle$$

where $$|\phi\rangle = \sum_{n=0}^{\infty} e^{in\phi}|n\rangle$$

Phase distribution

$$P(\phi) \equiv \frac{1}{2\pi}|\langle\phi|\psi\rangle|^2$$

$$\int_0^{2\pi} P(\phi)\, d\phi = 1$$

How to relate this phase to experimental measurements?

$|\phi\rangle$ probability to measure phase

Measuring phase difficult classically + quantum mechanically.

- Photon number states have uniform phase distribution over range 0 to 2π. No well defined phase

Distribution of Phase for $|n\rangle$

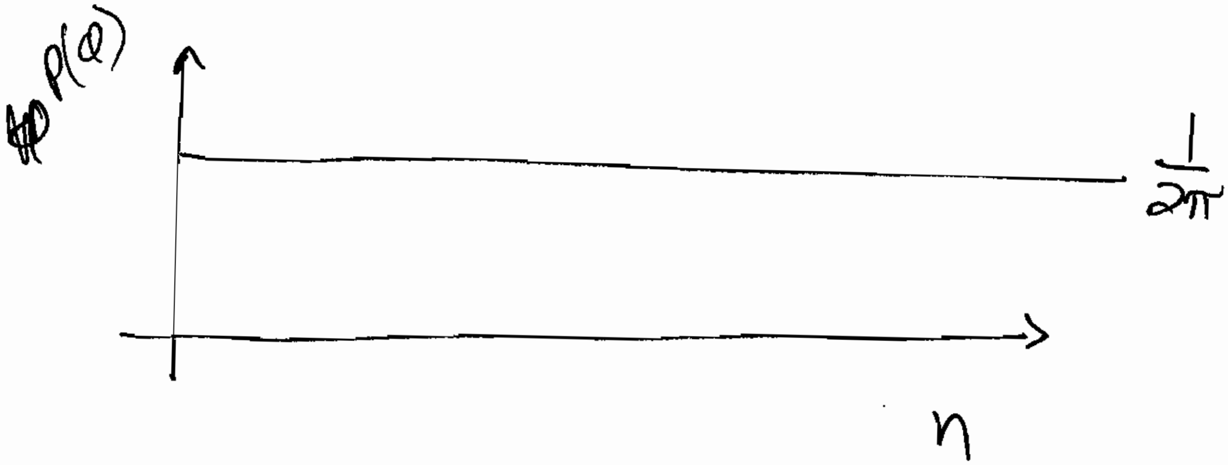
Uniform probability density

$$\Rightarrow P(\phi) = \frac{1}{2\pi} \qquad \text{all but } c_n = 1$$

with $\Delta\phi = \frac{\pi}{\sqrt{3}}$ $\qquad \langle\phi\rangle = \pi \qquad \langle\phi^2\rangle = \frac{4\pi^2}{3}$

Uniform distribution over 0 to 2π

True for all number states



niform phase distribution : Work it out

$$P(\phi) = \frac{1}{2} | \langle \phi | \psi \rangle |^2$$

$$= \frac{1}{2} \left| \sum_n \langle \phi | c_n | n \rangle \right|^2$$

$e^{i\phi}$

$$= \frac{1}{2\pi} \left| \sum_n \sum_m \langle m | e^{-im\phi} c_n | n \rangle \right.$$

$m = n$

$$= \boxed{\frac{1}{2\pi}}$$

Vacuum Fluctuations & zero pt. energy

$$\Delta E_x = \mathcal{E}_0 \sin(kz)$$

origin $\Rightarrow$ non commutability of $\hat{a}$ & $\hat{a}^\dagger$

Problem: Universe has infinite # of radiation modes each with energy $\hbar\omega/2$

$$E_{ZPE} = \frac{\hbar}{2} \sum_{\omega} \omega \rightarrow \infty$$

Energy differences $\Rightarrow$ Renormalization

ZPE $\Rightarrow$ Lamb shift, Casimir Effect, Spontaneous emission

Lamb Shift

— Theory predicts $2\,^2S_{1/2}$ & $2\,^2P_{1/2}$ levels should be degenerate.

— Experiment showed ~~they separate~~ $2\,^2S_{1/2}$ higher by 1000 MHz.

— Bethe explained this split due to the e^- interacting with ZPE

$$\Delta E/\hbar \sim 1000\,\text{MHz}$$

Electrons interact with flucuating zero point electric field and ~~grab~~ proton coulomb ~~force~~ potential

Result of Taylor expansion $\quad V(r+\Delta r) - V(r) = \Delta V$

$$\langle \Delta V \rangle = \frac{1}{6}\langle (\Delta r)^2 \rangle \nabla^2 V$$

$$\Delta E = \underbrace{\frac{1}{6}\langle (\Delta r)^2 \rangle}\ \underbrace{\langle n l m_l | \nabla^2 V | n l m_l \rangle}$$

Only contribute for wavefunction that $\psi(0) \neq 0$. Thus only S states ($l=0$) show a shift

Get $\langle (\Delta r)^2 \rangle$ by assuming that important field frequencies exceed atomic resonances, lower frequencies be sheilded by atomic binding an unable to influence the e^-. Solve for harmonic driving force

$$\boxed{\Delta E = \frac{2}{3}\left(\frac{e^2}{\hbar c}\right)^2 \frac{\hbar}{mc} \frac{hc}{\pi^2 n^3 a_0^3} \int_{\nu_0}^{\nu_f} \frac{1}{\nu}\, d\nu}$$

Freq of electron orbit $\quad \nu_0 = e^2/\hbar a_0^3 n^3 \quad \nu_f = mc^2/\hbar \quad$ (nonrelativistic)

Review

Show

$$\langle n | \hat{E}_x | n \rangle = 0 \qquad \langle \hat{E}(t) \rangle$$

We want some expectation value that varies sinusoidally with time

$$\langle \ddot{-} | E_x | \ddot{-} \rangle \simeq \sin(\omega t)$$

Need some state that looks more like the classical harmonic oscillator

Coherent States

— How to get classical limit?

We should get classical limit as $n \to \infty$

But $\langle n | E_x | n \rangle = 0$ even if $n \to \infty$!!

Fixed point in space in classical field oscillates sinusoidally

But $\langle n \rangle$ does not!

Coherent States

"most classical" quantum states of harmonic oscillator.

(Animal Farm: All states of light are quantum mechanical states, but some states are more quantum mechanical than others....)

— Want non zero expectation value of $\hat{E}_x$.

Need superposition of $|n\rangle$ differing by ± 1

Seek eigenstates of $\hat{a}$

$$\boxed{\hat{a}|\alpha\rangle = \alpha|\alpha\rangle}$$

"Right" eigenstate

$$\langle \alpha | \hat{a}^{\dagger} = \alpha^* \langle \alpha |$$

$|n\rangle \Rightarrow$ complete set so

$$|\alpha\rangle = \sum_{n=0}^{\infty} C_n |n\rangle$$

Operate by $\hat{a}$

$$\hat{a}|\alpha\rangle = \sum_{n=1}^{\infty} c_n \sqrt{n}\, |n-1\rangle = \alpha \sum_{n=0}^{\infty} c_n |n\rangle$$

so $\quad c_n \sqrt{n} = \alpha\, c_{n-1} \quad$ same $\underline{\underline{n}}$

$$c_n = \frac{\alpha}{\sqrt{n}} c_{n-1} = \frac{\alpha^2}{\sqrt{n(n-1)}} c_{n-2} = \cdots \frac{\alpha^n}{\sqrt{n!}} c_0$$

so $$\boxed{|\alpha\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle}$$

find c_0 by normalization

$$\langle\alpha|\alpha\rangle = 1 = |c_0|^2 e^{|\alpha|^2}$$

so $\quad c_0 = e^{-|\alpha|^2/2}$

Thus
$$\boxed{|\alpha\rangle = \exp(-\tfrac{1}{2}|\alpha|^2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle}$$

Write out electric field and expectation value

$$\langle\alpha|\hat{E}_x|\alpha\rangle = 2|\alpha|\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\sin(\omega t - \bar{k}\cdot\bar{r} - \theta)$$

ing

$$E = i\sqrt{\frac{\hbar}{2\varepsilon_0 V}}\left(\hat{a}\,e^{i\bar{k}\cdot\bar{r}-\omega t} + \hat{a}^\dagger e^{-i(\bar{k}\cdot\bar{r}-\omega t)}\right)$$

$$\alpha = |\alpha|e^{i\theta} \quad \left\{ \boxed{\langle\alpha|E_x|\alpha\rangle = 2|\alpha|\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{1/2}\sin(\omega t - \bar{k}\cdot\bar{r} - \theta)} \right.$$

And

$$\Delta E_x = \langle(\Delta\hat{E}_x)\rangle^{1/2} = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}$$

True for all n.

For quadrature operators

$$\langle(\Delta\hat{X}_1)^2\rangle_\alpha = \frac{1}{4} = \langle(\Delta\hat{X}_2)^2\rangle_\alpha$$

minimum uncertainty relation

Flucuation of vacuum

<u>hysical meaning of α?</u>

$|\alpha| \equiv$ related to field amplitude

$\bar{n} \equiv \langle\alpha|\hat{n}|\alpha\rangle = |\alpha|^2$ Average photon # $\boxed{\boxed{\bar{n} = |\alpha|^2}}$

$$\langle\alpha|\hat{n}^2|\alpha\rangle = \bar{n}^2 + \bar{n}$$

so $\Delta n = \sqrt{\langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2} = \sqrt{\bar{n}}$ $\Big\}$ Poissonian

(Not like thermal system)

Relationship between photons + Poisson distribution

- Bernolli variable : Photon? YES OR NO
- P_n is small
- n is large

} Detection of particles (shot noise)

Horse kick deaths : Poissonian distribution

- Bernolli varible : Death? YES/NO
- P_n is small ($\lambda = 0.61 = \frac{122}{200}$ fatality rate) deaths/corp-yr.
- n is large (200 here)

$$\boxed{P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!}}$$

λ is a rate

λT = # of events

1898 Bortkiewicz Horse kick deaths of Prussian Cavalry

10 corps
20 yrs.

# Deaths x	Observed Corp-yrs for x deaths	Poissonian with $\lambda = 0.61$
0	109	108.7
1	65	66.3
2	22	20.2
3	3	4.1
4	1	0.6

$\lambda = \frac{122}{200} = 0.61$ deaths per corp-yr.

$122 = 0(109) + 1(65) + 2(22) + 3(3) + 4(1)$

For case where $\Delta n = \sqrt{\bar{n}}$

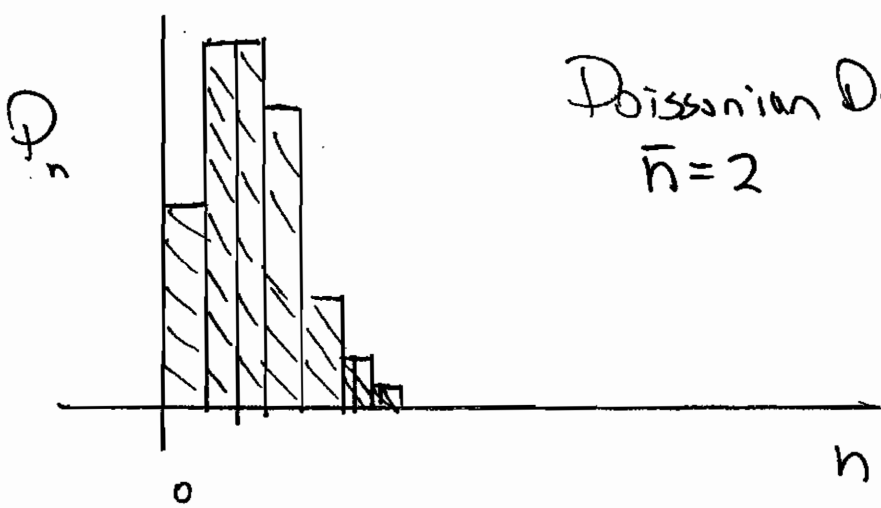
Variance = sq rt. of average

Poissonian distribution with mean $\bar{n}$

$$\frac{\Delta n}{\bar{n}} = \frac{1}{\sqrt{\bar{n}}}$$

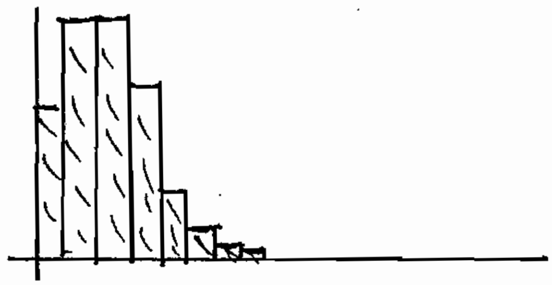
For n photons

$$\boxed{\begin{aligned} P_n = |\langle n|\alpha\rangle|^2 &= e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} \\ &= e^{-\bar{n}} \frac{\bar{n}^n}{n!} \end{aligned}}$$



Poissonian Distribution
$\bar{n} = 2$

oisson Distribution



xpectation value

$$E_x = i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\left(\hat{a}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \hat{a}^{\dagger} e^{-i(k\cdot r-\omega t)}\right)$$

$$\Downarrow$$

$$\langle\alpha|E_x|\alpha\rangle = \langle\alpha| \qquad\qquad |\alpha\rangle$$

$$\langle\alpha|\hat{a}|\alpha\rangle = \langle\alpha|\alpha\rangle\alpha$$

$$\langle\alpha|a^{\dagger}|\alpha\rangle = \alpha^{*}\langle\alpha|\alpha\rangle$$

Since
$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$
$\langle\alpha|\hat{a}^{\dagger} = \alpha^{*}\langle\alpha|$

o $$\langle\alpha|E_y|\alpha\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\left(\alpha e^{(+)} - \alpha^{*} e^{(-)}\right)$$

rite $\alpha = |\alpha| e^{i\theta}$

$$\langle\alpha|E_x|\alpha\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\,|\alpha|\left(e^{i\theta}e^{(+)} - e^{-i\theta}e^{(-)}\right)$$

$$\boxed{\langle\alpha|E_x|\alpha\rangle = 2|\alpha|\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\,\sin(\omega t - \vec{k}\cdot\vec{r} - \theta)}$$

$$\alpha|\hat{E}_x^2|\alpha\rangle = \frac{\hbar\omega}{2\varepsilon_0 V}\left(1 + 4|\alpha|^2\sin^2(\ )\right)$$

$$\Delta E_x)_\alpha = \sqrt{\langle(\Delta\hat{E}_x)^2\rangle} = \sqrt{\langle\hat{E}_x^2\rangle - \langle E_x\rangle^2} = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}$$

independent of n!!

$$\boxed{(\Delta E_x)_\alpha = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}}$$

efore

$$(\Delta E_x)_n = \sqrt{2\varepsilon_0}\,\sin(kz)\sqrt{n + 1/2}$$

- quadrature operators

$$\langle(\Delta\hat{X}_1)^2\rangle_\alpha = \frac{1}{4} = \langle(\Delta\hat{X}_2)^2\rangle_\alpha$$

oherent
States have the fluctuations of the vacuum!!

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3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER

COMET

Phase distribution of coherent states

$$P(\phi) = \frac{1}{2\pi} \left| \langle \phi | \alpha \rangle \right|^2$$

$$= \frac{1}{2\pi} e^{-|\alpha|^2} \left| \sum_{n=0}^{\infty} e^{i n(\phi-\theta)} \frac{|\alpha|^n}{\sqrt{n!}} \right|^2$$

For large n   Poissonian → Gaussian

$$P(\phi) = \sqrt{\frac{2|\alpha|^2}{\pi}} \exp\left(-2|\alpha|^2(\phi-\theta)^2\right)$$

peaked at $\phi = \theta$

Coherent states ⟺ "Near" Classical States

1) Expectation value of $\vec{E}$ has form of classical expression
2) Flucuations of $\vec{E}$ are same as vacuum.
3) Flucuations in fractional uncertainty for photon number decrease with increasing average photon #

$$\frac{\Delta n}{\bar{n}} = \frac{1}{\sqrt{\bar{n}}} \qquad \left(\frac{\Delta n}{\bar{n}} = \frac{\sqrt{\bar{n}+\bar{n}^2}}{\bar{n}} \text{ for thermal} \right)$$

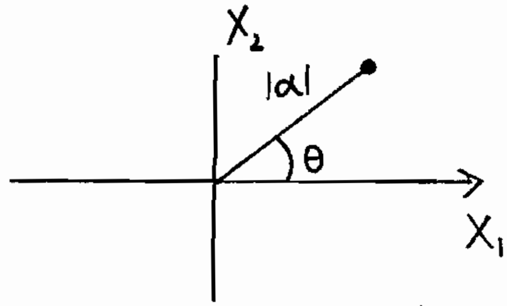
$$\bar{n} \to \infty \qquad \frac{\Delta n}{\bar{n}} \to 0 \qquad \left(\bar{n} \to \infty \quad \frac{\Delta n}{\bar{n}} \to 1 \quad \text{for thermal} \right)$$

4) States become well localized in phase space with increasing average photon number ($\bar{n}$).

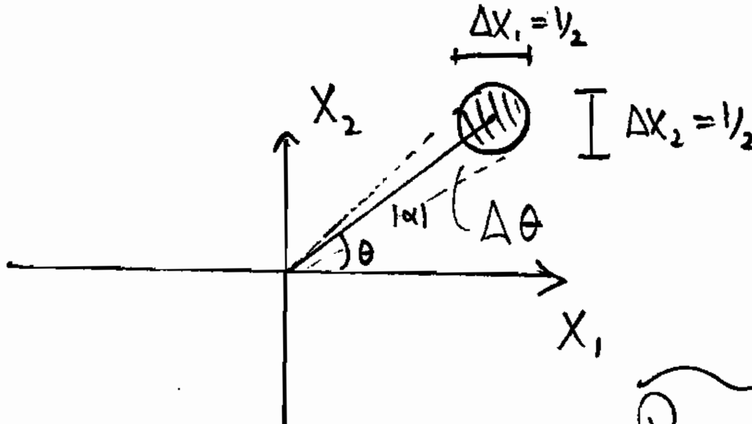
$\boxed{\text{See Fig 3.3} \Rightarrow \text{Fuzzy Sone}}$

)hase Space Pictures of Coherent States

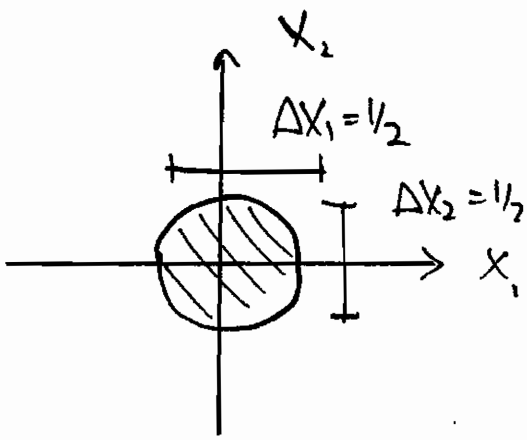
Classical Field



Quantum Field



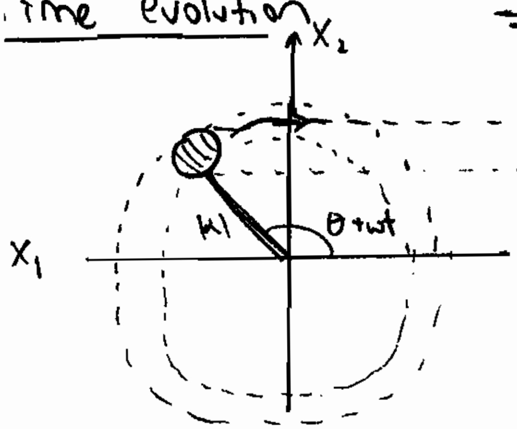
Vacuum

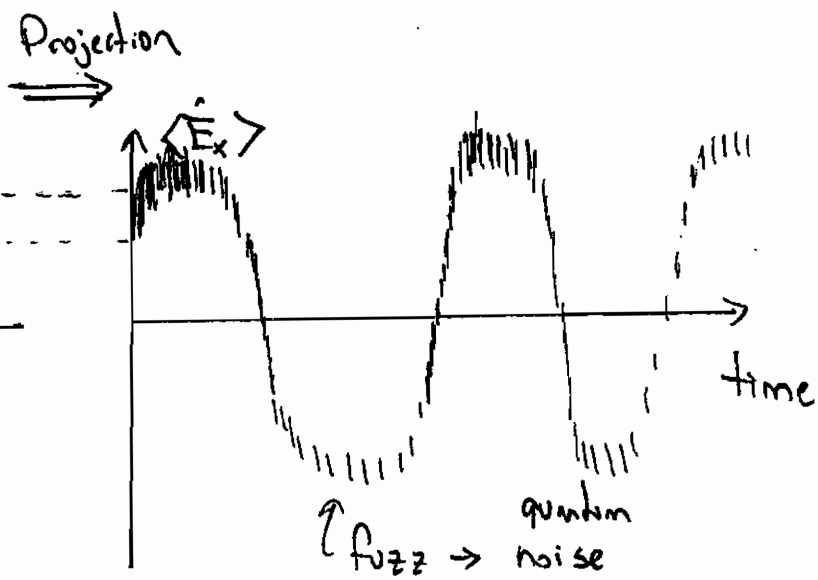


— Remember

$$\langle \alpha | E_x | \alpha \rangle \cong \sin(\omega t - \bar{k} \cdot \bar{r} - \theta)$$

Time evolution





lore properties on coherent states

Time evolution $\Rightarrow$ Coherent state remains a coherent state

$$|\alpha, t\rangle = \exp(-i \hat{H} t/\hbar) |\alpha\rangle \qquad \left\{ \begin{array}{l} \hat{H} = (\hat{a}^{\dagger}\hat{a} + 1/2)\hbar\omega \\ \hat{H} = (\hat{n} + 1/2)\hbar\omega \end{array} \right.$$

$$= e^{-i\omega t/2} e^{-i\frac{t\hbar\omega}{\hbar}\hat{n}} |\alpha\rangle$$

$$= e^{-i\omega t/2} e^{-i\omega t\hat{n}} |\alpha\rangle$$

$$= e^{-i\omega t/2} e^{-i\omega t|\alpha|^2} |\alpha\rangle \Rightarrow \text{another coherent state}$$

Coherent States are <u>not</u> orthogonal

number states are orthogonal and complete

$$|\langle\beta|\alpha\rangle|^2 = \exp(-|\beta-\alpha|^2) \neq 0 \qquad \boxed{\langle\alpha|\beta\rangle = \exp(-\tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2 + \alpha^*\beta)}$$

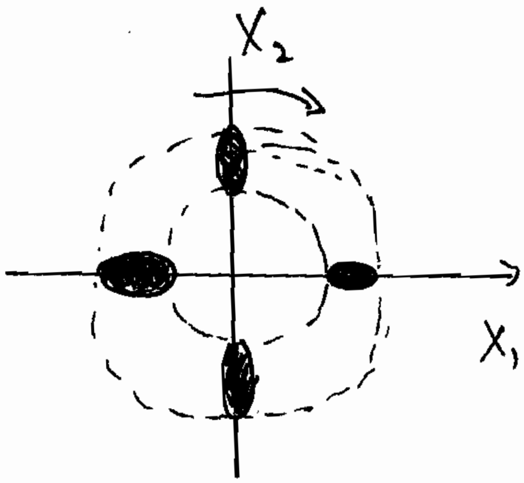
nearly orthogonal for large $|\beta-\alpha|^2$

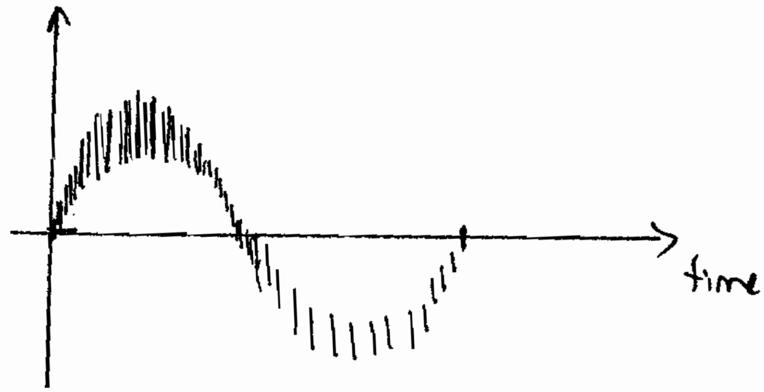
Completeness $\qquad \boxed{\int |\alpha\rangle\langle\alpha| \frac{d^2\alpha}{\pi} = 1}$

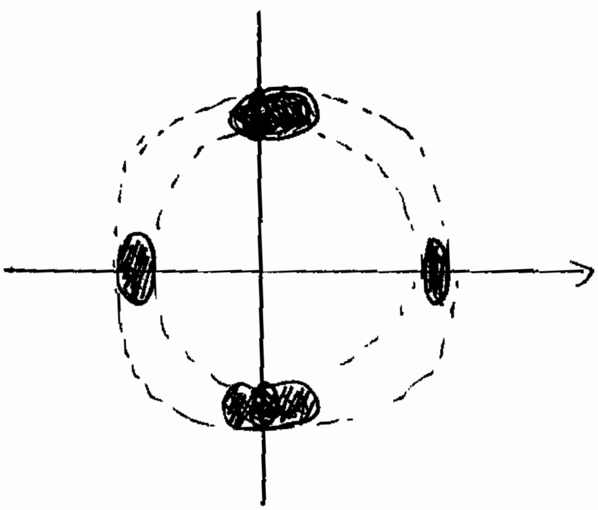
Coherent States not linearly independent

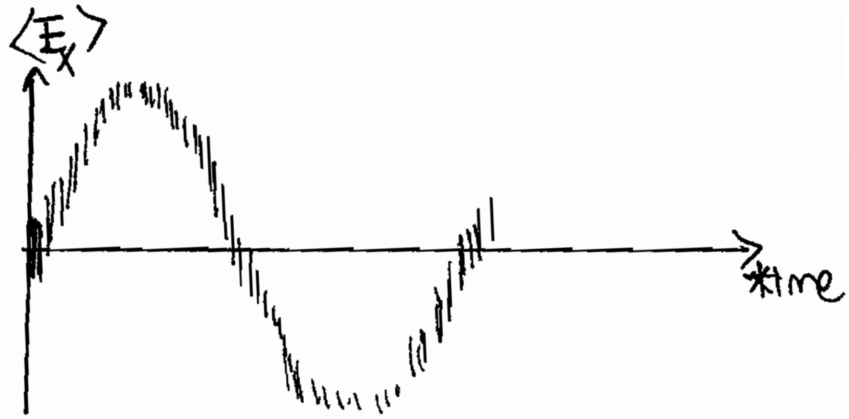
Overcomplete more than enough states

Squeezed States









$$\langle \alpha e^{-i\omega t}|\hat{E}_x|\alpha e^{i\omega t}\rangle = 2\mathcal{E}_0 \sin(kz)\,\alpha\cos\omega t$$

Time evolution + flucuations $\Rightarrow$ projection on $\langle X_1 \rangle$ axis

oherent States as Quantum Classical States

1) Expectation value has form of classical
2) Flucuations of E are same as vacum
3) Flucuations of fractional uncertainty for n decrease with increasing $\bar{n}$ $\quad \frac{\Delta n}{\bar{n}} = \frac{1}{\sqrt{\bar{n}}} \quad \Delta n = \sqrt{\overline{n^2} - \bar{n}^2}$
4) States become well localized in phase with $n \to \infty$.

ecture 35 More on coherent states

$$|\alpha\rangle = \exp\left(-\tfrac{1}{2}|\alpha|^2\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \qquad \text{Coherent States}$$

$$|\phi\rangle = \sum_{n=0}^{\infty} e^{in\phi}|n\rangle \qquad \text{Phase States}$$

10 Corps
20 yrs

oissonian Distribution

$$\boxed{\frac{e^{-\lambda}\lambda^x}{x!}}$$

# Deaths x	Observed Corp-years for x deaths
0	109
1	65
2	22
3	3
4	1

(For 1 corp 65 yrs for 1 death)

200 Corp years

122 deaths

Bortkiewicz $\Rightarrow$ Horse Kick Deaths of Prussian cavalry

n distribution

$$P(n) = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!} \Leftarrow \text{Poisson Distribution}$$

hase distribution

$$P(\phi) = \frac{1}{2\pi}|\langle\phi|\alpha\rangle|^2 = \frac{1}{2\pi}e^{-|\alpha|^2}\left|\sum_{n=0}^{\infty}e^{in(\phi-\theta)}\frac{|\alpha|^n}{\sqrt{n!}}\right|^2$$

Use $\alpha = |\alpha|e^{i\theta}$

For large $|\alpha|^2$ Poisson distribution becomes Gaussian

$$P(\phi) = \sqrt{\frac{2|\alpha|^2}{\pi}}\exp\left(-2|\alpha|^2(\phi-\theta)^2\right)$$

or n distribution   Poisson

$$\langle \alpha | \hat{n} | \alpha \rangle = \langle \alpha | a^{\dagger} a | \alpha \rangle$$

$$= \left(\left(\langle \alpha | a^{\dagger} \right) \left(a | \alpha \rangle \right) \right)$$

$$= \alpha^{*} \alpha = |\alpha|^{2} \equiv \bar{n}$$

$$\langle \alpha | \hat{n}^{2} | \alpha \rangle = |\alpha|^{4} + |\alpha|^{2} = \bar{n}^{2} + \bar{n}$$

So

$$\Delta n = \sqrt{\langle \hat{n}^{2} \rangle - \langle \hat{n} \rangle^{2}} = \sqrt{\left(\bar{n}^{2} + \bar{n} \right)^{0} - \bar{n}^{2}}$$

$$= \cancel{\bar{n}^{4} + 2\bar{n}\bar{n}^{2} + \bar{n}^{2} - \bar{n}^{2}}$$

$$= \sqrt{\bar{n}}$$

Average value $\bar{n}$

Standard deviation $\sqrt{\bar{n}}$

Binomial distribution as $n \to \infty$

Details

$$P(n) = |\langle n|\alpha\rangle|^2 = \left| \sum_{m=0}^{\infty} e^{-|\alpha|^2/2} \frac{\alpha^m}{\sqrt{m!}} \langle n|m\rangle \right|^2$$

$$= \frac{\alpha^n (\alpha^n)^*}{n!} e^{-|\alpha|^2}$$

$$= \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}$$

$$(|\alpha| e^{+i\theta})^2$$

$$P(\phi) = \frac{1}{2\pi} |\langle \phi|\alpha\rangle|^2 = \frac{1}{2\pi} \left| \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} e^{-im\phi} \langle m|n\rangle \frac{\alpha^n}{\sqrt{n!}} e^{-|\alpha|^2/2} \right|^2$$

$$= \frac{1}{2\pi} \left| \sum_{n=0} e^{+in(\phi+\theta)} \frac{|\alpha|^n}{\sqrt{n!}} \right|^2 e^{-|\alpha|^2/2 \cdot 2}$$

Lecture 11/16/10

Displacement operator

Beamsplitters + Interferometers (Chp. 6)

Read Einstein Podolsky Rosen (EPR) Phys Rev 47 1935

Lecture schedule

Nov. 16 Beamsplitters
Nov. 17 Finish Beamsplitters
Nov 18 Bell's Thm + quantum entanglement

Nov 30 Optical tests of EPR]
Dec 2 ~~Squeezed states~~ Optical tests of EPR
Dec 7 (Cancel?) Quantum coherence Functions

Dec 9]
Dec 14] Final projects

Ask:
Point of last lecture?

Review Coherent states $\langle\alpha|$
most classical of quantum states

$$\langle\alpha|\hat{E}_x|\alpha\rangle \approx 2\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}|\alpha| \sin(\omega t - \vec{k}\cdot\vec{r} - \theta)$$

$$\langle\alpha|n|\alpha\rangle = |\alpha|^2 \Big\} \quad \Delta E_x = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} \quad \text{(Same as vacuum)}$$

$$\frac{\Delta n}{\bar{n}} = \frac{1}{\sqrt{\bar{n}}}$$

$$P_n = e^{-\bar{n}} \frac{\bar{n}^n}{n!} = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} \quad \text{(Poissonian)}$$

$$P(\phi) \approx \exp(-2|\alpha|^2(\phi-\theta)^2) \quad \text{(Gaussian)}$$

<u>Review</u> : Coherent states $|\alpha\rangle$

most classical of quantum states

<u>Field</u>

$$\langle\alpha|\hat{E}_x|\alpha\rangle = 2\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\,|\alpha|\,\sin(\omega t - \vec{k}\cdot\vec{r} - \theta)$$

$$\Delta E_x = \sqrt{\hbar\omega/2\epsilon_0 V}$$

So
$$\frac{\Delta E_x}{\langle E_x\rangle} = \frac{\sqrt{\hbar\omega/2\epsilon_0 V}}{2\sqrt{\hbar\omega/2\epsilon_0 V}\,|\alpha|\sin(\,)} = \boxed{\frac{1}{2|\alpha|\sin(\,)}}$$

<u>Prob for n</u>

$$P_n = e^{-|\alpha|^2}\frac{(|\alpha|^2)^n}{\sqrt{n!}}$$

$$\langle\alpha|\hat{n}|\alpha\rangle = |\alpha|^2$$

so $\bar{n} = |\alpha|^2$

$$P_n = e^{-\bar{n}}\frac{\bar{n}^n}{\sqrt{n!}}$$

(Poissonian)

$$\Delta n = \sqrt{\bar{n}}$$

So
$$\boxed{\frac{\Delta n}{\bar{n}} = \frac{1}{\sqrt{\bar{n}}}}$$

<u>Prob for ϕ</u>

$$P(\phi) \simeq \exp(-2|\alpha|^2(\phi-\theta)^2) \quad \text{(Gaussian)}$$

width $\frac{1}{|\alpha|^2} = \frac{1}{\bar{n}}$

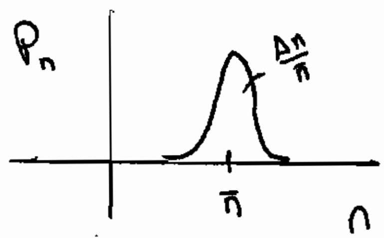
Classical limit $\bar{n} \to \infty$ for $|\alpha\rangle$

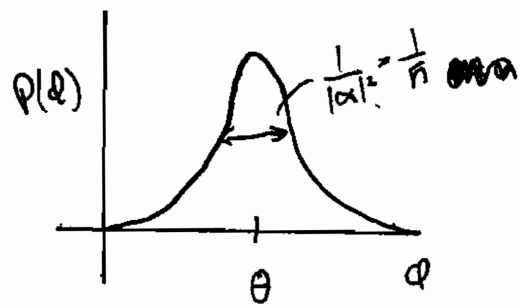
$$\frac{\Delta n}{\bar{n}} = \frac{1}{\sqrt{n}} \to 0 \qquad \Big\{ \text{Remember } |\alpha|^2 = n$$

$$P_n = e^{-\bar{n}} \frac{\bar{n}^n}{n!} \qquad \text{get narrower as } \bar{n} \to \infty$$

$$P(\phi) \cong \exp\left(-2|\alpha|^2(\phi-\theta)^2\right) \qquad \text{gets narrower as } \bar{n} \to \infty$$

The state $|\alpha\rangle$ looks more classical with a well defined amplitude & phase





The varience of the ~~...~~ electric field remains constant

$$\Delta E_x = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}$$

But

$$\frac{\Delta E_x}{\langle E_x \rangle} = \frac{\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}}{2\sqrt{\bar{n}}\,\sin(\ )\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}}$$

$$\cong \boxed{\frac{1}{\sqrt{\bar{n}}}} \qquad \text{goes to zero as } \bar{n} \to \infty$$

Becomes more 'classical'
well defined

Displaced vacuum states

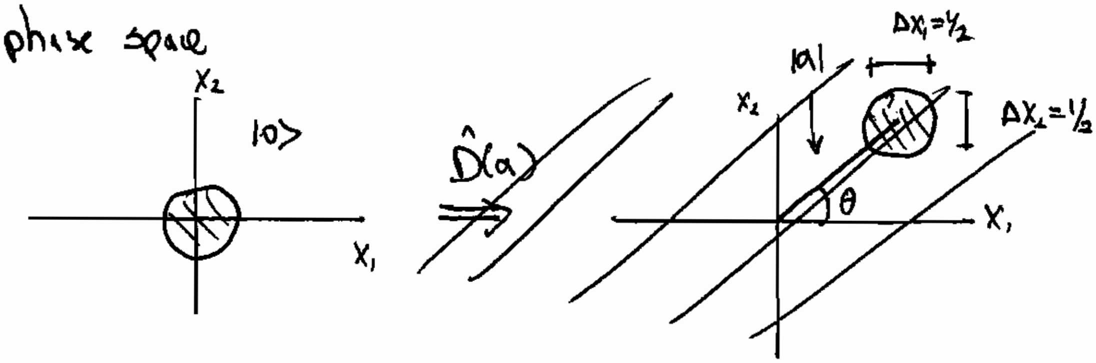
Displacement operator $\hat{D}(\alpha)$

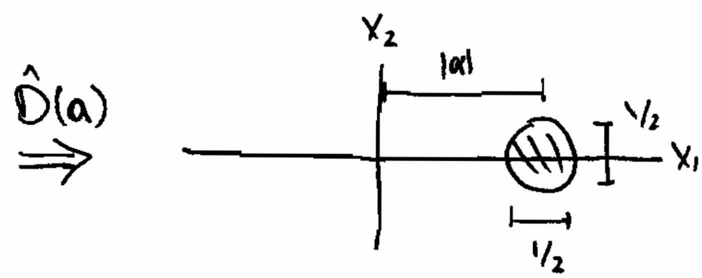
$$\hat{D}(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a})$$

Get coherent state $\quad |\alpha\rangle = \hat{D}(\alpha)|0\rangle$

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

In phase space

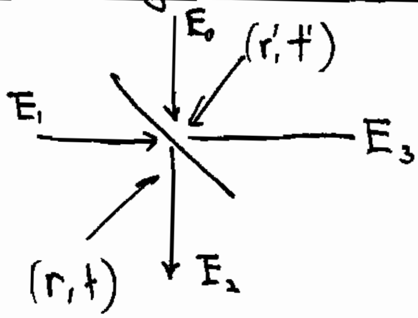




if α is real

Quantum mechanics of Beam splitters

Classically (Miniproject 1)



From Energy cons.

$|r| = |r'| \qquad |t| = |t'|$

$|r|^2 + |t|^2 = |r'|^2 + |t'|^2 = 1$

$r^* t + r' t'^* = 0$

$r^* t' + r' t^* = 0$

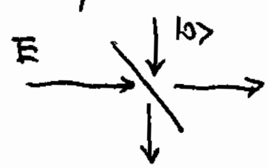
$r = |r| e^{i\theta_r} \qquad t = |t| e^{i\theta_t}$

$\theta_t - \theta_r + \theta_{t'} - \theta_r = \pm\pi$

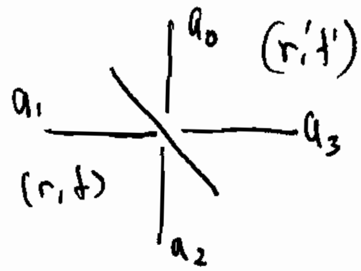
Scattering matrix

$$\begin{pmatrix} E_2 \\ E_3 \end{pmatrix} = \begin{pmatrix} t' & r \\ r' & t \end{pmatrix} \begin{pmatrix} E_0 \\ E_1 \end{pmatrix}$$

Need to write scattering matrix quantum mechanically. Keep all 4 ports even if one is vacuum state



Write scattering matrix in terms of $\hat{a}$



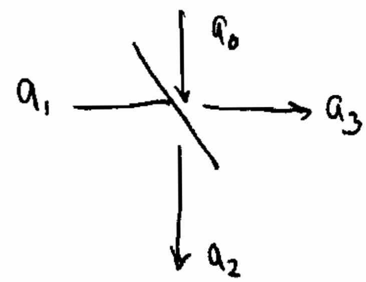
$$\hat{a}_2 = r\hat{a}_1 + t'\hat{a}_0$$

$$\hat{a}_3 = t\hat{a}_1 + r'\hat{a}_0$$

Have same relationships with r', t', r + t above.

$$\begin{pmatrix} \hat{a}_2 \\ \hat{a}_3 \end{pmatrix} = \begin{pmatrix} t' & r \\ r' & t \end{pmatrix} \begin{pmatrix} \hat{a}_0 \\ \hat{a}_1 \end{pmatrix}.$$

Example 50/50 Beamsplitter



$$\theta_r - \theta_t = \pi/2$$

$$t = t' \qquad r = r'$$

$$\left(\text{Let } t = \frac{1}{\sqrt{2}} \quad r = \frac{1}{\sqrt{2}} e^{i\pi/2}\right)$$

$$\hat{a}_2 = \frac{1}{\sqrt{2}}\hat{a}_0 + \frac{1}{\sqrt{2}} e^{i\pi/2}\hat{a}_1 = t\,\hat{a}_0 + r\,\hat{a}_1$$

$$\boxed{\hat{a}_2 = \frac{1}{\sqrt{2}}\left(\hat{a}_0 + i\,\hat{a}_1\right)}$$

$$\hat{a}_3 = r\,\hat{a}_0 + t\,a_1 = \frac{1}{\sqrt{2}} e^{i\pi/2}\hat{a}_0 + \frac{1}{\sqrt{2}}\hat{a}_1$$

$$\boxed{\hat{a}_3 = \frac{1}{\sqrt{2}}\left(\hat{a}_0 + i\,\hat{a}_1\right)}$$

If we solve for $\hat{a}_0$ + $\hat{a}_1$

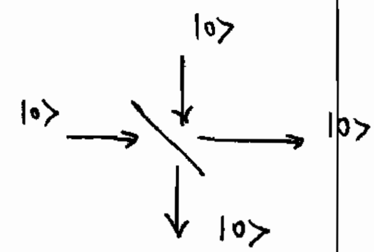
$$\hat{a}_1 = \frac{1}{\sqrt{2}}\left(-i\,\hat{a}_2 + \hat{a}_3\right)$$

$$\hat{a}_0 = \frac{1}{\sqrt{2}}\left(\hat{a}_2 + i\,\hat{a}_3\right)$$

$\dagger$

also

$$\hat{a}_1^\dagger = \frac{1}{\sqrt{2}}\left(i\,\hat{a}_2^\dagger + \hat{a}_3^\dagger\right)$$



For given input, what is the output?

Vacuum $\quad |0\rangle_0 \, |0\rangle_1 \xrightarrow{BS} |0\rangle_2 \, |0\rangle_3$

port port port port

One port numberstate

$$\cancel{|0\rangle_0 \, |1\rangle_1 \xrightarrow{BS} a_*^\dagger |0\rangle_0 |0\rangle}$$

$$|0\rangle_0 \, |1\rangle_1 = \hat{a}_1^\dagger |0\rangle_0 \, |0\rangle_0$$

use $\hat{a}_*^\dagger = \frac{1}{\sqrt{2}}(i a_2^\dagger + a_3^\dagger)$

So $$|0\rangle_0 \, |1\rangle_1 \xrightarrow{BS} \frac{1}{\sqrt{2}}\left(i \, \hat{a}_2^\dagger + \hat{a}_3^\dagger\right) |0\rangle_2 \, |0\rangle_3$$

$$\boxed{|0\rangle_0 \, |1\rangle_1 \xrightarrow{BS} \frac{1}{\sqrt{2}}\left(i \, |1\rangle_2 \, |0\rangle_3 + |0\rangle_2 \, |1\rangle_3\right)}$$

What does this say? A single photon (number state) with vacuum ~~oth~~ on the other port will be transmitted or reflected with equal probability $\Rightarrow$ Aspect Experiment!

This is an entangled state: cannot be written as simple product of states of individual modes 2 + 3.

Schrödinger Cat $\quad |\Psi_{cat}\rangle = \frac{1}{\sqrt{2}}\left(|\text{Dead}\rangle_1 \, |\text{Alive}\rangle_2 + |\text{Alive}\rangle_1 \, |\text{Dead}\rangle_2\right)$

Prob. to find photon at port 2 + 3.

$$\left| {}_2\langle 1| \, {}_3\langle 1| \left[\frac{1}{2} \left(i\, |1\rangle_2 |0\rangle_3 + |0\rangle_2 |1\rangle_3 \right) \right] \right|^2$$

$$= 0$$

Prob. to find photon at port 3

$$\left| {}_2\langle 0| \, {}_3\langle 1| \left[\tfrac{1}{2} \left(i\, |1\rangle_2 |0\rangle_3 + |0\rangle_2 |1\rangle_3 \right) \right] \right|^{1/2} = 1/2$$

~~Prob. to find photon at Port 2~~

$\cancel{\left| {}_2\langle 1| \right.}$

Probability to find photon in one arm

Let $$|\psi\rangle = \frac{1}{\sqrt{2}}\left(i\, |1\rangle_2 |0\rangle_3 + |0\rangle_2 |1\rangle \right)$$

$$\left| {}_2\langle 0|\, {}_3\langle 1|\psi\rangle \right|^2 = \frac{1}{2}\left| i\, {}_2\langle 0|1\rangle_2\, {}_3\langle 1|0\rangle_3 + {}_2\langle 0|0\rangle\, {}_3\langle 1|1\rangle \right|$$

$$= 1/2$$

$$\left| {}_2\langle 1|\, {}_3\langle 0|\psi\rangle \right|^2 = 1/2$$

photon has equal prob. to be transmitted or reflected

Aspect exp. with $A=0$

$\boxed{\text{Entangled state. No classical analog!}}$

Density Matrix for BS (mixed states)

Input
$|0\rangle$ at 0
$|1\rangle$ at 1

$$\boxed{\hat{\rho}_{01} = |1\rangle_1\, {}_1\langle 1| \quad |0\rangle_0\, {}_0\langle 0|}$$

Density matrix before BS

use this to compute the obvious....

~~Prob. to find photon @ port 1~~ Find $\hat{\rho}_0$ + $\hat{\rho}_1$

$$\hat{\rho}_0 = \mathrm{Tr}_1(\hat{\rho}_{01}) = \sum_{n=0}^{\infty} {}_1\langle n|\hat{\rho}_{01}|n\rangle_1 =$$ ← port 1

Sum over port 1 states

$$= {}_1\langle 1|1\rangle_1\, {}_1\langle 1|1\rangle_1 \quad |0\rangle_0\, {}_0\langle 0|$$

$$\boxed{\hat{\rho}_0 = |0\rangle_0\, {}_0\langle 0|}$$

Similarily
$$\boxed{\hat{\rho}_1 = \mathrm{Tr}_0(\hat{\rho}_{01}) = |1\rangle_1\, {}_1\langle 1|}$$

prob to find vacuum at port 0

$$\boxed{{}_0\langle 0|\hat{\rho}_0|0\rangle_0 = 1}$$

prob. to Find photon at port 1

$$\boxed{{}_1\langle 1|\hat{\rho}_1|1\rangle_1 = 1}$$

Compute the obvious

Density matrix ~~AA~~ after BS

output

$$\hat{\rho}_{23} = \frac{1}{2}\Big[|1\rangle_2 |0\rangle_3 \, {}_2\langle 1| \, {}_3\langle 0| + |0\rangle_2 |1\rangle_3 \, {}_2\langle 0| \, {}_3\langle 1|$$

$$+ i \left(|1\rangle_2 |0\rangle_3 \, {}_2\langle 0| \, {}_3\langle 1| - |0\rangle_2 |1\rangle_3 \, {}_2\langle 1| \, {}_3\langle 0| \right) \Big]$$

Make measurement on Port 2

$$\hat{\rho}_2 = \mathrm{Tr}_3(\hat{\rho}_{23}) = \sum_{n=0}^{\infty} {}_3\langle n|\hat{\rho}_{23}|n\rangle_3$$

sum over port 3 states

$$\hat{\rho}_2 = \frac{1}{2}\left[|0\rangle_2 \, {}_2\langle 0| + |1\rangle_2 \, {}_2\langle 1| \right]$$

OR $$\hat{\rho}_2 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$ No off diagonal terms

Probability to find photon at Port 2

$$\left|{}_2\langle 1 \,|\, \hat{p}_2 \,|\, 1\rangle_2\right| = 1/2$$

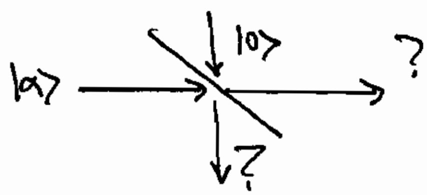
Same for Port 1

Beamsplitting a coherent state (most classical quantum state)

What to expect? Beam splitter 'splits' fields by $1/\sqrt{2}$ with $\pi/2$ phase shift

The <u>classical</u> result.

Initial State



$$|0\rangle_0 \, |\alpha\rangle_1 = \hat{D}_1(\alpha)\, |0\rangle_0 \, |0\rangle_1$$

where $\boxed{\hat{D}_1(\alpha) = \exp(\alpha \hat{a}_1^\dagger - \alpha^* \hat{a}_1)}$

$$|0\rangle_0 \, |\alpha\rangle_1 \xrightarrow{BS}$$

$$\exp\left[\frac{\alpha}{\sqrt{2}}\left(i\hat{a}_2^\dagger + \hat{a}_3^\dagger\right) - \frac{\alpha^*}{\sqrt{2}}\left(i a_2 + \hat{a}_3\right)\right] |0\rangle_2 |0\rangle_3$$

$$= \sum_{n=0}^{\infty} \left(\frac{i\alpha}{\sqrt{2}}\right)^n \frac{1}{\sqrt{n!}} \exp\left(-\tfrac{1}{2} |i\alpha/\sqrt{2}|^2\right) |n\rangle_2$$

$$* \sum_{m=0}^{\infty} \left(\frac{\alpha}{\sqrt{2}}\right)^m \frac{1}{\sqrt{m!}} \exp\left(-\tfrac{1}{2} |\alpha/\sqrt{2}|^2\right) |m\rangle_3$$

Note: use derivation below

$$= |i\alpha/\sqrt{2}\rangle_2 \; |\alpha/\sqrt{2}\rangle_3 \Rightarrow$$ output is <u>not</u> entangled.

So $\quad |0\rangle_0 \, |\alpha_1\rangle \xrightarrow{BS} \left|e^{i\pi/2}\frac{\alpha}{\sqrt{2}}\right\rangle_2 \left|\frac{\alpha}{\sqrt{2}}\right\rangle_3$

- Not entangled
- Both are coherent states with amplitude $\alpha/\sqrt{2}$ + are $\pi/2$ out of phase

Classical result

<u>Details from above</u>

$$|0\rangle_0 |\alpha\rangle_1 \xrightarrow{BS} \exp\left[\frac{\alpha}{\sqrt{2}}\left(i\hat{a}_3^\dagger + \hat{a}_3^\dagger\right) - \frac{\alpha^*}{\sqrt{2}}\left(i\hat{a}_2 + \hat{a}_3\right)\right] |0\rangle_2 |0\rangle_3$$

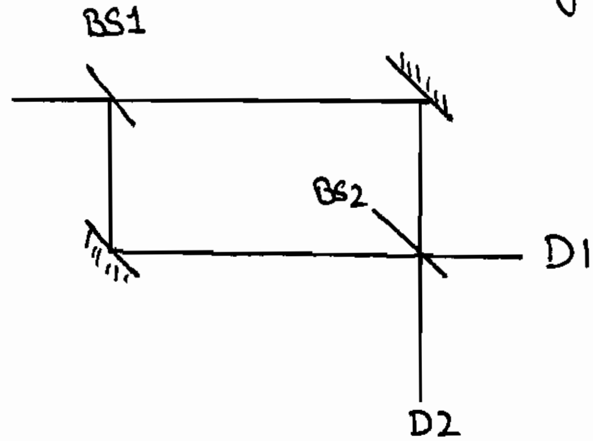
$$\xrightarrow{BS} \exp\left[\left(\frac{i\alpha}{\sqrt{2}}\right)\hat{a}_2^\dagger - \left(\frac{-i\alpha^*}{\sqrt{2}}\right)\hat{a}_2\right] \exp\left[\frac{\alpha}{\sqrt{2}}\hat{a}_3^\dagger - \left(\frac{\alpha^*}{\sqrt{2}}\right)\hat{a}_3\right] |0\rangle_2 |0\rangle_3$$

$$\xrightarrow{BS} D_2(i\alpha)\, D_3(\alpha)\, |0\rangle_2 |0\rangle_3$$

$$\xrightarrow{BS} |i\alpha/\sqrt{2}\rangle_2 \, |\alpha/\sqrt{2}\rangle_3$$

Two photons in Beam splitter

We wish to describe the experiment of Aspect et al



We have describe a BS with one photon, now we must consider a BS with two photons. $|1\rangle_0 |1\rangle_1$

$$|1\rangle_0 |1\rangle_1 \xrightarrow{BS} \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left(\hat{a}_2^\dagger + i\hat{a}_3^\dagger\right)\left(i\hat{a}_2^\dagger + \hat{a}_3^\dagger\right)|0\rangle_2 |0\rangle_3$$

Use commutation results

$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$

$[\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0$

$i,j = \{1,2,3\}$

$$= i/2\left(\hat{a}_2^\dagger \hat{a}_2^\dagger + \hat{a}_3^\dagger \hat{a}_3^\dagger\right)|0\rangle_2 |0\rangle_3$$

$$= i/2\left(|2\rangle_2 |0\rangle_3 + |0\rangle_2 |2\rangle_3\right)$$

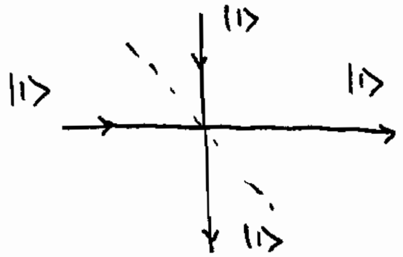
$\boxed{\text{Two photons out one port only}}$

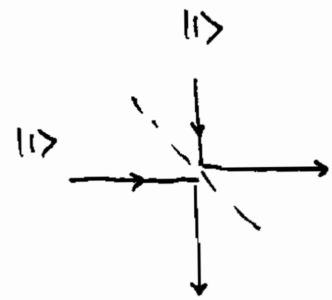
Here, two indistinguishable processes causes interference

D1 & D2 will not register simultaneous counts

- Not result of particle-like nature of photons
- Caused by interference between two ways to get the absent $|1\rangle_2 |1\rangle_3$ output state where both photons are transmitted.

Two states





Both photons transmitted

$$|1\rangle_0 |1\rangle_1 \xrightarrow{BS} |1\rangle_2 \, |1\rangle_3$$

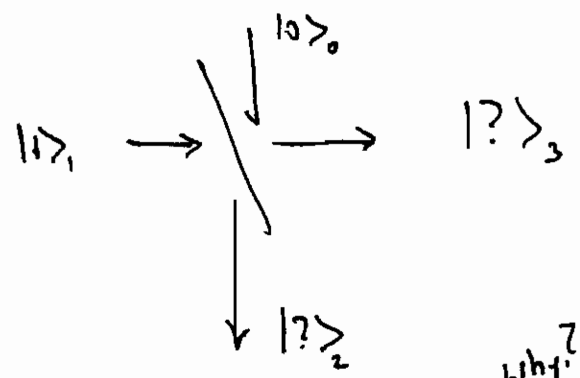
Both photons reflected

$$|1\rangle_2 \, |1\rangle_3$$

Two states interfer destructively with each other

Lecture 11/17/20

Review



Why?

$$\boxed{|0\rangle_0 \, |1\rangle_1 \xrightarrow{BS} \frac{1}{\sqrt{2}}\left(i\,|1\rangle_2 |0\rangle_3 + |0\rangle_2 |1\rangle_3 \right)}$$

~~why?~~

Why i?

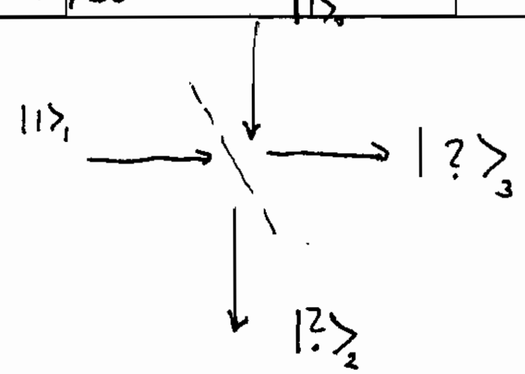
phase shift $\pi/2$

$$e^{i\pi/2} = i$$

$$50/50 \Rightarrow 1/\sqrt{2}$$

This is an entangled state

Review



$$|1\rangle_0 \, |1\rangle_1 \xrightarrow{BS} \frac{i}{2}\left(|2\rangle_2 |0\rangle_3 + |0\rangle_2 |2\rangle_3\right)$$

entangled state

So $\left| {}_2\langle 2| \, {}_3\langle 0| \psi\rangle \right|^2 = \frac{1}{2}$

$\left| {}_2\langle 0| \, {}_3\langle 2| \psi\rangle \right|^2 = \frac{1}{2}$

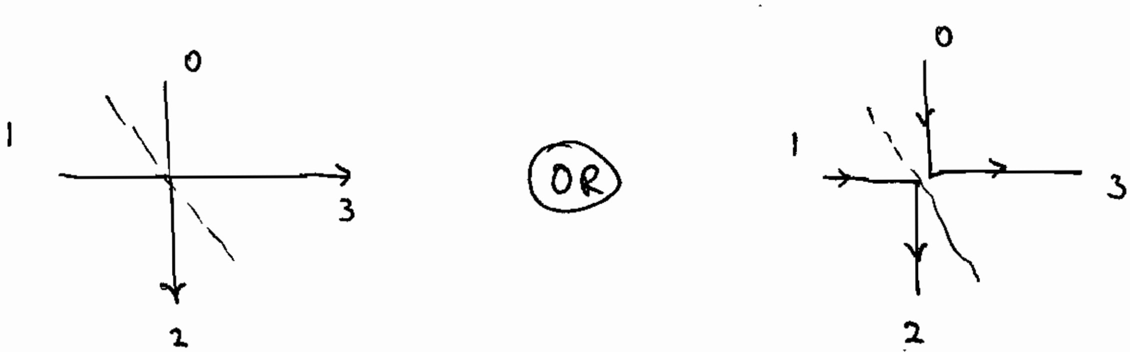
Two photons out one port

$\left| {}_2\langle 1| \, {}_3\langle 1| \psi\rangle \right|^2 = 0$

$\left| {}_2\langle 1| \, {}_3\langle 1| \psi\rangle \right|^2 = 0$

"Missing" states $|1\rangle_2 |1\rangle_3$ indistinguishable

Can get $|1\rangle_2 |1\rangle_3$ by two ways



See p. 11 Next page

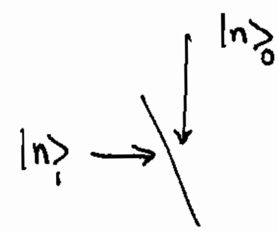
interfere to give two photons out Port 2 or 3.

Not a result of particle nature of light but of interference

Due to light being Bosons ?

B-E $\Rightarrow$ statistical properties

For n photons



the output state is <u>not</u> $\sim \;|2n\rangle_2 |0\rangle_3 + e^{i\theta} |0\rangle_2 |2n\rangle_3$

~~onephoton~~ For indistinguishable process: (Feynman)

$$P_{11} = \Big| \overbrace{A_T \; A_T}^{\text{both transmitted}} + \overbrace{A_R \cdot A_R}^{\text{both reflected}} \Big|^2$$

prob for photon at each detector

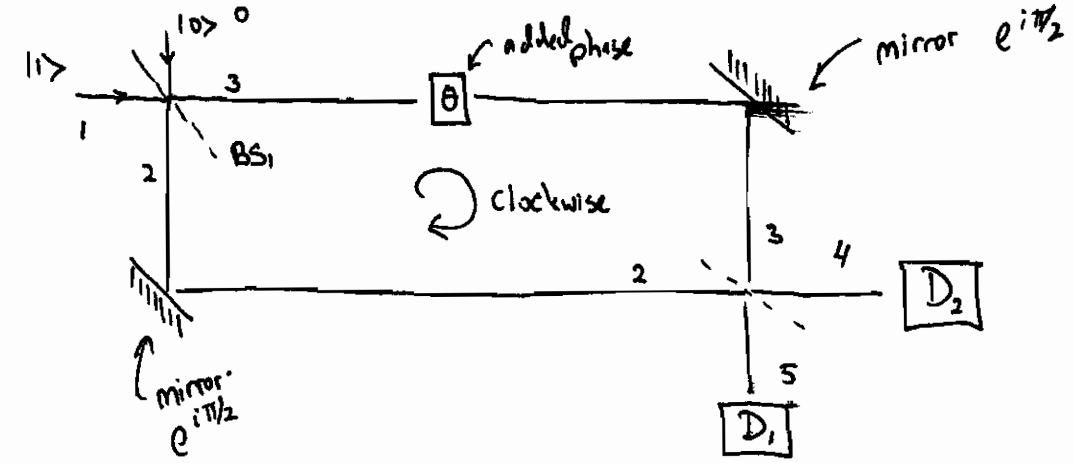
$$= \left| \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \frac{i}{\sqrt{2}} \right|^2 = 0$$

true for 50/50 ~~not~~ only

All amplitudes of Prob & compute sq of modulus

Like spectral interferometry $\quad I_{tot} = |E_1 + E_2|^2$

Aspect Experiment #2: Mach-Zehnder Interferometer



Look at action of BS1 + BS2

Input $|0\rangle_0 |1\rangle_1$

BS1 gives $\quad |0\rangle_0 |1\rangle_1 \xrightarrow{BS_1} \frac{1}{\sqrt{2}}\left(i\,|1\rangle_2|0\rangle_3 + |0\rangle_2|1\rangle_3\right)$

Clockwise path causes phase change on Part 3 components only

$$\frac{1}{\sqrt{2}}\left(|0\rangle_2 |1\rangle_3 + i\,|1\rangle_2|0\rangle_3\right) \xrightarrow{\theta} \frac{1}{\sqrt{2}}\Big(\underbrace{e^{i\theta}|0\rangle_2|1\rangle_3}_{\text{phase shift on clockwise arm only (photon on 3)}} + \underbrace{i\,|1\rangle_2|0\rangle_3}_{\text{counterclockwise}}\Big)$$

BS2 gives Individually

$$|0\rangle_2 |1\rangle_3 \xrightarrow{BS_2} \frac{1}{\sqrt{2}}\left(|0\rangle_4|1\rangle_5 + i|1\rangle_4|0\rangle_5\right)$$

$$|1\rangle_2 |0\rangle_3 \xrightarrow{BS_2} \frac{1}{\sqrt{2}}\left(|1\rangle_4|0\rangle_5 + i|0\rangle_4|1\rangle_5\right)$$

$5 \rightarrow D_1$
$4 \rightarrow D_2$

Put it all together

$$\frac{1}{\sqrt{2}}\left(e^{i\theta}|0\rangle_2 |1\rangle_3 + i|1\rangle_2 |0\rangle_3 \right) \xrightarrow{BS_2}$$

$$\frac{1}{2}\left[(e^{i\theta}-1)|0\rangle_4 |1\rangle_5 + i(e^{i\theta}+1)|1\rangle_4 |0\rangle_5 \right]$$

Define $|4\rangle = \frac{1}{2}\left[(e^{i\theta}-1)|0\rangle_4 |1\rangle_5 + i(e^{i\theta}+1)|1\rangle_4 |0\rangle_5 \right]$

Probability to see click @ D_1

$$\left| {}_5\langle 1| \, {}_4\langle 0|4\rangle \right|^2 = \left| \frac{1}{2}(e^{i\theta}-1) \right|^2$$

$$= \frac{1}{4}(e^{i\theta}-1)(e^{-i\theta}-1)$$

$$= \frac{1}{4}(1 - e^{-i\theta} - e^{+i\theta} + 1)$$

$$= \frac{1}{4}\left(2 - (e^{i\theta}+e^{-i\theta})\right)$$

$$= \frac{1}{4}(2 - 2\cos\theta)$$

$$= \frac{1}{2}(1 - \cos\theta)$$

from $e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow e^{i\theta} + e^{-i\theta} = 2\cos\theta$

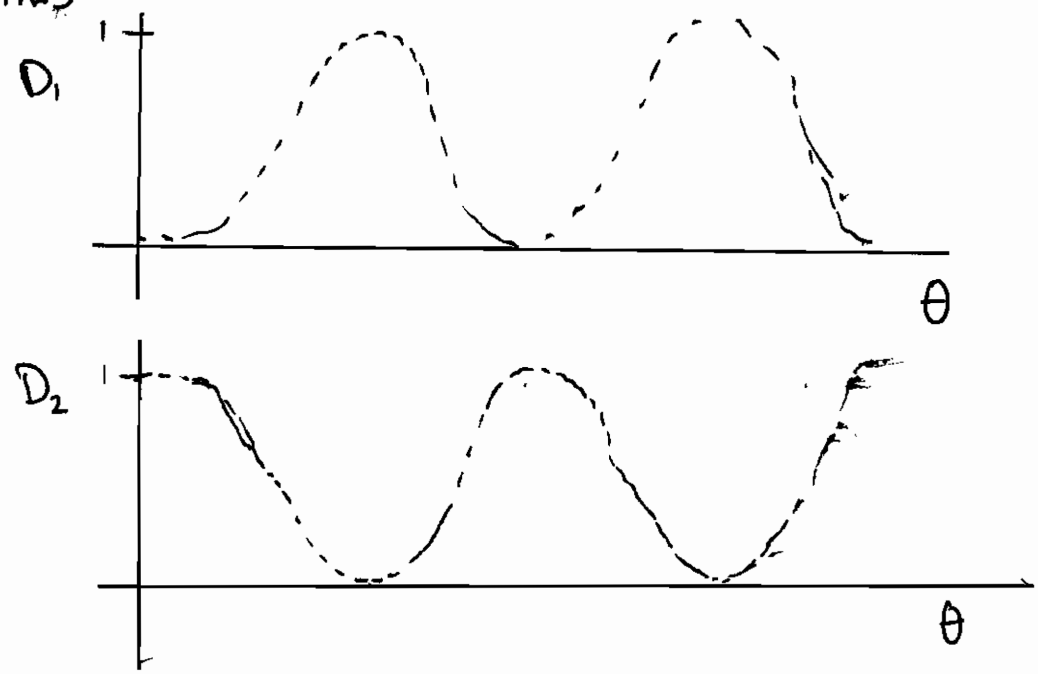
Prob to see click at D_1 + D_2

$$\left| {}_5\langle 1| \, {}_4\langle 1|4\rangle \right|^2 = 0$$

Prob to see click at D_2

$$\left| {}_5\langle 0| \, {}_4\langle 1|4\rangle \right|^2 = \frac{1}{2}(1 + \cos\theta)$$

Thus



Results from Aspect experiment

Interaction-free measurement

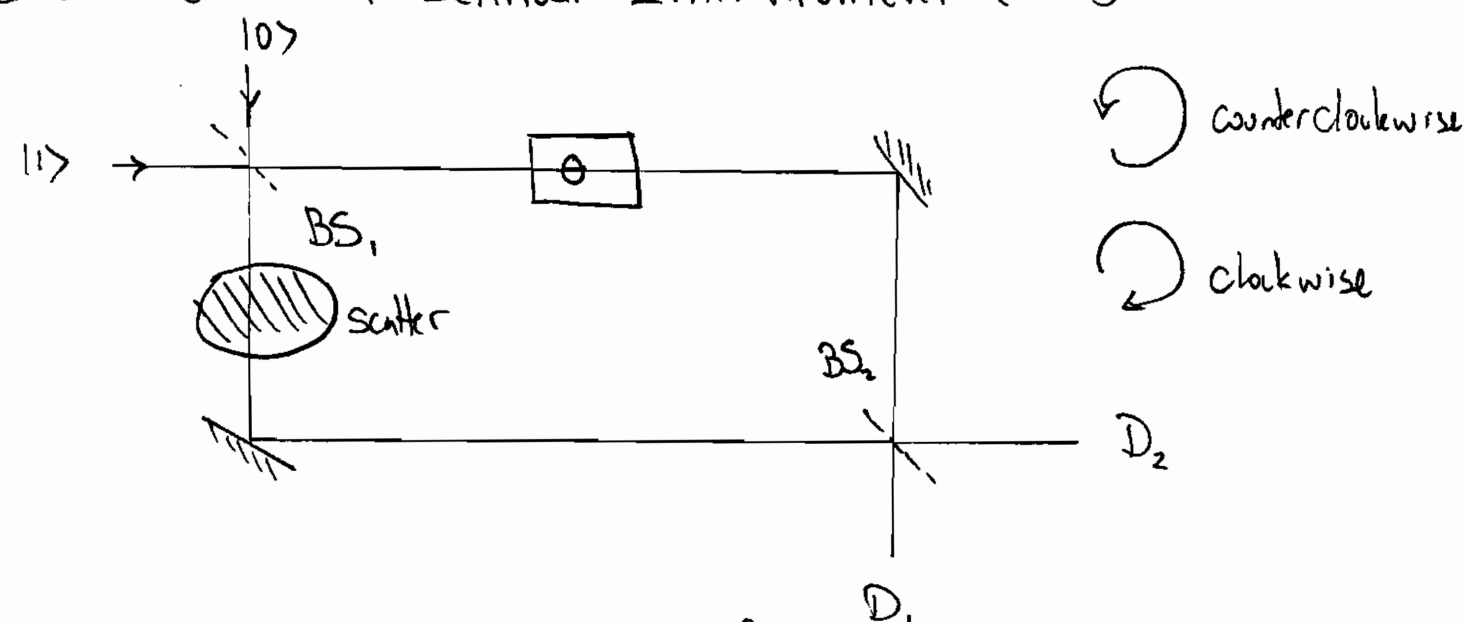
The capability of detecting the presence of a object without scattering any quanta off it.

Exposes feature of quantum mechanics

$\Rightarrow$ Nonlocality

- the apparent instanteous effects of certain kinds of influences.

Consider the Mach Zehnder Interferometer (Hong, Mandel, Ou)



1) Set $\theta = 0$ Expect probability for D_1 to fire $= 1$, prob for D_2 to fire $= 0$

2) Put in the scattering object

3) If we put a detector around the object we can detect "which-path" the photo took thus destroying the interference.

4) However, we do not need a detector. The experiment is set up to determine which path irregardless whether we measure the scattered photons or <u>not</u>.

5) After beamsplitter 1, a photo may be

50% in clockwise arm

50% in counterclockwise arm

After beamsplitter 2 the same

But if the clockwise path is "choosen", there is a 50% chance id will go to either D_1 or D_2

6) In the end (with scatterer)

Probability 50% that neither detector fires

Probability 25% that D_1 or D_2 fires

7) Initially the interferometer was set up so D_1 fires 100%

~~If D_1 does not fire 100%~~ then

So when D_2 fires <u>at</u> <u>all</u> then we know that something is in the arm of the interferometer.

But the photon we measure <u>has</u> <u>not</u> been scattered!! $\Rightarrow$ Nonlocal behavior
It never comes in contact with the scatter!!

The Einstein, Podosky, + Rosen Argument (EPR) 1932

Einstein never liked quantum mechanics because he belived it was an incomplete theory. He posed a gedenken experiment to illustrate a possible fault with QM. Here, I will discuss David Bohm's version of the EPR argument. Bohm's version is structured around entangled <u>electrons</u> but a similar argument can be constructed for photons.

Bohm's version $\Rightarrow$ Electron spins $(\pm 1/2)$

But first, some definitions

3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER
COMET

David Bohm (1917 - 1992)

Grad School Berkeley
Worked on Manhattan project (request of Oppenheimer)

Faculty at Princeton (QM book)

1949 Testified infront of House Un-American activities
Committee ⇒ Pleaded 5th (decline to testify)

Princeton did not renew contract

Moved to Brazil, took US passport

Eventually became British citizen.

In ~~school~~ grad school
Committee for Peace Mobilization
⇒ Branded Communist by FBI

Young Communist League.
Committee against conscription

Bohm's PhD work became classified, but he could not get a security clearance so he could not finish his PhD!

Oppenheimer certified his work & he got PhD.

Locality + ~~[illegible]~~ Realism (Garrison)

Locality : a measurement occuring in a finite volume of space in a given time interval could not influence - or be influenced - by measurements in a distante volume of space time before any light signal could connect the two localidies.

Space - time separated

Realism : - Physical properties exist independent of any measurements or observation

- Spatial separability : the physical properties of spatially separated systems are mutually independent.

local realism : Bell's inequality test this.

violation of Bell's inequality : Must give up locality or realism or both?

Bell's Ineq
Philosophy ⟹ to physics (testable)

Definitions in EPR

1) Elements of physical reality

"If, without any way disturbing a system, we can predict with certainty (i.e. with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity."

2) Criterion of completeness for physical theory

"every element of a physical reality must have a counter part in the physical theory"

Assumptions in EPR

1) Realism

2) Locality

From QM we have a diclemma

S_x^B + S_y^B are represented by non commuting operators

$$[S_x^B, S_y^B] = i\hbar S_z^B \neq 0$$

So the cannot be simultaneously predicted or measured.

This leaves two alternatives:

1) If S_x^B + S_y^B are both elements of physical reality, then quantum mechanics - which cannot predict values for both - is incomplete.

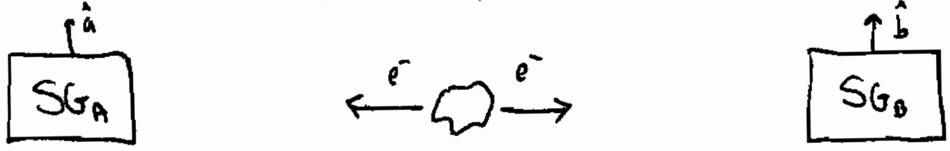
2) Two physical quantities that are associated with non commuting operators cannot be simultaneously real.

Replacement of $|\Phi\rangle_{AB}$ by $|\Phi\rangle_{AB}^x$ or $|\Phi\rangle_{AB}^y$ occurs as soon as measurement is completed independent of distance from A to B.

Global change in state vectors occur <u>before</u> any signal could travel from A to B.

$\Rightarrow$ violation of locality

EPR Paradox (Bohm) (Garrison)



Ⓐ Alice

Ⓑ BOB

Bohm singlet state $\quad |\Phi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B\right)$

fermions

component of spin is measured by Stern Gerlach (SG) magnets

<u>Orientate to $\hat{x}$</u> ($\hat{a} = \hat{x}$)

S_x^A upper counter click $\to +1/2$

$$\boxed{|\Phi\rangle_{AB}^x = |\uparrow_x\rangle_A \, |\downarrow_x\rangle_B}$$

S_x^B lower counter click $\to -1/2$

Get definite value of S_x^B without measuring it at all!

Thus $\boxed{S_x^B \text{ is a physical reality at } B.}$ (before ~~BOB~~ Alice measured it)

<u>Orientate ~~all~~ to $\hat{y}$ ($\hat{a} = \hat{y}$)</u> } Assume large distance $\Rightarrow$ locality

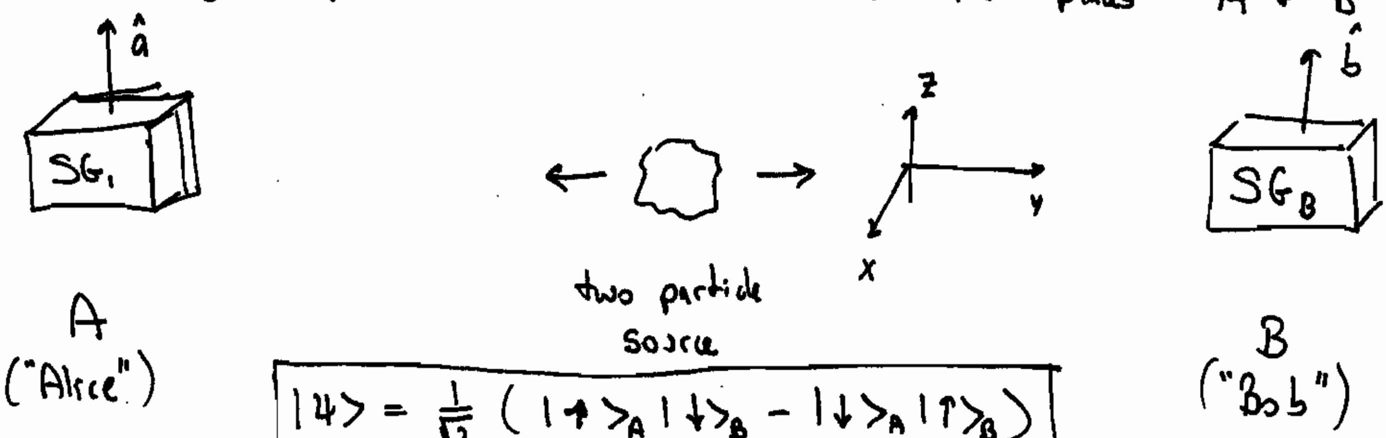
$$|\Phi\rangle_{AB}^y = |\uparrow_y\rangle_A \, |\downarrow_y\rangle_B$$

S_y^A gives $+1/2$ then S_y^B gives $-1/2$

So $\boxed{S_y^B \text{ is a physical reality at } B}$

EPR Argument for electrons

Consider a process that produces two particles with opposite spin. A Stern-Gerlach analyzer, which measures the component of spin along a specific axis is located at two places A + B



$$\boxed{|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B\right)}$$

For spin, it is important to note that $[S_x, S_z] \neq 0$

Alice orients here SG along $\hat{z}$ $\qquad \hat{a} \parallel \hat{z}$

she reads "spin up" or "spin down" say spin up

Since the two particles are pairs then Bob's particle must be spin down along $\hat{z}$.

Alice has measured the component of Bob's particle.

. Point A can be very far from point B so what goes on at A cannot have no effect on B (this is the locality assumption). Even though Alice experiment may have had an effect on her particle it should not affect Bob's!

Thus EPR conclude that Bob's particle must had spin down before Alice made her measurement!

3. Now Alice aligns her SG along $\hat{x}$. The same argument can be made about Bob's particle.

Thus the two complementary variables S_z and S_x exist and have definite values

Conclusions from EPR Argument

The EPR argument is based on locality.

(~~justified if "something" can move faster than light~~)

he EPR Argument ~~experimentally demonstrates~~ trys to show that quantum mechanics is an incomplete theory since there are "hidden variables" to quantum mechanics that are well defined (according to the EPR Argument). Thus a better theory, a local hidden variable theory is needed.

lowever, the EPR argument is in error due to its locality assumption. Can we set up a condition to test the EPR argument? Thus we re testing locality.

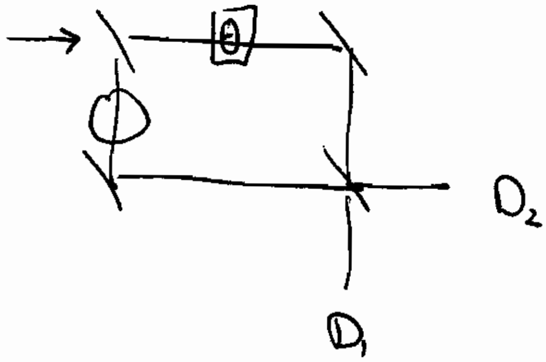
ell's theorem (Bell's inequality) 1964

John Bell devised a logical argument in the form of an inequality as a method to test the conclusions from the EPR argument.

Review

Nonlocality

Example 1



Set $\theta =$ so D_1 only clicks

~~BS, 50%~~

~~BS2 to scatter~~

Example 2 EPR

~~NDAB~~ $|\Phi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B\right)$

State reduction

Alice measures $\hat{x}$ $|\Phi\rangle^x_{AB} = |\uparrow\rangle_A |\downarrow\rangle_B$

but this instantaneously ~~[illegible]~~ sets the spin of Bob!

EPR + local hidden variable theory

If you accept the assumptions of EPR argument, QM is not complete. Thus there must be another local theory that has hidden variables that satisfies the elements of physical reality.

These variables are hidden since QM does not give any information about them, but they are real.

<u>Back to electron spins</u>

Alice $\overset{A}{\boxed{SG_A}}$ • $\overset{B}{\boxed{SG_B}}$ Bob

Let $\hat{a}$ be the orientation of SG_A, $\hat{b}$ be the orientation of SG_B

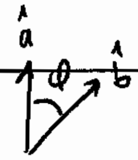
@ A $\begin{cases} A = +1 & \text{if Alice measures } \uparrow \\ A = -1 & \text{if Alice measures } \downarrow \end{cases}$ Component of spin

@ B $\begin{cases} B = +1 & \text{if Bob measures } \uparrow \\ B = -1 & \text{if Bob measures } \downarrow \end{cases}$

Look at product of AB

$AB = +1$ Alice + Bob got same answer

$AB = -1$ Alice + Bob got different answers



QM gives us

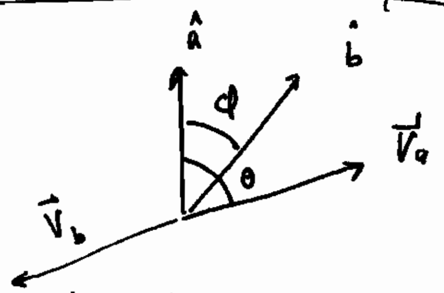
$$E_{QM}(\hat{a}, \hat{b}) = -\hat{a}\cdot\hat{b} = -\cos\phi$$

A hidden variable theory will give

$$E_{HV}(\hat{a}, \hat{b})$$

Bell Thm states: $E_{HV}(\hat{a}, \hat{b}) \neq E_{QM}(\hat{a}, \hat{b})$ for all $\hat{a}$ + $\hat{b}$

<u>Example of hidden variable theory</u>



$$\vec{V}_b = -\vec{V}_a$$

Say particles are actually rotating: naïve spin

spin vector $\vec{V}_a$, $\vec{V}_b$

$\theta_a \equiv$ Hidden variable

Each detector records sign of projection of $\vec{V}$ along its axis ($\hat{a}$ or $\hat{b}$)

$$A = \text{sign}[\cos\theta]$$

$$B = \text{sign}[\cos(180 - \theta - \phi)] = \text{sign}(-\cos(\theta - \phi))$$

$$\boxed{AB = \text{sign}[-\cos\theta\,\cos(\theta-\phi)]}$$

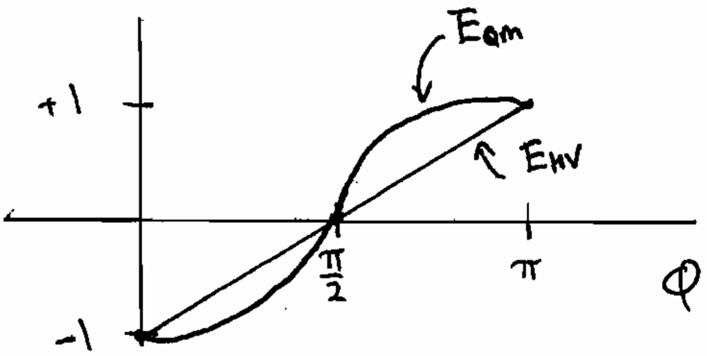
Compute AB over all possible values of hidden variable θ

$$E_{HV}(\hat{a},\hat{b}) = \frac{1}{2\pi}\int_{-\pi/2}^{3\pi/2} AB \, d\theta$$

$$\boxed{E_{HV}(\hat{a},\hat{b}) = \frac{2}{\pi}\varphi - 1}$$

~~compare~~ but $\underset{qm}{E}(\hat{a},\hat{b}) = -\cos\varphi$

So



$E_{qm}(\hat{a},\hat{b}) \neq E_{HV}(\hat{a},\hat{b})$ for all $\hat{a}$ + $\hat{b}$

Logic arguments towards Bell's inequalities

$E \Rightarrow$ expectation (or #)

Argument

If we have three properties A, B, C then if we consider the joint properties we can say

$$\boxed{E(A,\bar{B}) + E(B,\bar{C}) \geq E(A,\bar{C})}$$

$\bar{C}$ = Not C

Proof

$$E(A,\bar{B},C) + E(\bar{A},B,\bar{C}) = 0 \text{ or positive integer}$$

$$E(A,\bar{B},C) + E(\bar{A}\, B,\bar{C}) \geq 0$$

(either No members of the group have these combinations of properties or some members do)

Add $E(A,\bar{B},\bar{C}) + E(A,B,\bar{C})$

$$①\left(E(A\,\bar{B},C) + E(A,\bar{B},\bar{C})\right) + ②\left(E(\bar{A},B\bar{C}) + E(A\,B\,\bar{C})\right) \geq 0 + \underbrace{E(A,\bar{B},\bar{C}) + E(A\,B\,\bar{C})}_{E(A,\bar{C})}$$

(For everyone either B or $\bar{B}$ must be true. everyone has property $\bar{B}$ or B)

ON LHS ① $= E(A,\bar{B})$ ② $= E(B\,\bar{C})$

So

$$\boxed{E(A,\bar{B}) + E(B,\bar{C}) \geq E(A,\bar{C})}$$

Example : People

			YES	No
Traits	A:	Male?	male	female
	B:	Happy?	happy	sad
	C:	Glasses?	glasses	no glasses

In pictures

YES NO

A:

B:

C:

All people will fit into at least one of eight groups

1 2 3 4

sad men glasses	sad men No glasses	happy men glasses	happy men no glasses
1	1, 3		2, 3

5 6 7 8

Sad Women Glasses	Sad Women no glasses	happy Women Glasses	happy Women no glasses
			2,

So

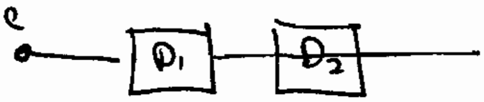
$$E(\text{male}, \text{sad}) + E(\text{happy}, \text{no glasses}) \geq E(\text{male}, \text{no glasses})$$

$$(\text{Group } 1 + \text{Group } 2) + (\text{Group } 4 + \text{Group } 8) \geq (\text{Group } 2 + \text{Group } 4)$$

1 2 3

This inequality will be satisfied by people

What about electrons?



A: spin up 0°?

B: spin up 45°?

C: spin up 90°?

$$E(\uparrow 0°, \downarrow 45°) + E(\uparrow 45°, \downarrow 90°) \geq E(\uparrow 0°, \downarrow 90°)$$

Not spin up = spin down

For one detector 0° ½ will be up / ½ will be ↓

45° ½ will be ↑ / ½ will be ↓

Measure 0° + 45° ⇒ problem

First 0° ½ ↑ / ½ down ↓ (Pass ↑)

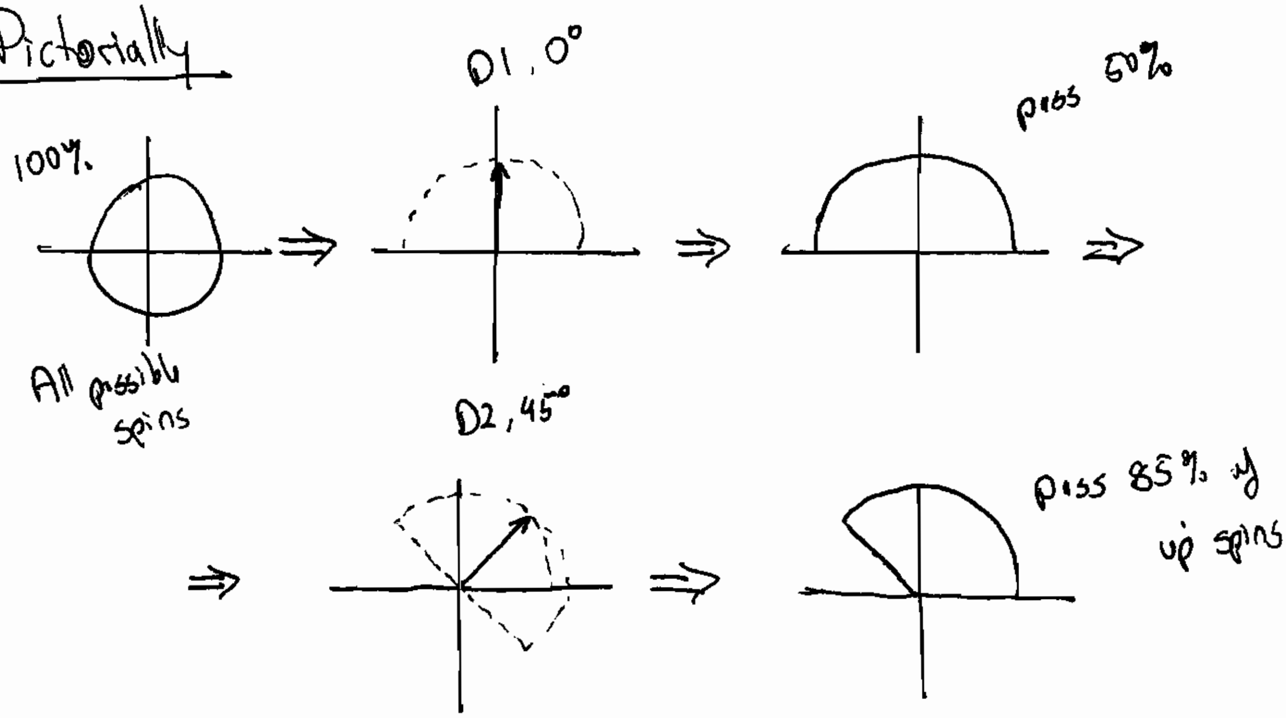
For # that pass D1 : 45° 85% will pass 2nd detector

Thus $\boxed{\uparrow 0° \rightarrow 15\% \downarrow 45°}$

Measuring spin at 0° changes # of electrons ↓ at 45°!

Determining [illegible] happen? changes their sex!

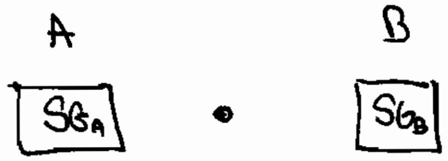
Pictorially



Thus we have 15% of spin down

Act of measuring changes outcome

EPR setup



Meas Let A 45°, B 0°

If we measure ↑ 45° at A, then we will have ↓ 45° at B

[illegible]

$$\boxed{\text{Thus we have determined whether or not } e^- \text{ at B is } \uparrow 0°, \downarrow 45° \text{ by measuring its spin at } 0° \text{ at B \& spin at } 45° \text{ at A.}}$$

$$E(\uparrow 0°_B, \uparrow 45°_A) + E(\uparrow 45_B, \uparrow 90°_A) \geq E(\uparrow 0°_B, \uparrow 45_A)$$

QM will violate this!

Bell's Inequality (Phys. vol 1 1964 p 195 200)

John Bell devised a logical argument in the form of an inequality as a method to test the conclusions of the EPR argument.

There are many versions of the inequality. We will look at two

1) Bell's original inequality

$$\boxed{\left| E_{HV}(\hat{a},\hat{b}) - E_{HV}(\hat{a},\hat{c}) \right| \leq 1 + E_{HV}(\hat{b},\hat{c})}$$

2) Clauser Horn Shimony + Holt version (CHSH)

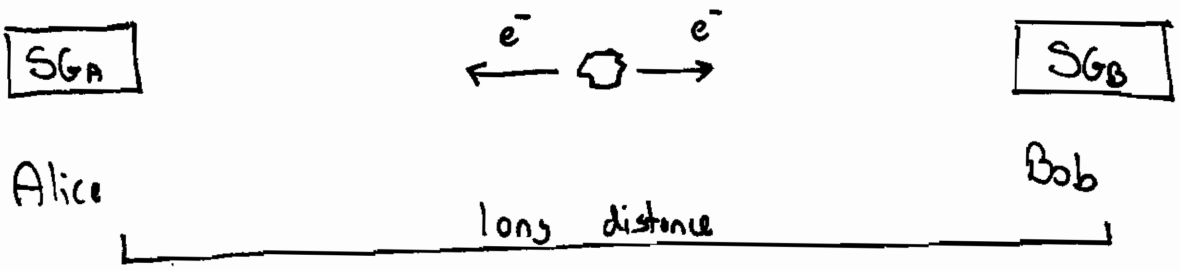
$$\boxed{-2 \leq E_{HV}(\theta,\phi) + E(\theta',\phi) + E(\theta,\phi') - E(\theta',\phi') \leq 2}$$

Look at 1)

$$\boxed{\text{Quantum optical measurements show a violation of Bell's inequality (up to } 242\sigma)!}$$

This demonstrates that no local hidden variable theory can predict the measured results which are predicted by Quantum Mechanics

Revisit EPR



$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right)$$

1) Alice orientates her SG along $\hat{z}$

Measures up $S_z = +1$

2) So Bob's particle must have $S_z = -1$ even if he does not measure it.

3) But Bob has a choice, so instead he aligns his SG along $\hat{x}$ and measures $+1$

4) Thru Alice's measurement and Bob's we have measured both S_x and S_z precisely.

~~By Quantum mechanics Sx and Sz do not commute, but we have measured both of them~~

But QM tells us that S_x and S_z do not commute so we cannot measure them precisely!

In the language of EPR, since we have determined S_x and S_z precisely, they exist <u>before</u> the measurement. But QM cannot tell us values of both S_x & S_z so QM must be an incomplete theory.

Bohr's Response to EPR

- Complementarity applied after long distances
- The context needed to think about the $\hat{z}$ component of B is not compatible with what is needed to think about $\hat{x}$ component

- Even though we can predict B without disturbing B there is no experimental situation ~~measurement~~ where both S_x & S_z have meaning.

Nonlocality "hidden" in Bohr's Response

The measurement at A "collapses the wavefunction" which predetermines the result a B without the experiment of A interacting with B.

Is there a way to test this nonlocality $\Rightarrow$ Bell's inequality

Back to Alice & Bob

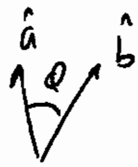
Alice orientates her SG along $\hat{a}$

Bob orientates his SG along $\hat{b}$

$A = 1$ up
$A = -1$ down

$B = 1$
$B = -1$

Look at product AB



$$\langle AB \rangle_{qm} = -\hat{a}\cdot\hat{b} = -\cos\varphi$$

$$E(\hat{a},\hat{b}) = -\cos\varphi$$

Bell's inequality shows

$$E_{HV}(\hat{a},\hat{b}) \neq E_{qm}(\hat{a},\hat{b}) \quad \text{for all } \hat{a} \,\&\, \hat{b}$$

"No physical theory of local hidden variables can ever produce the predictions of quantum mechanics"

The next Fact about the Bell's inequality is that it can be tested in the lab.

Proof of inequality

Assume locality A does not depend on $\hat{b}$

B does not depend on $\hat{a}$

$A = A(\hat{a}, \lambda)$ not $A(\hat{a}, \hat{b}, \lambda)$

$B = B(\hat{b}, \lambda)$ not $B(\hat{a}, \hat{b}, \lambda)$

$\lambda \equiv$ hidden variable

Compute expectation value from hidden variable theory

$$\boxed{E_{HV}(\hat{a}, \hat{b}) = \int AB \, d\lambda}$$

Technically

$$E_{HV}(\hat{a}, \hat{b}) = \int d\lambda \; P_A(\hat{a}, \lambda) P_B(\hat{b}, \lambda) \rho(\lambda)$$

Derivation

$$E_{HV}(\hat{a}, \hat{b}) - E_{HV}(\hat{a}, \hat{c}) = \int \left(A(\hat{a}, \lambda) B(\hat{b}, \lambda) - A(\hat{a}, \lambda) B(\hat{c}, \lambda) \right) d\lambda$$

$\Rightarrow$ EPR case $A(\hat{a}, \lambda) = -B(\hat{a}, \lambda)$ opposite spins

$$E_{HV}(\hat{a}, \hat{b}) - E(\hat{a}, \hat{c}) = -\int \left(A(\hat{a}, \lambda) A(\hat{b}, \lambda) - A(\hat{a}, \lambda) A(\hat{c}, \lambda) \right) d\lambda$$

$\Rightarrow$ Using $|A(\hat{b}, \lambda)|^2 = 1$

$$= -\int \underbrace{A(\hat{a}, \lambda) A(\hat{b}, \lambda)} \left[1 - A(\hat{b}, \lambda) A(\hat{c}, \lambda) \right] d\lambda$$

$\Rightarrow$ Note that $A(\hat{a}, \lambda) A(\hat{b}, \lambda) = +1$ or -1

$$\left| E_{HV}(\hat{a},\hat{b}) - E_{HV}(\hat{a},\hat{c}) \right| \le \left| \int \left(1 - A(\hat{b},\lambda)\, A(\hat{c},\lambda)\right) d\lambda \right|$$

$$\le \left| \int \left(1 + A(\hat{b},\lambda)\, B(\hat{c},\lambda)\right) d\lambda \right|$$

$\Rightarrow$ from $A(\hat{c},\lambda) = -B(\hat{c},\lambda)$

$$\le 1 + \int A(\hat{b},\lambda)\, B(\hat{c},\lambda)\, d\lambda$$

So $$\boxed{\left| E_{HV}(\hat{a},\hat{b}) - E_{HV}(\hat{a},\hat{c}) \right| \le 1 + E_{HV}(\hat{b},\hat{c})}$$

Show QM violates the inequality

$$\left| E(\hat{a}, \hat{b}) - E(\hat{a}, \hat{c}) \right| \leq 1 + E(\hat{b}, \hat{c})$$

Case I $\quad \hat{a} = \hat{b} = \hat{c}$

$\phi = 0$

So $E_{QM}(\hat{a}, \hat{b}) = -\cos\phi = -1$

$E_{QM}(\hat{a}, \hat{c}) = -1$

$E_{QM}(\hat{b}, \hat{c}) = -1$

Into eqn inequality

$$\left| (-1) - (-1) \right| \overset{?}{\leq} 1 + (-1)$$

$0 \leq 0$ ✓ OK No violation!

Case II $\quad \hat{b}$ makes $60°$ angle with $\hat{a}$, $\hat{c}$ makes $60°$ angle with $\hat{b}$

$E_{QM}(\hat{a}, \hat{b}) = -\cos 60 = -1/2$

$E_{QM}(\hat{a}, c) = -\cos(120) = +1/2$

$E_{QM}(\hat{b}, \hat{c}) = -\cos(60) = -1/2$

$$\left| (-1/2) - (1/2) \right| \overset{?}{\leq} 1 + (-1/2)$$

$1 \overset{?}{\leq} 1/2$ NOT OK Violation!

Objections on the concept of nonlocality

Newton	"philosophical absurdity"
Einstein	"Spooky" action at a distance
Bohm	"cannot see any well-founded reason for such objections..."
Aspect	See Nature Paper

Lecture 40 Entanglement

Generation of Entangled States (Polarization Entangled states)

1. Spontaneous Parametric Down-conversion in a $\chi^{(2)}$ crystal

   (Degenerate form of difference frequency generation $\omega_s = \omega_i$)

$$\hat{H}_1 \sim \chi^{(2)} \hat{a}_p \hat{a}_s^\dagger \hat{a}_i^\dagger + \chi^{(2)*} \hat{a}_p^\dagger \hat{a}_s \hat{a}_i \qquad \text{Nondegenerate case}$$

Generate signal and idler photon from pump

$$|1\rangle_p |0\rangle_s |0\rangle_i \xrightarrow{\chi^{(2)}} \hat{a}_p \hat{a}_s^\dagger \hat{a}_i^\dagger |1\rangle_p |0\rangle_s |0\rangle_i = |0\rangle_p |1\rangle_s |1\rangle_i$$

- process is spontaneous since modes are originally from vacuum.
- Signal and idler photons are generated simultaneously
- Must satisfy both energy conservation and momentum conservation (i.e. phase matching)

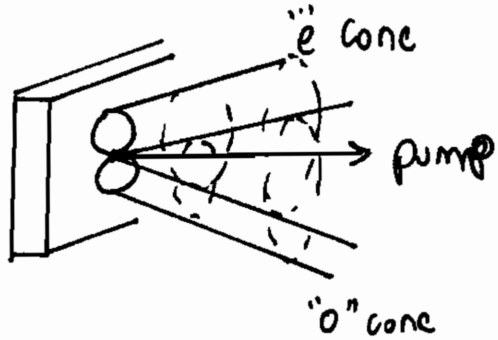
Type II Down conversion (in BBO or KDP)

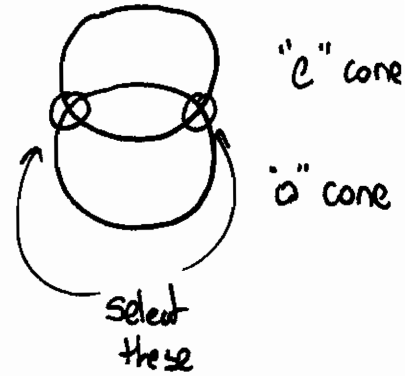
This crystal exhibits birefringence

Photons are emitted in two different cones

one for "o" axis

one for "e" axis





- Intersection of cones produce polarization entangled states

Use notation to represent polarization of single photon states

$$|V\rangle \quad + \quad |H\rangle$$

$$\hat{H} \cong \chi^{(2)} \left(\hat{a}^{+}_{Vs} \hat{a}^{+}_{Hi} + \hat{a}^{+}_{Hs} \hat{a}^{+}_{Vi} \right)$$

$$+ \quad \chi^{(2)*} \left(\hat{a}_{Vs} \hat{a}_{Hi} + \hat{a}_{Hs} \hat{a}_{Vi} \right)$$

Initial state

$$|\psi_0\rangle = |0\rangle_{Vs} |0\rangle_{Hs} |0\rangle_{Vi} |0\rangle_{Hi}$$

Final states (see text for renormalization procedure)

$$|\psi(t)\rangle = \exp(-i\, H_1 t/\hbar)\, |\psi_0\rangle = \boxed{\frac{1}{\sqrt{2}} \left(|H\rangle_1 |V\rangle_2 + e^{i\theta} |V\rangle_1 |H\rangle_2 \right)}$$

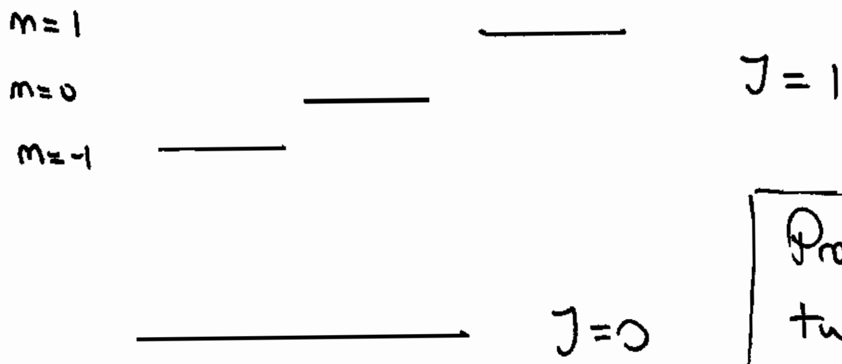
$\cancel{|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}} \left(|H\rangle_s |V\rangle_i \pm |V\rangle_s |H\rangle_i \right)}$ ~~are of the Bell States~~

$$\boxed{\begin{aligned} |\psi^{\pm}\rangle &= \frac{1}{\sqrt{2}} \left(|H\rangle_1 |V\rangle_2 \pm |V\rangle_1 |H\rangle_2 \right) \\ |\Phi^{\pm}\rangle &= \frac{1}{\sqrt{2}} \left(|H\rangle_1 |H\rangle_2 \pm |V\rangle_1 |V\rangle_2 \right) \end{aligned}}$$

By a choice of phase θ one ~~complete set~~ can complete the set.

2. Cascade Emission for generation of polarization entangled states

Transition $J \to 0 \to 1 \to 0$



Process produces two photons

Output State

$$\boxed{|\psi\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle_1 |H\rangle_2 + |V\rangle_1 |V\rangle_2 \right)}$$

rewrite in terms of circularly polarized light $|+\rangle$ $|-\rangle$

$$\boxed{|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle_1 |-\rangle_2 + |-\rangle_1 |+\rangle_2 \right)}$$

$|\Psi^{\pm}\rangle$ + $|\Phi^{\pm}\rangle$ form a complete set (basis) in the Hilbert space

They are known as <u>Bell</u> <u>states</u> (for reasons we will come to later)

Type II down conversion is a very important process since it can produce <u>all</u> <u>4</u> <u>Bell</u> <u>states</u>. This process provides an experimental optical tool to test quantum mechanics.

. . . . But what shall we test?

<u>locality</u> + <u>the EPR argument</u>

The Einstein Podolsky + Rosen Argument (EPR) (1932)

Einstein never liked quantum mechanics because he believed it was an incomplete theory. He posed a gedanken experiment to illustrate a possible fault with QM. Here, we will discuss David Bohm's version of the EPR argument. Bohm's argument is structured around entangled <u>electrons</u> but a similar argument can be constructed for photon states.

We will express ~~the~~ Bohm's version of the EPR argument in terms of ~~polarization~~ ~~entangled states of light~~ electron spins. ~~These~~ This is isomorphic to polarization states of light (see page 228).

Optics Letters vol 30, 8 p 813-813 (2005)

From the Board of Editors: on Plagiarism

Dear Colleagues:

There has been a significant increase in the number of duplicate submissions and plagiarism cases reported in all major journals, including the journals of the Optical Society of America. Duplicate submissions and plagiarism can take many forms, and all of them are violations of professional ethics, the copyright agreement that an author signs along with the submission of a paper, and OSA's published Author Guidelines. There must be a significant component of new science for a paper to be publishable. The copying of large segments of text from previously published or in-press papers with only minor cosmetic changes is not acceptable and can lead to the rejection of papers.

Duplicate submission: Duplicate submission is the most common ethics violation encountered. Duplicate submission is the submission of substantially similar papers to more than one journal. There is a misperception in a small fraction of the scientific community that duplicate submission is acceptable because it sometimes takes a long time to get a paper reviewed and because one of the papers can be withdrawn at any time. This is a clear violation of professional ethics and of the copyright agreement that is signed on submission. Duplicate submission harms the whole community because editors and reviewers waste their time and in the process compound the time it takes to get a paper reviewed for all authors. In cases of duplicate submission, the Editor of the affected OSA journal will consult with the Editor of the other journal involved to determine the proper course of action. Often that action will be the rejection of both papers.

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>
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OSA identifies an act of plagiarism in a published document to be the substantial replication, without appropriate attribution, of significant elements of another document already published by the same or other authors. OSA has implemented a process for dealing with cases of plagiarism. When the Editor-in-Chief of a journal is notified of an instance of either of the two possible forms of plagiarism discussed above, he or she will make a preliminary investigation of the allegations, including a request for the accused authors to explain the situation. If further action is justified, then the Editor-in-Chief will convene a panel consisting of the Editor-in-Chief of the OSA journal involved, the Chair of the Board of Editors, and the Senior Director of Publications. Their unanimous decision confirming that an act of plagiarism has occurred requires the insertion of the following statement in the official OSA electronic record of the plagiarizing article:

> "It has come to the attention of the Optical Society of America that this article should not have been submitted owing to its substantial replication, without appropriate attribution, of significant elements found in the following previously published material: [citation data – including the authors, journal title, full citation of the earlier published material.]"

The same statement shall be added to the next available print run of the journal in an appropriate location such as a "Notice to Readers."

The OSA Board of Editors

The Plagiarism Resource Site

Charlottesville, Virginia

"The Importance of Writing"

by Louis Bloomfield. Professor of Physics, University of Virginia. Charlottesville. VA 22904

Originally published on the Commentary Page of the ***Philadelphia Inquirer*** on Sunday, April 4, 2004. edited by John Timpane.

Writing is hard work and all the marvels of modern technology haven't made it any easier. Vast resources now lie just keystrokes away. but the basic art of assembling one's thoughts into engaging prose is little changed since the days of paper and pencil. While mindless information doubles every three years. thoughtful writing still proceeds at an old fashioned pace.

Unfortunately. the timeless nature of writing isn't shared by its fraudulent imitation: plagiarism. Though nearly as ancient as writing itself, plagiarism adapts quickly to new technology. With a web full of seemingly ownerless prose. plagiarism is as easy as cut-and-paste. And if you don't see exactly what you want for free, you can buy it online at any number of "paper mills."

But a more insidious way in which technology has fostered plagiarism is by shifting our attention from content to appearance. A well-written student paper is no longer "A" work unless it's printed in color on glossy paper, with fonts and images and an accompanying multimedia presentation. Students feel expected to turn in the best papers ever written, not the best papers they can write themselves. So they assemble those papers. With hours invested in the decorations, students feel justified in stealing some or all of the text. After all, they "couldn't have said it any better" themselves.

In addition to its easy rationalization by people seeking the rewards of writing without the associated effort, plagiarism is also widely misunderstood. It isn't limited to the theft of another person's words; it also includes the theft of their ideas. More generally, plagiarism is any form of dishonesty about authorship. A reader or listener should always know whose thoughts they're hearing.

Plagiarism isn't a victimless crime. It deprives its readers of their time and trust. and its true authors of their good names. In academia, plagiarism inflates grades relative to education and devalues honest scholarship. Among authors and journalists, plagiarism cheapens the very art of writing, much as performance enhancing drugs cheapen so many sports. Plagiarism is as much a problem of morale as it is of ethics.

Prosecuting plagiarists is a miserable undertaking. It brings joy to no one, as I know from sad experience at the University of Virginia. After uncovered extensive plagiarism in my large introductory physics class in 2001, I spent two years dealing with endless honor cases. But I view that episode as an anti-scandal—as an enlightened community taking action against a misbehaving few in order to maintain its own intellectual integrity. Eliminating plagiarism isn't about the plagiarists; it's about supporting the honest people by giving them a fair environment.

Plagiarism isn't an obscure tweed-collar crime. It's a sorry fact of life everywhere and any school or organization that feels untainted is probably in denial. With plagiarism so commonplace, an organization that deals openly with it deserves our support, not our condemnation. There is no scandal in cleaning house. The scandal is in tolerating or covering up plagiarism.

Unfortunately, plagiarism is openly tolerated in the most public sectors of modern life. It wasn't always that way. Lincoln didn't just perform his Gettysburg address; he actually wrote it. What happened to that tradition of intellectual honesty in public speech? With ghostwriting so ubiquitous among the rich and powerful, it's no wonder that young people see little value in learning to write well. They view writing the way they view cleaning their rooms—an unpleasant chore they'll do only until they can afford to hire someone else.

When students believe that writing assignments are merely hazing rituals, hurdles on the path to success in life, some will inevitably plagiarize. And when instructors assign writing that has no clear educational goals, how can the students value it? Having explicitly stated goals is both good discipline and a way to avoid misunderstandings. If students believe an assignment is "busy work," some will be busy cheating.

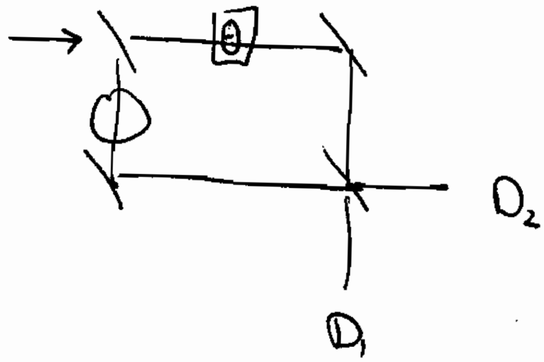
Finally, students need to be taught that the act of writing is intrinsically valuable *to them*. It crystallizes one's thoughts in a way that nothing else can. As a physicist, I find that I often learn more from writing papers and proposals than I do from working in the laboratory. I rarely find writing easy. but I always find it rewarding.

Page Last Updated: April 12. 2004

Review

Nonlocality

Example 1



Set $\theta =$ so D_1 only clicks

~~50%~~

~~scatter~~

Example 2 EPR

$$|\Phi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B\right)$$

State reduction

Alice measures $\hat{x}$ $\qquad |\Phi\rangle^{x}_{AB} = |\uparrow\rangle_A \, |\downarrow\rangle_B$

but this instantaneously sets the spin of Bob!

EPR + local hidden variable theory

If you accept the assumptions of EPR argument, QM is not complete. Thus there must be another local theory that has hidden variables that satisfies the elements of physical reality.

These variables are hidden since QM does not give any information about them, but they are real.

Back to electron spins

Alice A $\boxed{SG_A}$ • $\boxed{SG_B}$ B Bob

Let $\hat{a}$ be the orientation of SG_A, $\hat{b}$ be the orientation of SG_B

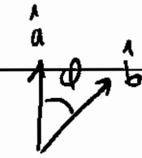
$$@A \left\{ \begin{array}{ll} A = +1 & \text{if Alice measures } \uparrow \\ A = -1 & \text{if Alice measures } \downarrow \end{array} \right\} \text{component of spin}$$

$$@B \left\{ \begin{array}{ll} B = +1 & \text{if Bob measures } \uparrow \\ B = -1 & \text{if Bob measures } \downarrow \end{array} \right.$$

Look at product of AB

$AB = +1$ Alice + Bob got same answer

$AB = -1$ Alice + Bob got different answers



QM gives us

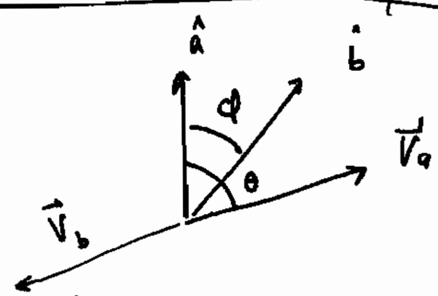
$$E_{QM}(\hat{a},\hat{b}) = -\hat{a}\cdot\hat{b} = -\cos\phi$$

A hidden variable theory will give

$$E_{HV}(\hat{a},\hat{b})$$

$$\boxed{\text{Bell Th}^{\underline{m}}\text{ states: } E_{HV}(\hat{a},\hat{b}) \neq E_{QM}(\hat{a},\hat{b}) \text{ for all } \hat{a} + \hat{b}}$$

<u>Example of hidden variable theory</u>



$$\vec{V}_b = -\vec{V}_a$$

Say particles are actually rotating : naïve spin

spin vector $\vec{V}_a$, $\vec{V}_b$

$\theta_a \equiv$ Hidden variable

Each detector records sign of projection of $\vec{V}$ along its axis ($\hat{a}$ or $\hat{b}$)

$$A = \text{sign}[\cos\theta]$$

$$B = \text{sign}[\cos(180-\theta-\phi)] = \text{sign}(-\cos(\theta-\phi))$$

$$\boxed{AB = \text{sign}[-\cos\theta\cos(\theta-\phi)]}$$

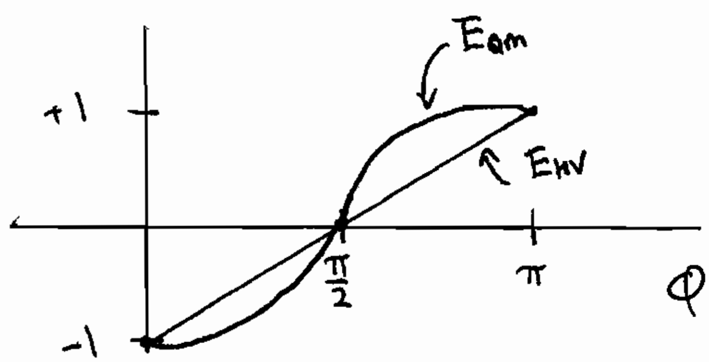
Compute AB over all possible values of hidden variable θ

$$E_{HV}(\hat{a},\hat{b}) = \frac{1}{2\pi}\int_{-\pi/2}^{3\pi/2} AB \, d\theta$$

$$\boxed{E_{HV}(\hat{a},\hat{b}) = \frac{2}{\pi}\phi - 1}$$

~~compare~~ but $E_{am}(\hat{a},\hat{b}) = -\cos\phi$

So



$E_{am}(\hat{a},\hat{b}) \neq E_{HV}(\hat{a},b)$ for all $\hat{a}$ + $\hat{b}$

Logic arguments towards Bell's inequalities

$E \Rightarrow$ expectation (or #)

Argument If we have three properties A, B, C then if we consider the joint properties we can say

$$\boxed{E(A,\bar{B}) + E(B,\bar{C}) \geq E(A,\bar{C})}$$

$\bar{C} =$ NOT C

Proof

$$E(A,\bar{B},C) + E(\bar{A},B,\bar{C}) = 0 \text{ or positive integer}$$

$$E(A,\bar{B},C) + E(\bar{A}\ B,\bar{C}) \geq 0$$

(either No members of the group have these combinations of properties or some members do)

Add $E(A,\bar{B},\bar{C}) + E(A,B,\bar{C})$

$$① \left(E(A\ \bar{B},C) + E(A,\bar{B},\bar{C})\right) + ② \left(E(\bar{A},B\bar{C}) + E(A\,B\,\bar{C})\right) \geq 0 + \underbrace{E(A,\bar{B},\bar{C}) + E(A\,B\,\bar{C})}_{E(A,\bar{C})}$$

(For everyone either B or $\bar{B}$ must be true. everyone has property $\bar{B}$ or B)

ON LHS ① $= E(A,\bar{B})$ ② $= E(B\bar{C})$

So

$$\boxed{E(A,\bar{B}) + E(B,\bar{C}) \geq E(A,\bar{C})}$$

Example : People

Traits			YES	No
	A:	Male?	male	female
	B:	Happy?	happy	sad
	C:	Glasses?	glasses	no glasses

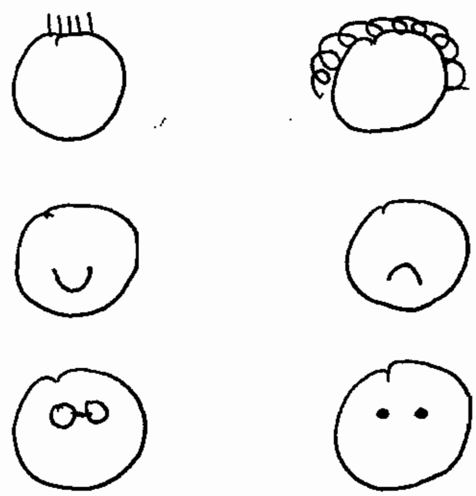
In pictures

YES NO

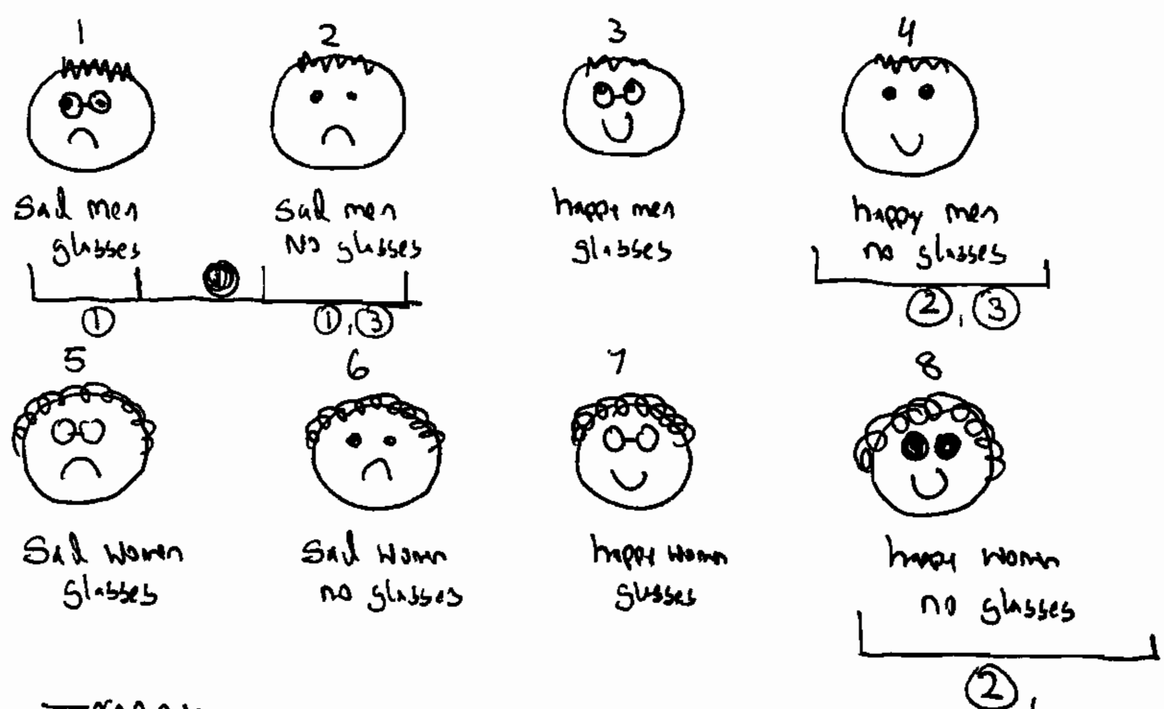
A :

B :

C :



All people will fit into at least one of eight groups



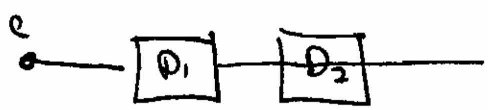
So

$$\underbrace{E(\text{male}, \text{sad})}_{} + \underbrace{E(\text{happy}, \text{no glasses})}_{} \geq \underbrace{E(\text{male}, \text{no glasses})}_{}$$

$$\underset{1}{(\text{Group 1} + \text{Group 2})} + \underset{2}{(\text{Group 4} + \text{Group 8})} \geq \underset{3}{(\text{Group 2} + \text{Group 4})}$$

This inequality will be satisfied by people

<u>What about electrons?</u>



A: spin up 0°?

B: spin up 45°?

C: spin up 90°?

$$E(\uparrow 0°, \downarrow 45°) + E(\uparrow 45°, \downarrow 90°) \geq E(\uparrow 0°, \downarrow 90°)$$

Not spin up = spin down

For one detector . . . 0° ½ will be up / ½ will be ↓

45° ½ will be ↑ / ½ will be ↓

Measure 0° + 45° ⇒ problem

First 0° ½ ↑ / ½ down ↓ (Pass ↑)

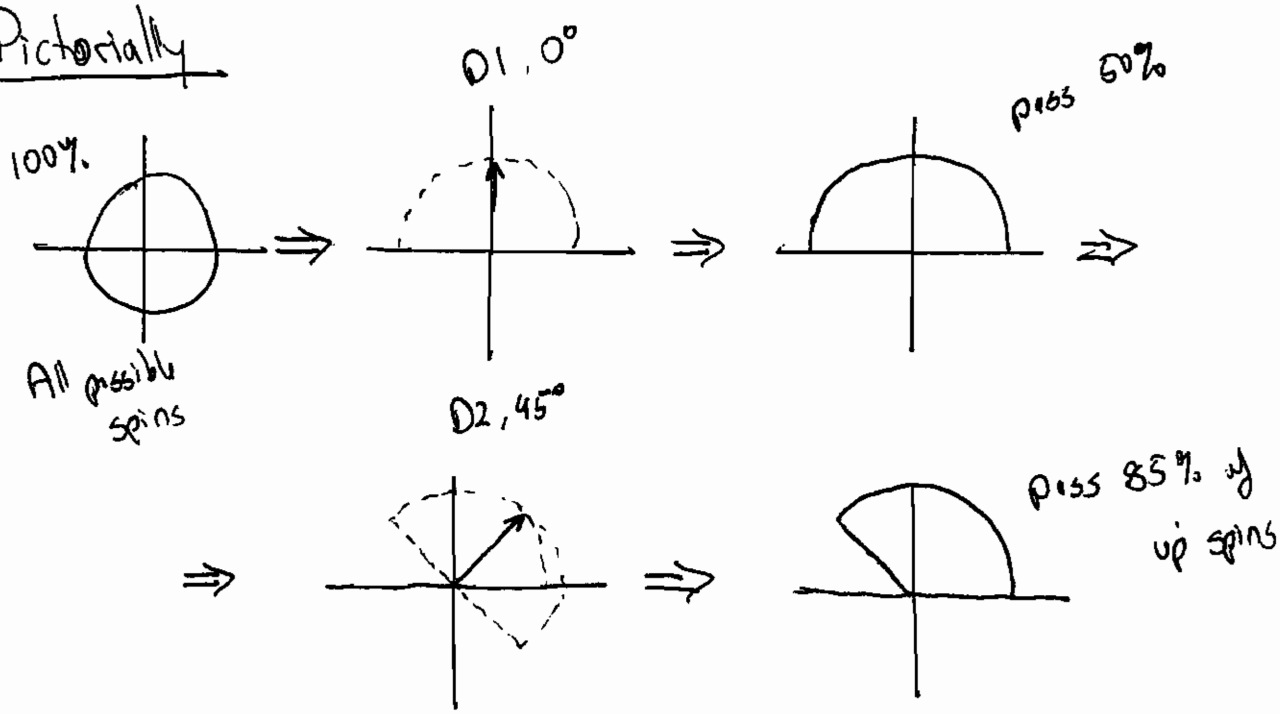
For # that pass D1 : 45° 85% will pass 2nd detector

Thus $\boxed{\uparrow 0° \rightarrow 15\% \downarrow 45°}$

Measuring spin at 0° changes # of electrons ↓ at 45°!

Determining ~~[illegible]~~ happy? changes their sex!

Pictorially



Thus we have 15% of spin down

Act of measuring changes outcome

EPR setup

~~Measu~~ Let A 45°, B 0°

If we measure ↑ 45° at A, then we will have ↓ 45° at B

[illegible]

$$\boxed{\text{Thus we have determined whether or not } e^- \text{ at B is } \uparrow 0^\circ, \downarrow 45^\circ \text{ by measuring its spin at } 0^\circ \text{ at B + spin at } 45^\circ \text{ at A.}}$$

$$E(\uparrow 0^\circ_B, \uparrow 45^\circ_A) + E(\uparrow 45_B, \uparrow 90^\circ_A) \geq E(\uparrow 0^\circ_B, \uparrow 45_A)$$

QM will violate this!

3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER

COMET

Bell's Inequality (Phys. vol 1 1964 p 195-200)

John Bell devised a logical argument in the form of an inequality as a method to test the conclusions of the EPR argument.

There are many versions of the inequality. We will look at two

1) Bell's original inequality

$$\boxed{\left| E_{HV}(\hat{a},\hat{b}) - E_{HV}(\hat{a},\hat{c}) \right| \leq 1 + E_{HV}(\hat{b},\hat{c})}$$

2) Clauser Horn Shimony + Holt version (CHSH)

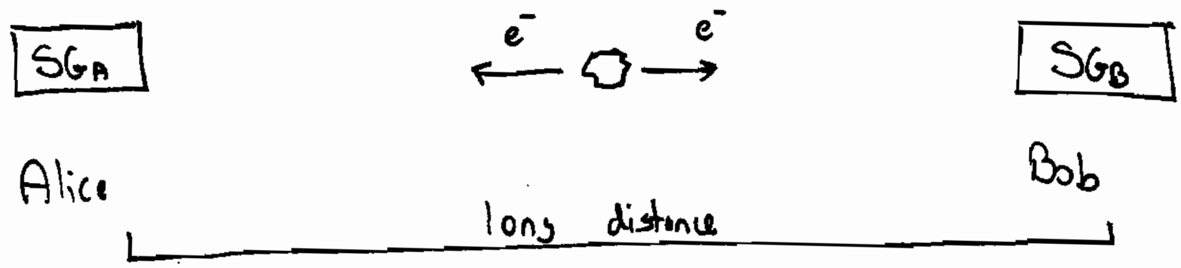
$$\boxed{-2 \leq E_{HV}(\theta,\phi) + E(\theta',\phi) + E(\theta,\phi') - E(\theta',\phi') \leq 2}$$

Look at 1)

> Quantum optical measurements show a violation of Bell's inequality (up to 242σ)!

This demonstrates that no local hidden variable theory can predict the measured results, which are predicted by Quantum Mechanics

Revisit EPR



$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right)$$

1) Alice orientates her SG along $\hat{z}$

   Measures up $S_z = +1$

2) So Bob's particle must have $S_z = -1$ even if he does not measure it.

3) But Bob has a choice, so instead he aligns his SG along $\hat{x}$ and measures $+1$

4) Thru Alice's measurement and Bob's we have measured both S_x and S_z precisely.

~~By Quantum mechanics S_x and S_z do not commute but we have measured both of them~~

But QM tells us that S_x and S_z do not commute so we cannot measure them precisely!

In the language of EPR, since we have determined S_x and S_z precisely, they exist before the measurement. But QM cannot tell us values of both S_x & S_z so QM must be an incomplete theory.

Bohr's Response to EPR

- Complementarity applied after long distances
- The context needed to think about the $\hat{z}$ component of B is not compatible with what is needed to think about $\hat{x}$ component
- Even though we can predict B without disturbing B there is no experimental situation where both S_x & S_z have meaning.

Nonlocality "hidden" in Bohr's Response

The measurement at A "collapses the wavefunction" which predetermines the result a B without the experiment of A interacting with B.

Is there a way to test this nonlocality $\Rightarrow$ Bell's inequality

Back to Alice + Bob

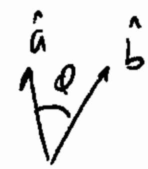
Alice orientates her SG along $\hat{a}$

Bob orientates his SG along $\hat{b}$

$A = 1$ up
$A = -1$ down

$B = 1$
$B = -1$

Look at product AB



$$\langle AB \rangle_{qm} = -\hat{a} \cdot \hat{b} = -\cos\varphi$$

$$E(\hat{a}, \hat{b}) = -\cos\varphi$$

Bell's inequality shows

$$E_{HV}(\hat{a}, \hat{b}) \neq E_{qm}(\hat{a}, \hat{b}) \quad \text{for all } \hat{a} + \hat{b}$$

"No physical theory of local hidden variables can ever produce the predictions of quantum mechanics"

The next ~~a~~ fact about the Bell's inequality is that it can be tested in the lab.

Proof of inequality

Assume locality A does not depend on $\hat{b}$

B does not depend on $\hat{a}$

$A = A(\hat{a}, \lambda)$ not $A(\hat{a}, \hat{b}, \lambda)$

$B = B(\hat{b}, \lambda)$ not $B(\hat{a}, \hat{b}, \lambda)$

} $\lambda \equiv$ hidden variable

Compute expectation value from hidden variable theory

$$\boxed{E_{HV}(\hat{a}, \hat{b}) = \int AB \, d\lambda}$$

Technically

$$E_{HV}(\hat{a}, \hat{b}) = \int d\lambda \; P_A(\hat{a}, \lambda) P_B(\hat{b}, \lambda) \, \rho(\lambda)$$

Derivation

$$E_{HV}(\hat{a}, \hat{b}) - E_{HV}(\hat{a}, \hat{c}) = \int \left(A(\hat{a}, \lambda) B(\hat{b}, \lambda) - A(\hat{a}, \lambda) B(\hat{c}, \lambda) \right) d\lambda$$

$\Rightarrow$ EPR case $A(\hat{a}, \lambda) = -B(\hat{a}, \lambda)$ opposite spins

$$E_{HV}(\hat{a}, \hat{b}) - E(\hat{a}, \hat{c}) = -\int \left(A(\hat{a}, \lambda) A(\hat{b}, \lambda) - A(\hat{a}, \lambda) A(\hat{c}, \lambda) \right) d\lambda$$

$\Rightarrow$ using $|A(\hat{b}, \lambda)|^2 = 1$

$$= -\int A(\hat{a}, \lambda) A(\hat{b}, \lambda) \left[1 - A(\hat{b}, \lambda) A(\hat{c}, \lambda) \right] d\lambda$$

$\Rightarrow$ Note that $A(\hat{a}, \lambda) A(\hat{b}, \lambda) = +1$ or -1

$$\left| E_{HV}(\hat{a},\hat{b}) - E_{HV}(\hat{a},\hat{c}) \right| \leq \left| \int \left(1 - A(\hat{b},\lambda) A(\hat{c},\lambda)\right) d\lambda \right|$$

$$\leq \left| \int \left(1 + A(\hat{b},\lambda) B(\hat{c},\lambda)\right) d\lambda \right|$$

$$\Rightarrow \text{from } A(\hat{c},\lambda) = -B(\hat{c},\lambda)$$

$$\leq 1 + \int A(\hat{b},\lambda) B(\hat{c},\lambda)\, d\lambda$$

So $$\boxed{\left| E_{HV}(\hat{a},\hat{b}) - E_{HV}(\hat{a},\hat{c}) \right| \leq 1 + E_{HV}(\hat{b},\hat{c})}$$

Show QM violates the inequality

$$\left| E(\hat{a},\hat{b}) - E(\hat{a},\hat{c}) \right| \leq 1 + E(\hat{b},\hat{c})$$

Case I $\hat{a} = \hat{b} = \hat{c}$

$\phi = 0$

So $E_{QM}(\hat{a},\hat{b}) = -\cos\phi = -1$

$E_{QM}(\hat{a},\hat{c}) = -1$

$E_{QM}(\hat{b},\hat{c}) = -1$

Into QM inequality

$$\left| (-1) - (-1) \right| \overset{?}{\leq} 1 + (-1)$$

$0 \leq 0$ ✓ OK No violation!

Case II $\hat{b}$ makes $60°$ angle with $\hat{a}$, $\hat{c}$ makes $60°$ angle with $\hat{b}$

$E_{QM}(\hat{a},\hat{b}) = -\cos 60 = -1/2$

$E_{QM}(\hat{a},c) = -\cos(120) = +1/2$

$E_{QM}(\hat{b},\hat{c}) = -\cos(60) = -1/2$

$$\left| (-1/2) - (1/2) \right| \overset{?}{\leq} 1 + (-1/2)$$

$1 \overset{?}{\leq} 1/2$ NOT OK Violation!

Objections on the concept of nonlocality

Newton	"philosophical absurdity"
Einstein	"Spooky" action at a distance
Bohm	" cannot see any well-founded reason for such objections ... "
Aspect	See Nature Paper

EPR + Bell's Th$^{\underline{m}}$

Review

Bell's Th$^{\underline{m}}$

No local realistic hidden variable theory will give the same results as QM

$$\boxed{\begin{array}{c} E_{HV}(\hat{a}, \hat{b}) \neq E_{QM}(\hat{a}, \hat{b}) \\ \text{for all } \hat{a} + \hat{b} \end{array}}$$

Bell's Inequality (with respect to EPR)

$$\left| E_{HV}(\hat{a}, \hat{b}) - E_{HV}(\hat{a}, \hat{c}) \right| \leq 1 + E_{HV}(\hat{b}, \hat{c})$$

QM violates this inequality

We will now look at a more useful version of the Bell's inequality

Clauser Horne Shimony Holt

11/30/10

Optical Test of EPR : Experimental violations of the Bell's inequality

We will discuss ~~two~~ three experiments that perform Bohm's version of EPR with photons

1) Aspect et al.

   Three papers here

   a) PRL 47 (7) 1981 Single channel detectors

   b) PRL 49 (2) 1982 Double channel detectors

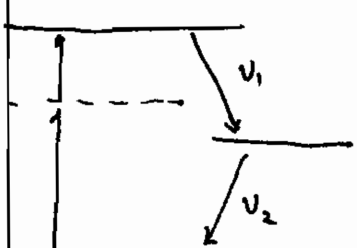
   c) PRL 49 (25) 1982 Time gated ~~[illegible]~~ ~~single~~ channel detectors

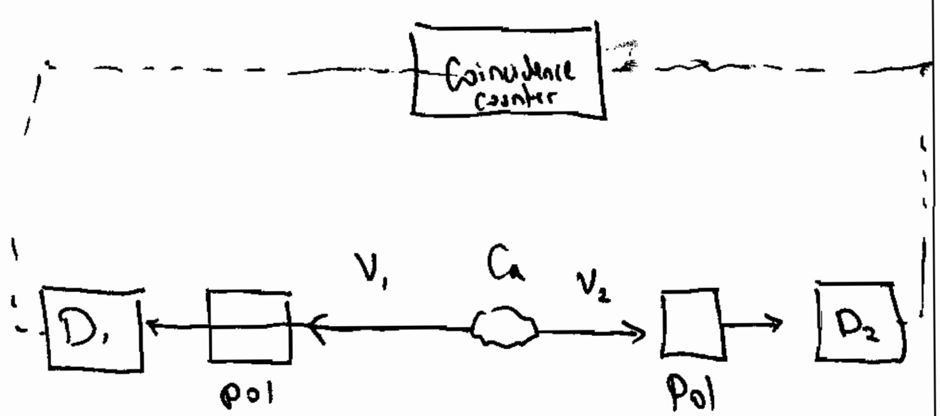
2) Ou + Mandel PRL 61 (1) 1988

3) Hong Ou + Mandel PRL 59 (18) 1988

a) Aspect et al PRL 47 (7) 1981

Used Cascaded Ca Source with single channel detectors



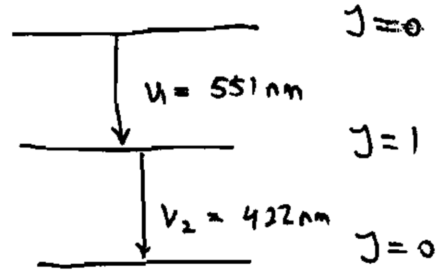


Problem : single channel detectors. If you don't detect a photon
Was it due to polarization or detector efficiency?

Important to note:

photon polarization states are isomorphic to spin 1/2 states (EPR)

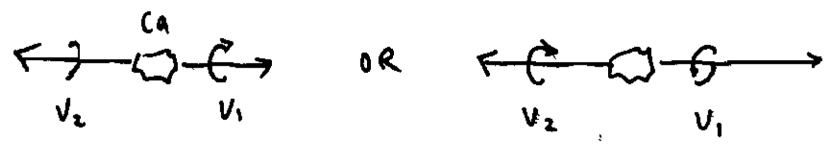
Decay path



angular momentum conserved : photons have opposite angular momenta

spin angular momentum $\Rightarrow$ related to polarization

if photons in opposite directions $\Rightarrow$ Handedness must be same



$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|R\rangle_1 |R\rangle_2 + |L\rangle_1 |L\rangle_2 \right)$$

Aspect experiment $|H\rangle$ + $|V\rangle$

$$|R\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle + i\,|V\rangle\right) \qquad |L\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle - |V\rangle\right)$$

So

$$\boxed{|\psi\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle_1 |H\rangle_2 - |V\rangle_1 |V\rangle_2 \right) \quad (\text{Bell State } |\Phi^-\rangle)}$$

After measurement ! If photon 1 is horizontal photon 2 will be verticle

Define A + B

$A = +1$ pol $\parallel$ to alice $\hat{a}$

$A = -1$ pol $\perp$ to alice $\hat{a}$

$B = +1$ pol $\parallel$ to Bob's $\hat{b}$

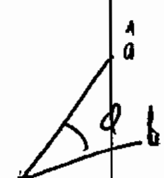
$B = -1$ pol $\perp$ to Bob's $\hat{b}$

$$E(\hat{a},\hat{b}) = AB(++)\,P_{++}(\hat{a},\hat{b}) + AB(--)\,P_{--}(\hat{a},\hat{b}) + AB(+-)\,P_{+-}(\hat{a},\hat{b}) + AB(-+)\,P_{-+}(\hat{a},\hat{b})$$

AB $= 1$ if Bob + Alice get same result

-1 if Bob + Alice get different results

So
$$\boxed{E(\hat{a},\hat{b}) = P_{++}(\hat{a},\hat{b}) + P_{--}(\hat{a},\hat{b}) - P_{+-}(\hat{a},\hat{b}) - P_{-+}(\hat{a},\hat{b})}$$



From QM
$$P_{++}(\hat{a},\hat{b}) = P_{--}(\hat{a},\hat{b}) = \tfrac{1}{2}\cos 2\phi$$
$$P_{+-}(\hat{a},\hat{b}) = P_{-+}(\hat{a},\hat{b}) = \tfrac{1}{2}\sin(2\phi)$$

$$E_{QM}(\hat{a},\hat{b}) = \cos 2\phi$$

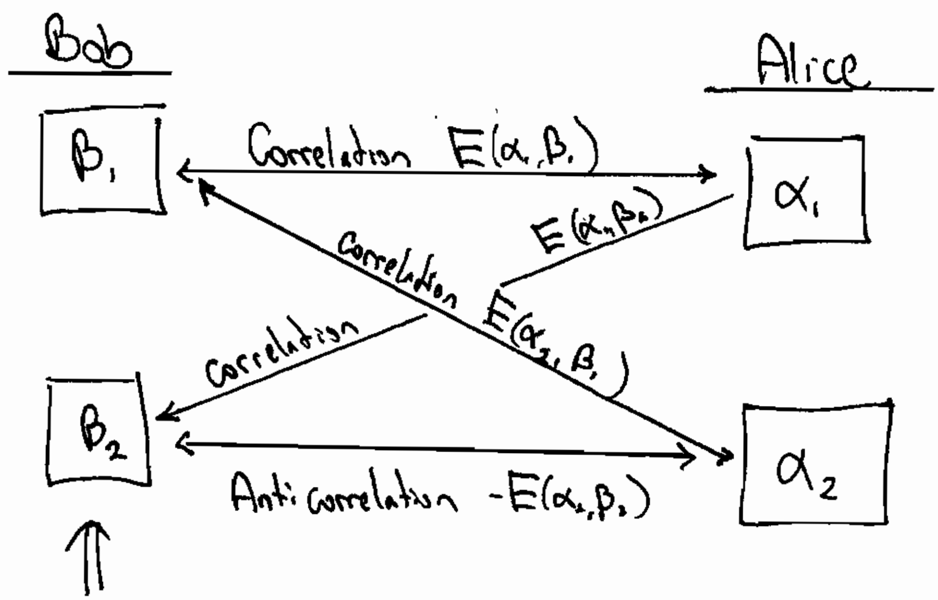
<u>Generalization of Bells' Th$^{\underline{m}}$ (Clauser Horne Shimony Holt PRL 1969)</u>

Bell's inequality (Based on Bohm's version of EPR gedankenexperiment)

Define $S_{HV} = E_{HV}(\alpha_1,\beta_1) + E_{HV}(\alpha_1,\beta_2) + E_{HV}(\alpha_2,\beta_1) - E_{HV}(\alpha_2,\beta_2)$

Then

$$-2 \leq S_{HV} \leq 2$$



Two choices of Bob's settings

Two choices of Alice's settings

Proof

First start with Mermin's lemma

Mermin's lemma: If $x_1\ x_2\ y_1 + y_2$ are real #'s in interval $[-1,+1]$ then $S \equiv x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2$ lies in interval $[-2,2]$

Proof S is a linear function of each variable so it takes an extreme value when one of the arguments is extreme.

$$(x_1\ x_2\ y_1\ y_2) = (\pm 1, \pm 1, \pm 1, \pm 1)$$

$\boxed{\text{Four terms so } |S| \le 4}$ (1)

Rewrite $S = \underbrace{(x_1+x_2)(y_1+y_2)} - 2x_2y_2$

x_1+x_2 Extrema 0 or ± 2

$(x_1+x_2)(y_1+y_2)$ Extrema 0 or ± 4

$2x_2y_2$ Extrema ± 2

$\boxed{S \quad \text{Extrema} \quad \pm 2 \text{ or } \pm 6}$

But from (1) the extrema of S cannot be ± 6

Thus. Extrema of S are ± 2

$$|S| \le 2$$

$$-2 \le S \le 2$$

Now For CHSC Bell's Inequality

Let
$$x_1 = E(\lambda, \alpha_1)$$
$$x_2 = E(\lambda, \alpha_2)$$
$$y_1 = E(\lambda, \beta_1)$$
$$y_2 = E(\lambda, \beta_2)$$

From Mermin's lemma

$$-2 \le E(\lambda,\alpha_1)E(\lambda,\beta_1) + E(\lambda,\alpha_1)E(\lambda,\beta_2) + E(\lambda,\alpha_2)E(\lambda,\beta_1) - E(\lambda,\alpha_2)E(\lambda,\beta_2) \le 2$$

Assuming locality $\quad E(\lambda,\alpha,\beta) = E(\lambda,\alpha)E(\lambda,\beta)$

$$-2 \le E(\lambda,\alpha_1,\beta_1) + E(\lambda,\alpha_1,\beta_2) + E(\lambda,\alpha_2,\beta_1) - E(\lambda,\alpha_2,\beta_2) \le 2$$

Integrate over λ $\quad \int \rho(\lambda)\, d\lambda$

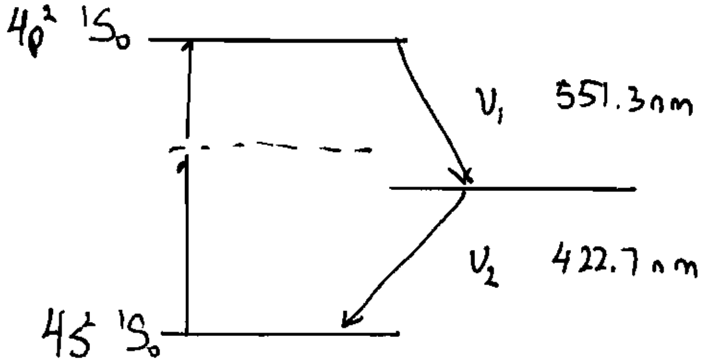
Thus $\quad \boxed{-2 \le S \le 2}$

b) Aspect et al PRL 49(2) 1982

Use gated single photon source described before (Ca)

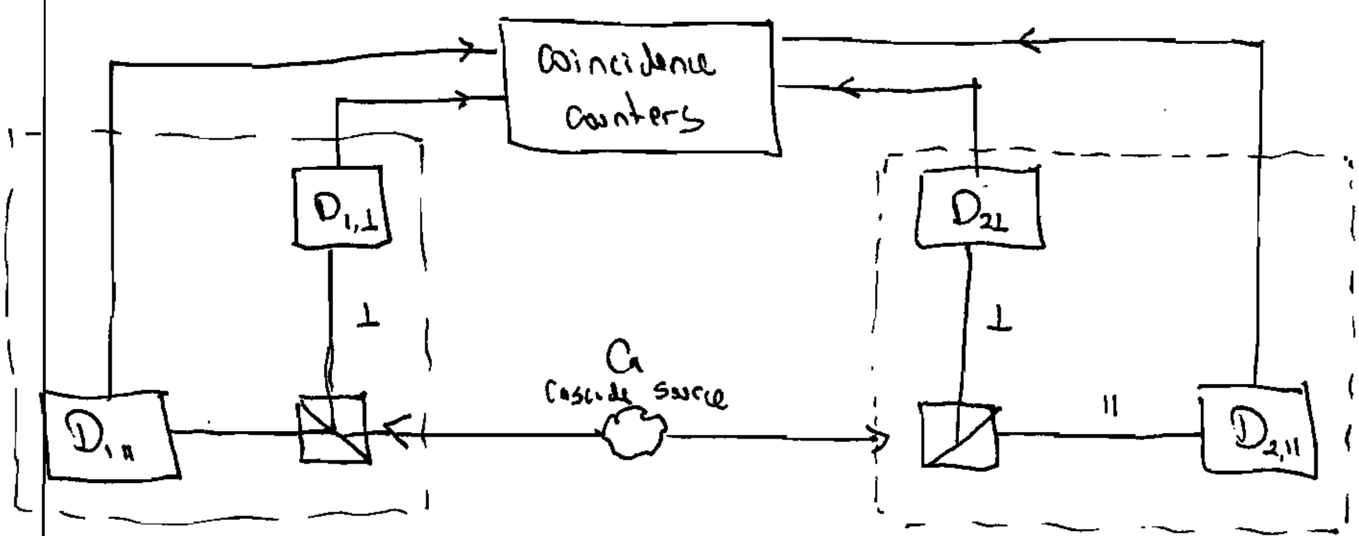
Europhys Let

PRL 47 (7) 1981



Ca cascade source

Experimental setup



rotate stage "Alice"

rotate stage "Bob"

Measurements

$P_{\pm\pm}(\hat{a}, \hat{b})$ Prob of measuring $\pm$ along $\hat{a}$
$\pm$ along $\hat{b}$

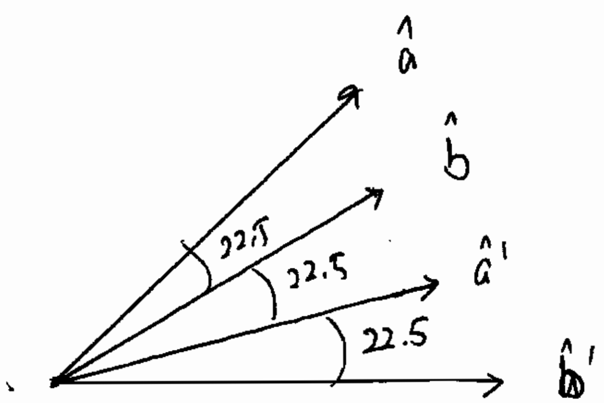
Measure coincident rates $R_{\pm\pm}(\hat{a}, \hat{b})$

$\Longrightarrow$ problem : low detector efficiency

Measured

$$E(\vec{a}, \vec{b}) = \frac{R_{++}(\hat{a}, \hat{b}) + R_{--}(\hat{a}, \hat{b}) - R_{+-}(\hat{a}, \hat{b}) - R_{-+}(\hat{a}, \hat{b})}{\text{total counts}}$$

do for orientations $\vec{a}, \vec{b}$
pick different orientations



Local hidden variable theory says $\quad -2 \le S \le 2$

QM gives

$$E(\hat{a}, \hat{b}) = F \frac{(T_1^{\parallel} - T_1^{\perp})(T_2^{\parallel} - T_2^{\perp})}{(T_1^{\parallel} + T_1^{\perp})(T_2^{\parallel} + T_2^{\perp})} \cos(2\theta)$$

F: efficiency, solid angles

$\boxed{S_{QM} = 2.70 \pm 0.05}$ } For perfect detectors + lossless cubes $\boxed{S_{QM} = 2\sqrt{2}}$

Measured $\boxed{S_{exp} = 2.697 \pm 0.015}$ (violates LHV by 40σ).

Issues with measurement : loopholes

$\Rightarrow$ Low detector efficiency

For non ideal detectors

$$E(\hat{a},\hat{b}) = -\eta_{det}^2 \cos 2\theta$$

QM gives $\quad S_{QM} = \eta_{det}^2\, 2\sqrt{2}$

Bell's inequality violated when $\quad S > 2$

So the detector efficiency must be larger than

$$\eta_{det} \geq 1/\sqrt[4]{2} \simeq 0.84$$

$\boxed{84\%!}$

Thus not all pairs will register counts

Fair sampling Assumption

Detectors perform fair sampling of ensemble of all events

When photons <u>are</u> detected they represent the ensemble.

$$E(\hat{a},\hat{b}) = \frac{\text{Average}(\ A(\hat{a})\ B(\hat{b})\ )}{\text{Average}\ (\ N_A\ N_B\ )} \sim \frac{\eta_{eff}^2}{\eta_{eff}^2}$$

$$\text{Average}(N_A, N_B) = P(\hat{a},\hat{b}) + P(\hat{a}^{\perp},\hat{b}^{\perp}) + P(\hat{a},\hat{b}^{\perp}) + P(\hat{a}^{\perp},\hat{b})$$

Dectection Loophole $\underline{\underline{\text{NOT}}}$ resolved

Loopholes

$\Rightarrow$ Static experiment / Locality Loophole

possible orientation at Bob could be signaled to Alice.

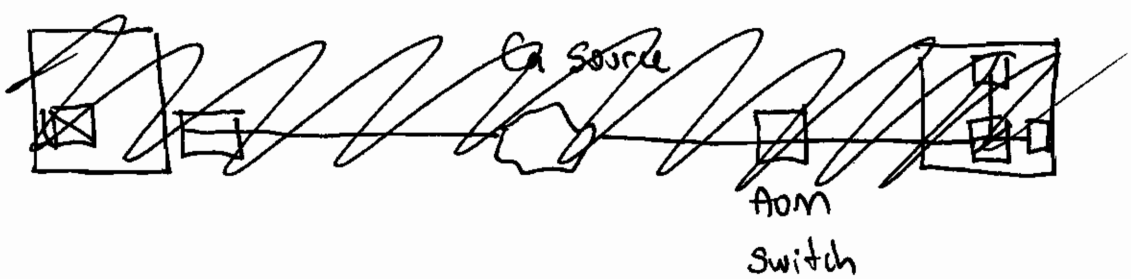
$$A = A(\hat{a}, \lambda) \quad \underline{\underline{\text{not}}} \quad A(\hat{a}, \hat{b}, \lambda)$$

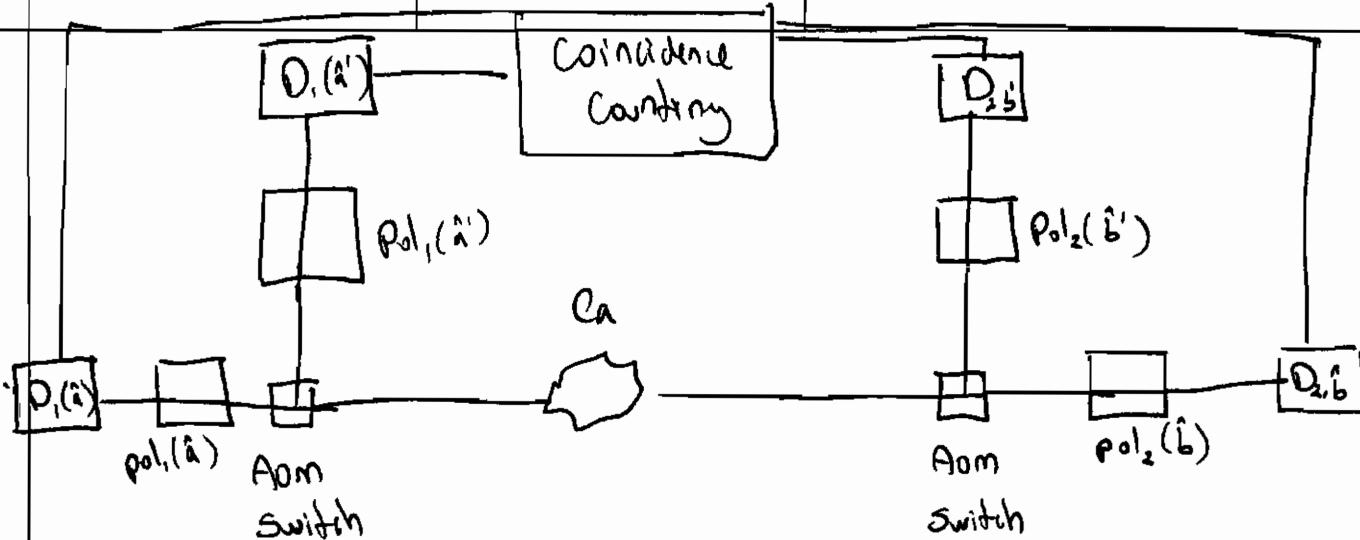
(?) Resolved in Aspect. PRL 49 1982

- use AOM as switch
- Change on time ~~[illegible]~~ shorter than time for light to travel from Alice to Bob (40 ft)

  $\Rightarrow$ 40ns

- switch orientation of flight path of each photon between the two polarizers.





Schematic of Aspect PRL 49 1982

$\boxed{\text{Loophole resolved}}$

also by further experiments

~~Single Photon Sources~~

Generation of Polarization Entangled States

1. Spontaneous parametric down conversion

Nonlinear optics with quantized pump electric field

Interaction Hamiltonian

$$\boxed{\hat{H}_I \sim \chi^{(2)} \hat{a}_p \hat{a}_s^{\dagger} \hat{a}_i^{\dagger} + \chi^{*(2)} \hat{a}_p^{\dagger} \hat{a}_s \hat{a}_i}$$

H.C. = Hermitian conjugate

post selection after apeture

(destroy ~~pho~~ pump photon to create signal + idler photon)

spontaneous

$$|1\rangle_p |0\rangle_s |0\rangle_i \overset{\chi^{(2)}}{\Longrightarrow} \hat{a}_p \hat{a}_s^{\dagger} \hat{a}_i^{\dagger} |1\rangle_p |0\rangle_s |0\rangle_i$$

$$= |0\rangle_p |1\rangle_s |1\rangle_i$$

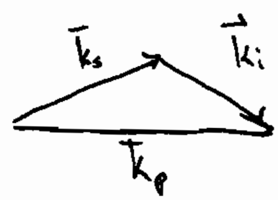
$$\omega_p = \omega_s + \omega_i$$
$$\vec{k}_p = \vec{k}_s + \vec{k}_i$$

Two types : Type I + Type II

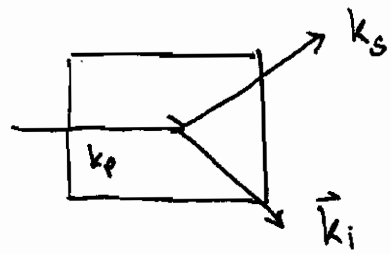
- Type I : signal + idler have the same polarization.

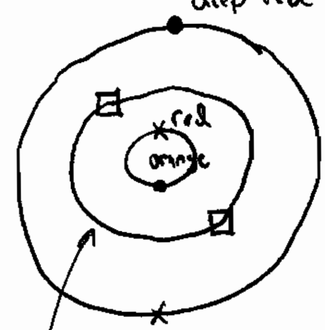
$$\hat{H}_I = \hbar \eta \hat{a}_s^{\dagger} \hat{a}_i^{\dagger} \qquad \eta \sim \chi^{(2)} \mathcal{E}_p$$

phasematching
noncolinear



Signal + idler emmerge on opposite sides of concentric cones





Symbols ⇒ conjugate pairs

Type II down conversion

Signal + idler have ⊥ polraizations

Becuase of crystal birefringence get two cones

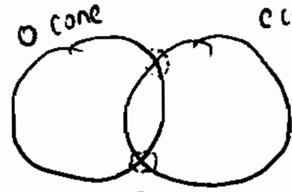
Ordinary (o) cone

extraoridinary (e) cone

$|V\rangle$ Horizontal + verticle polarized single photon states

$|H\rangle$

* photons at intersection will have ambiguity to whether the signal or idler photons will be horizontally or vertically polarized.



Cambility in $|H\rangle$ or $|V\rangle$

For Type II

$$\hat{H}_I = \hbar\eta\left(\hat{a}^\dagger_{Vs}\,\hat{a}^\dagger_{Hi} + \hat{a}^\dagger_{Hs}\,a^\dagger_{Vi}\right) + H.C.$$

post selection after pinholes

Start with $|\psi_0\rangle = |\{0\}\rangle$ collective vacuum

$$|\psi(t)\rangle = \exp(-it\hat{H}_I/\hbar)\,|\psi_0\rangle$$

H_I has no explicit time dependence

$$|\psi(t)\rangle \simeq \left[1 - it\hat{H}_I/\hbar + \tfrac{1}{2}(-it\hat{H}_I/\hbar)^2\right]|\psi_0\rangle$$

For Type I

$$\boxed{|\psi(t)\rangle \cong \left(1-(\eta t)^2/2\right)|0\rangle_s|0\rangle_i - i(\eta t)\,|1\rangle_s|1\rangle_i}$$

For Type II

$$\boxed{\begin{aligned}|\psi(t)\rangle \cong\ & \left(1-(\eta t)^2/2\right)|0\rangle_{Vs}|0\rangle_{Hs}|0\rangle_{Vi}|0\rangle_{Hi} \\ & -i\eta t\,\frac{1}{\sqrt{2}}\left(|1\rangle_{Vs}|0\rangle_{Hs}|0\rangle_{Vi}|1\rangle_{Hi} + |0\rangle_{Vs}|1\rangle_{Hs}|1\rangle_{Vi}|0\rangle_{Hi}\right)\end{aligned}}$$

Define $|0\rangle := |0\rangle_V|0\rangle_H$ $\quad |V\rangle := |1\rangle_V|0\rangle_H$ $\quad |H\rangle := |0\rangle_V|1\rangle_H$

$$\begin{aligned}|\psi(t)\rangle =\ & \left(1-(\eta t)^2/2\right)|0\rangle_s|0\rangle_i \\ & -i(\eta t)\ |V\rangle_s|H\rangle_i + |H\rangle_s|V\rangle_i\end{aligned}$$

2nd term call it $|\psi^+\rangle$

$$\boxed{|\psi^+\rangle = \frac{1}{\sqrt{2}}\left(|V\rangle_s|H\rangle_i + |H\rangle_s|V\rangle_i\right)}$$

Get 1 out of 4 Bell state's

$$|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle_1 |V\rangle_2 \pm |V\rangle_1 |H\rangle_2\right)$$

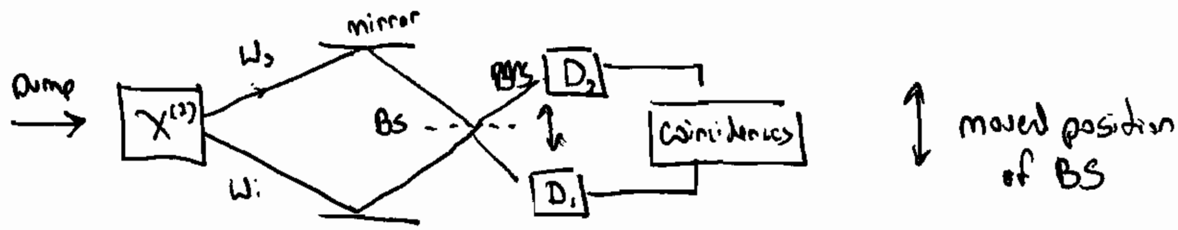
$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle_1 |H\rangle_2 \pm |V\rangle_1 |V\rangle_2\right)$$

Hong-Ou-Mandel Interferometer

Two single photon states on Beamsplitter

$$\boxed{\begin{array}{l} |1\rangle_1 |1\rangle_0 \overset{BS}{\Rightarrow} \frac{1}{\sqrt{2}}\left(|2\rangle_2 |0\rangle_3 + |0\rangle_2 |2\rangle_3\right) \\ \text{Do } \underline{\text{not}} \text{ see simultaneous clicks at } D_1 + D_2 \end{array}}$$

- Use Type I source KDP : Measure time separation two photons hitting BS



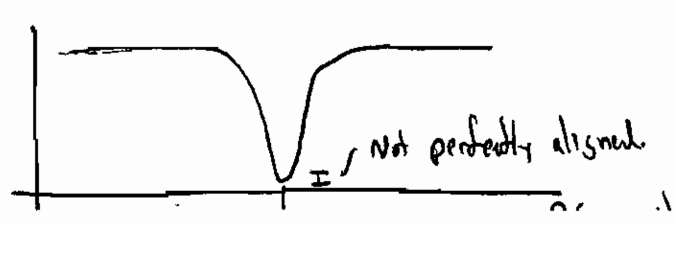
- Changing BS position causes a slight time delay when signal or idler arrive at BS.

- With a slight delay the photons independently reflect or transmit thru the beam splitter, causing both detectors to fire within a short time of each other.

$$R_{coincidence} \sim 1 - e^{-(\Delta\omega)^2(\tau_s - \tau_i)^2} \qquad \left\{ \begin{array}{l} l_s = c\tau_s \\ l_i = c\tau_i \end{array} \right.$$

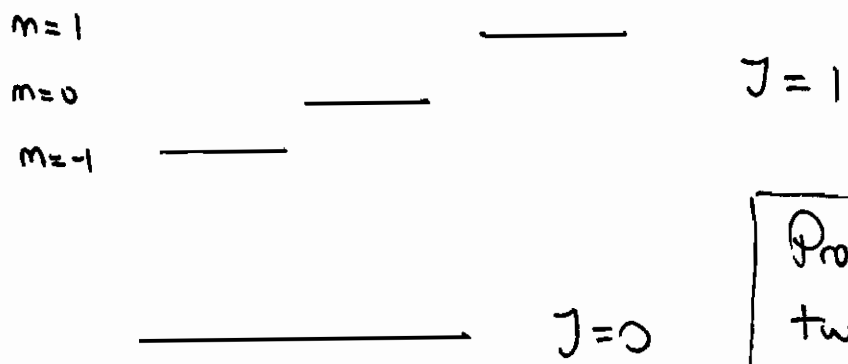
for $\tau_s - \tau_i = 0$ $R_{coinc} = 0$

Rate of of coincidences is max at $|\tau_s - \tau_i| \gg \tau_{corr}$ $\quad \tau_{corr} = \frac{1}{\Delta\omega}$



2. Cascade Emission for generation of polarization entangled states

Transition $J \rightarrow 0 \rightarrow 1 \rightarrow 0$



Process produces two photons

Output State

$$\boxed{|\psi\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle_1 |H\rangle_2 + |V\rangle_1 |V\rangle_2 \right)}$$

rewrite in terms of circularly polarized light $|+\rangle$ $|-\rangle$

$$\boxed{|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle_1 |-\rangle_2 + |-\rangle_1 |+\rangle_2 \right)}$$

Photons Like Spins

<u>More optical tests of local realistic theories</u> : EPR with photons from parametric sources

Type I + Type II processes can generate the Bell States

$$|\psi^{\pm}\rangle, \quad |\Phi^{\pm}\rangle$$

Start with $\quad |\psi^{-}\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle_1 |V\rangle_2 - |V\rangle_1 |H\rangle_2\right)$

Define θ + ϕ : directions for mode 1 + 2

For $\phi = \theta$

$$\boxed{|\psi^{-}\rangle = \frac{1}{\sqrt{2}}\left(|\theta_a\rangle_1 |\theta^{\perp}\rangle_2 - |\theta^{\perp}\rangle |\theta\rangle\right)}$$

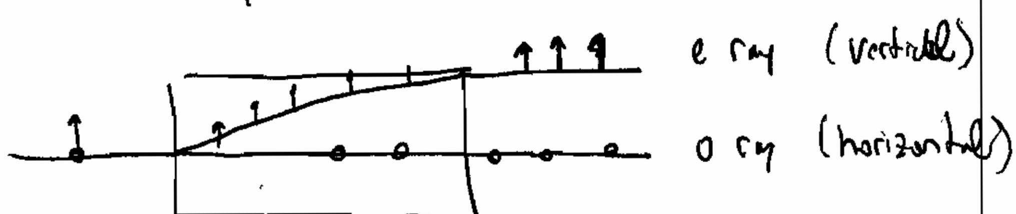
where $\quad |\theta\rangle_1 \equiv \cos\theta\, |H\rangle_1 + \sin\theta\, |V\rangle_1$

$\quad |\theta^{\perp}\rangle_1 \equiv -\sin\theta\, |H\rangle_1 + \cos\theta\, |V\rangle_1$

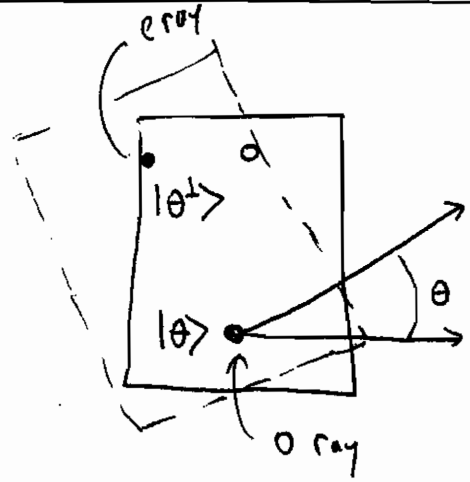
$|\psi^{-}\rangle \equiv$ equivalent to singlet spin state

<u>For testing Bell's inequality</u>

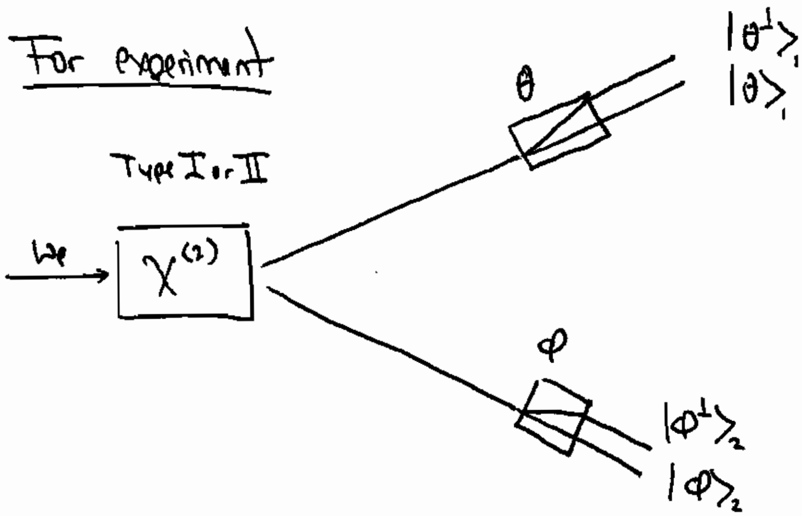
Use calcite crystal



(Like Stern Gerlach)



For experiment

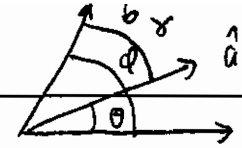


Consider

$$|H\rangle \equiv \text{o ray} \qquad |V\rangle \equiv \text{e ray}$$

For rotation by θ $\qquad |\theta\rangle$ o ray

$\qquad\qquad\qquad\qquad |\theta^\perp\rangle$ e ray



$\gamma = \theta - \varphi$

Define correlation
(Show we get the same correlation)

$$E(\theta,\varphi) = \text{Ave}\left[\ A(\theta)B(\varphi)\ \right]$$

$A(\theta) = +1$ photon out of "o" beam $|\theta\rangle_1$

$\quad\;\; = -1$ photon out of "e" beam $|\theta^\perp\rangle_1$

$B(\varphi) = +1$ photon out of o beam $|\varphi\rangle_2$

$\qquad\qquad$ " $\qquad\qquad$ " $|\varphi^\perp\rangle_2$

Product $A(\theta)B(\varphi)$

$+1 \quad |\theta\rangle_1 |\varphi\rangle_1$ or $|\theta^\perp\rangle_1 |\varphi^\perp\rangle_2$

$-1 \quad |\theta\rangle_1 |\varphi^\perp\rangle_2$ or $|\theta^\perp\rangle |\varphi\rangle_2$

$$\boxed{E(\theta,\varphi) = -\cos 2(\theta-\varphi) = -\cos 2\gamma}$$

Get same result as for spin ½ states

Can get this also using

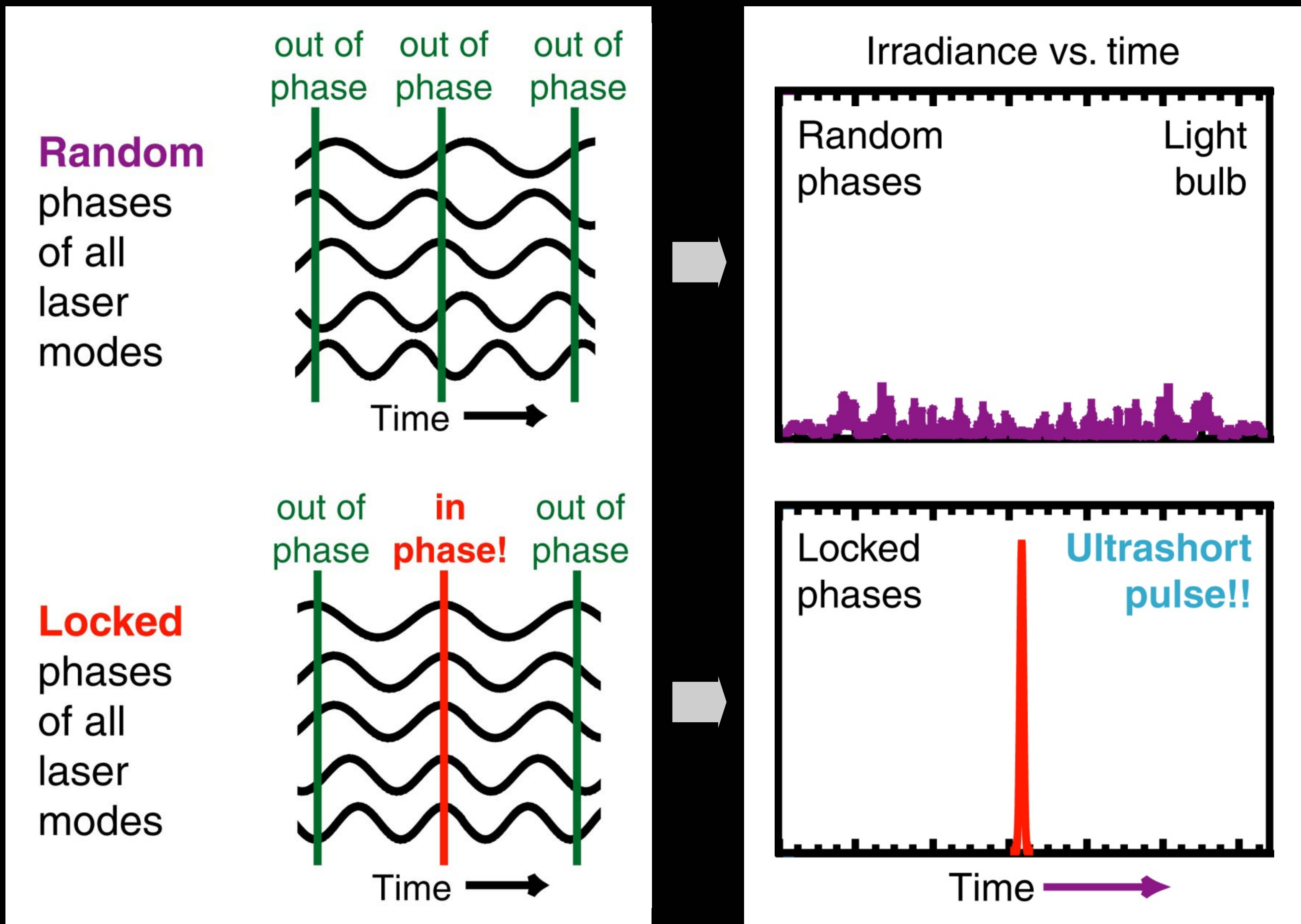
$$\boxed{E(\theta,\varphi) = \langle\psi^-|\hat{\Sigma}_3^{(1)}(\theta)\,\hat{\Sigma}_3^{(2)}(\varphi)|\psi^-\rangle = -\cos(2(\theta-\varphi))}$$

<u>where</u> $\hat{\Sigma}_3^{(1)}(\theta) = |\theta\rangle_1\ {}_1\langle\theta| - |\theta^\perp\rangle_1\ {}_1\langle\theta^\perp|$

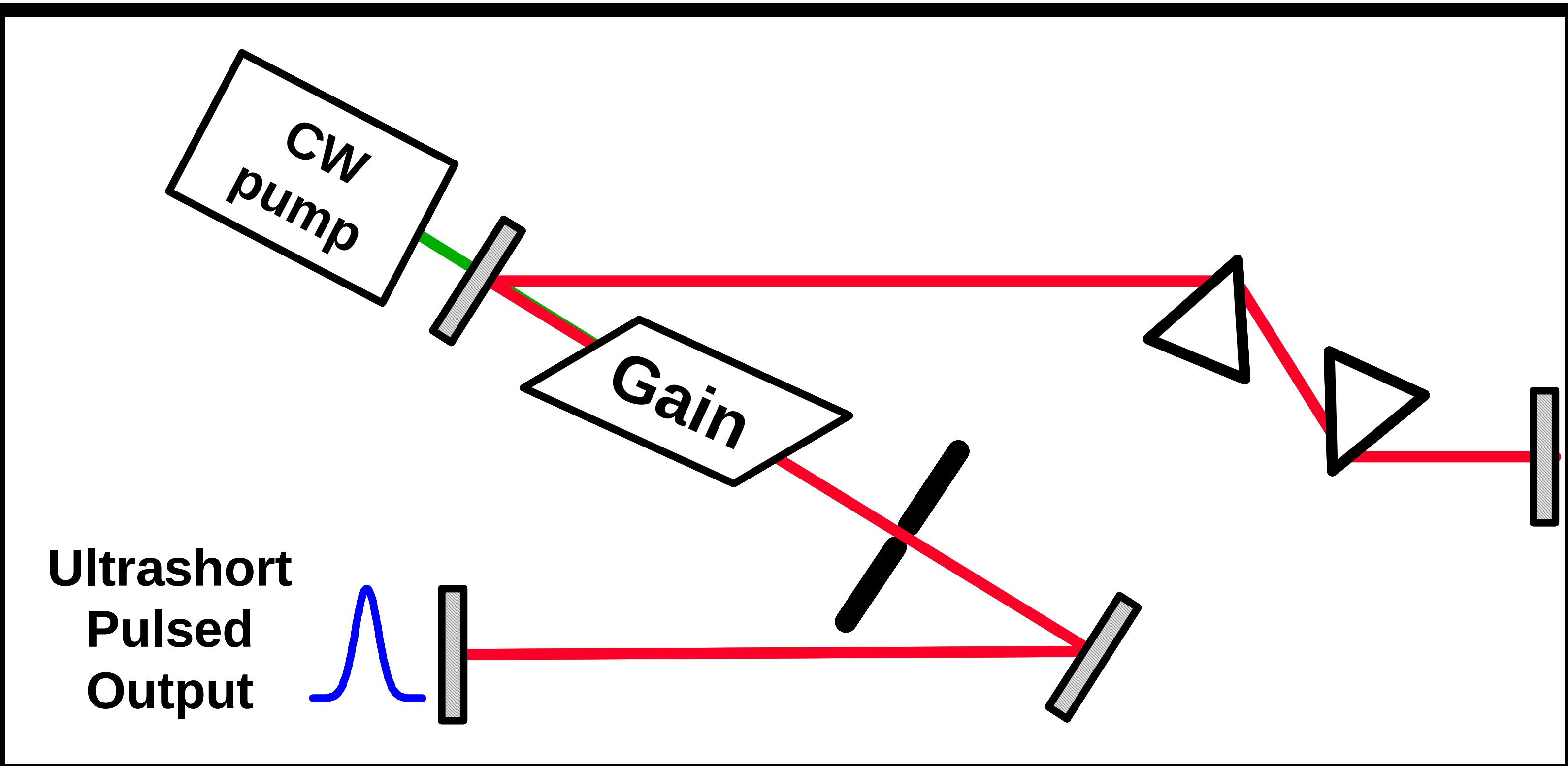
$\hat{\Sigma}_3^{(2)}(\varphi) = |\varphi\rangle_2\ {}_2\langle\varphi| - |\varphi^\perp\rangle_2\ {}_2\langle\varphi^\perp|$

Lecture Power Point Slides

Generating short pulses = mode-locking



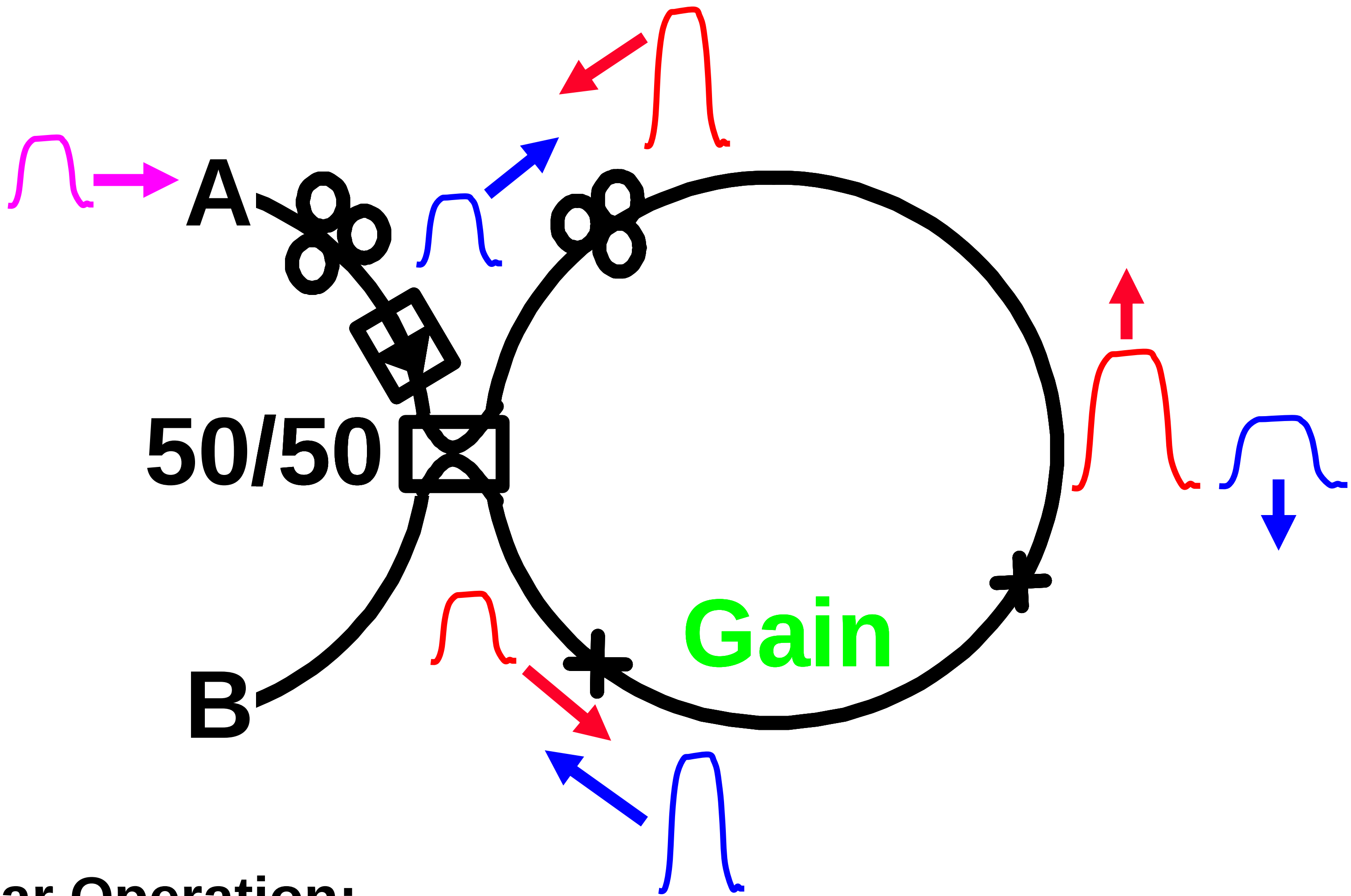
Kerr Lens Modelocked Laser



Elements of mode-locked lasers

- Feedback
- Pump source
- Gain element
- Saturable absorber
- Dispersion compensation

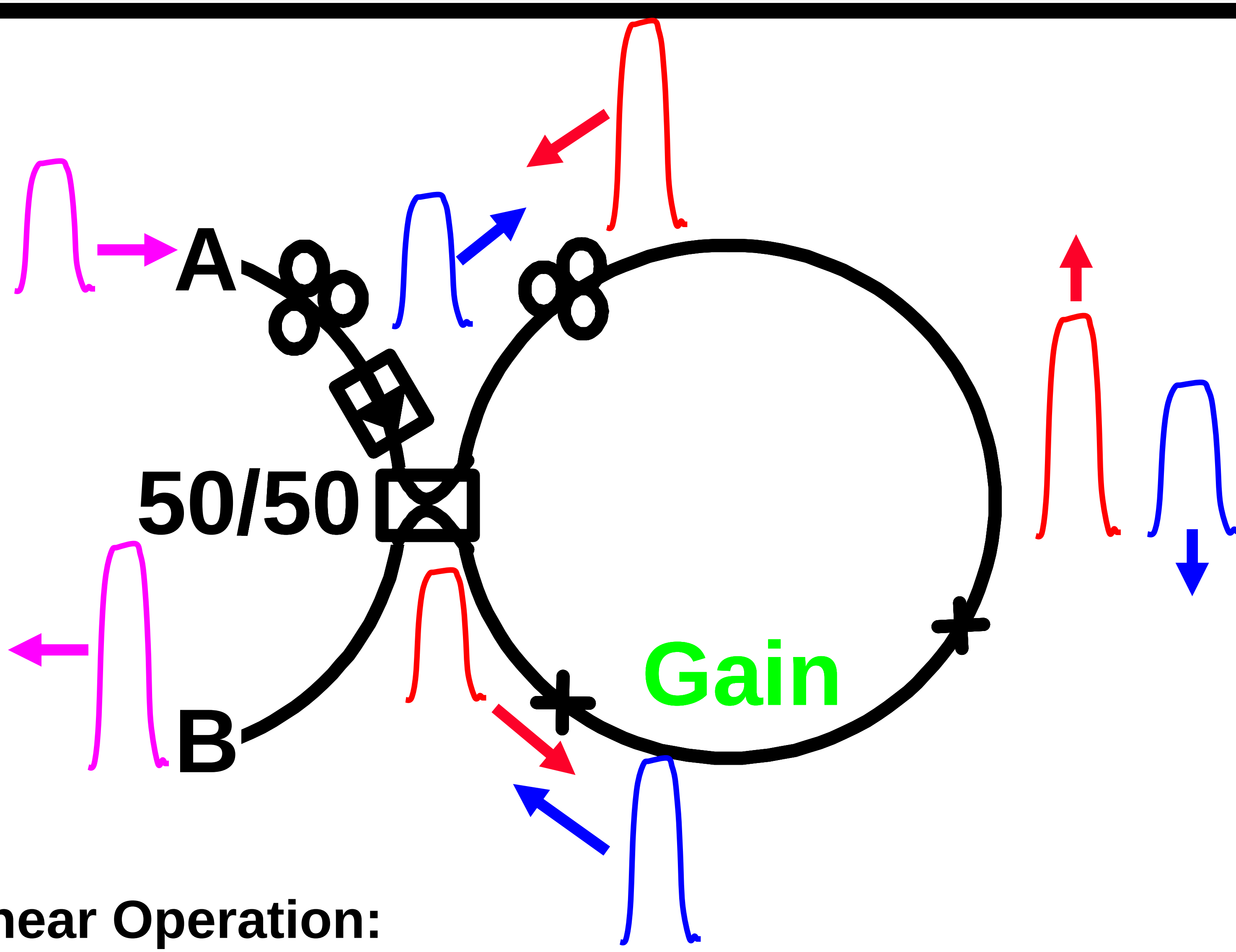
Nonlinear Loop Mirror: Linear Operation



Linear Operation:

No phase shift between interferometer arms

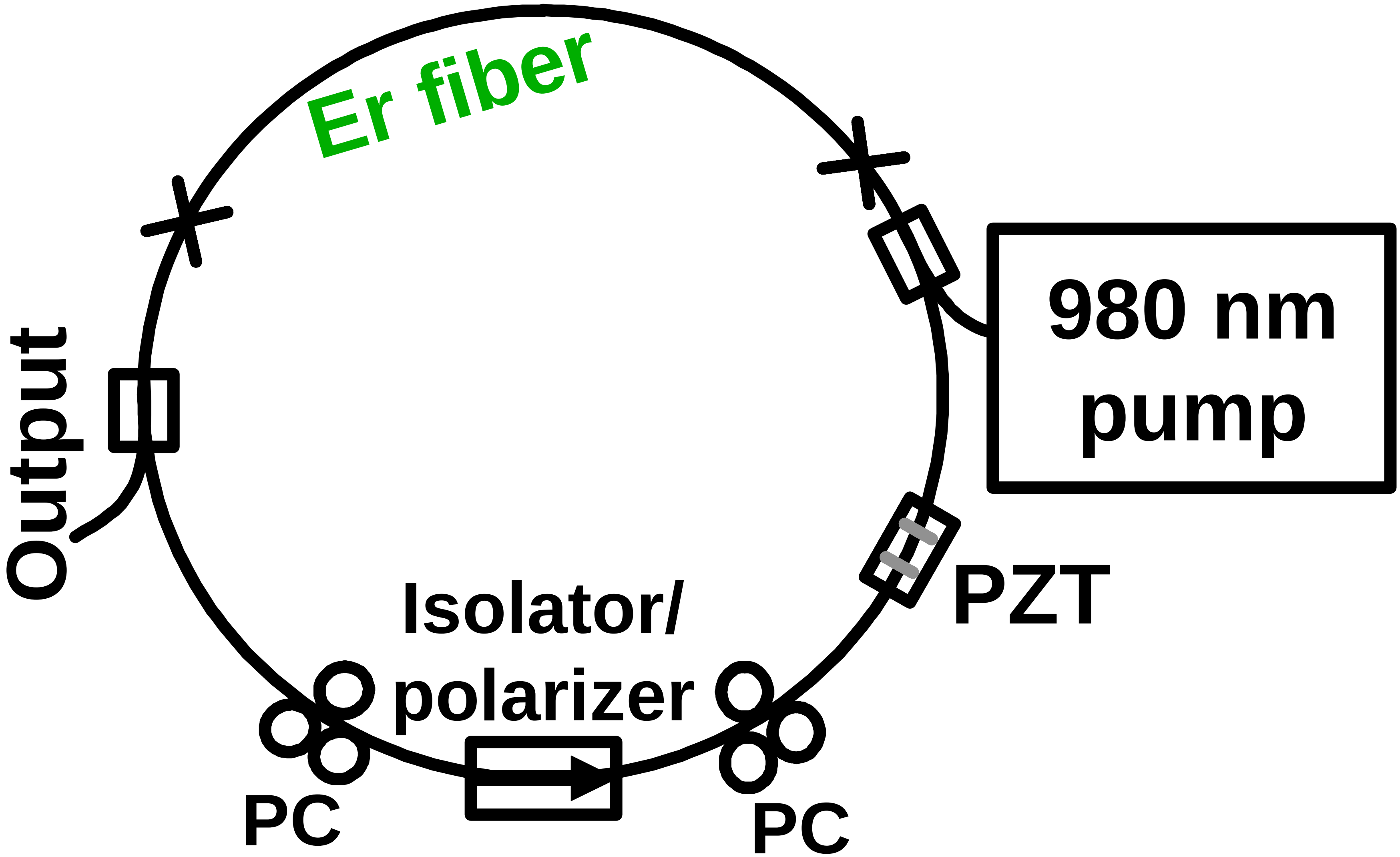
Nonlinear Loop Mirror: Nonlinear Operation



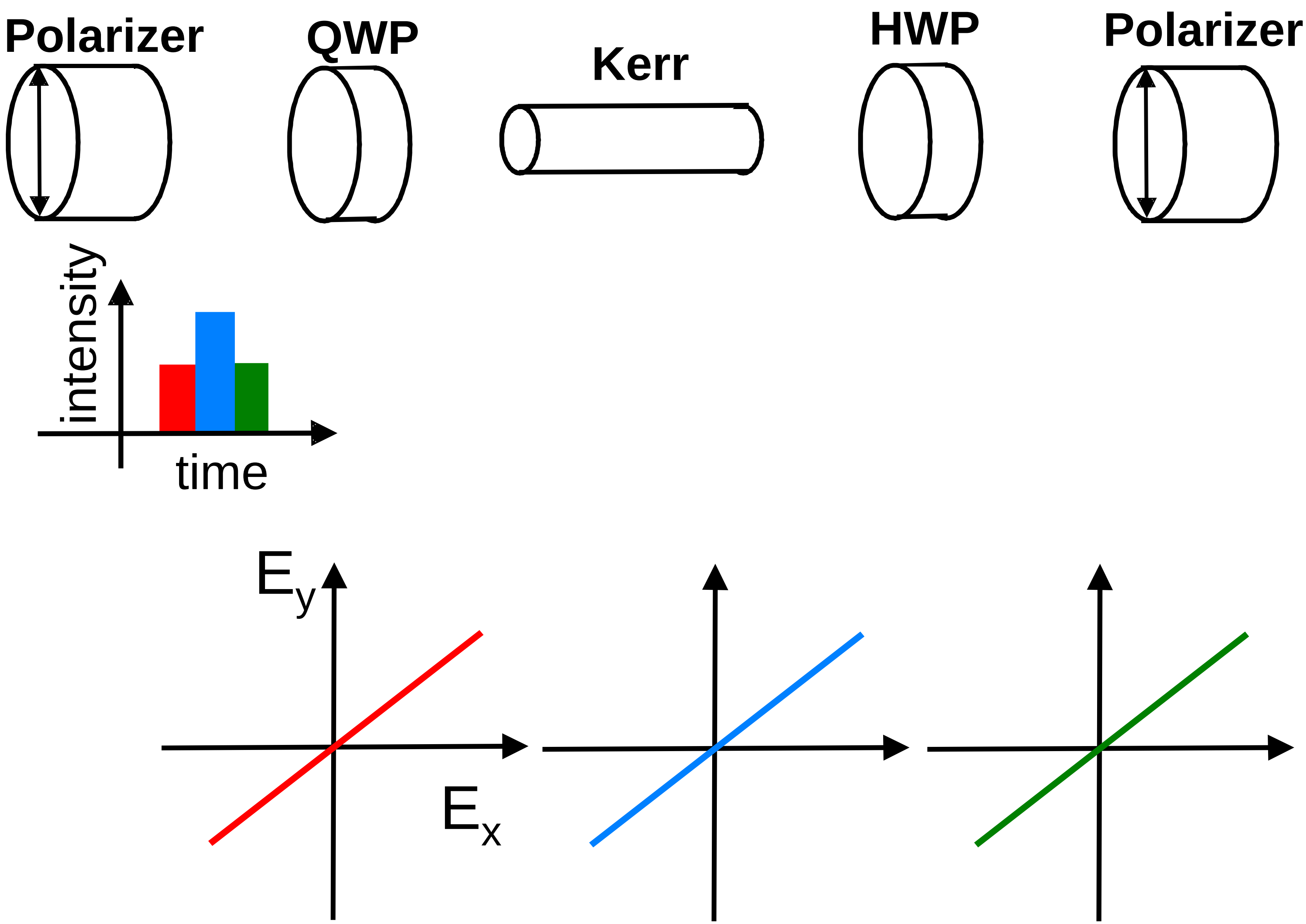
Nonlinear Operation:

$$\text{Phase Difference: } \Delta\varphi \propto n_2 (G-1) I(t) L$$

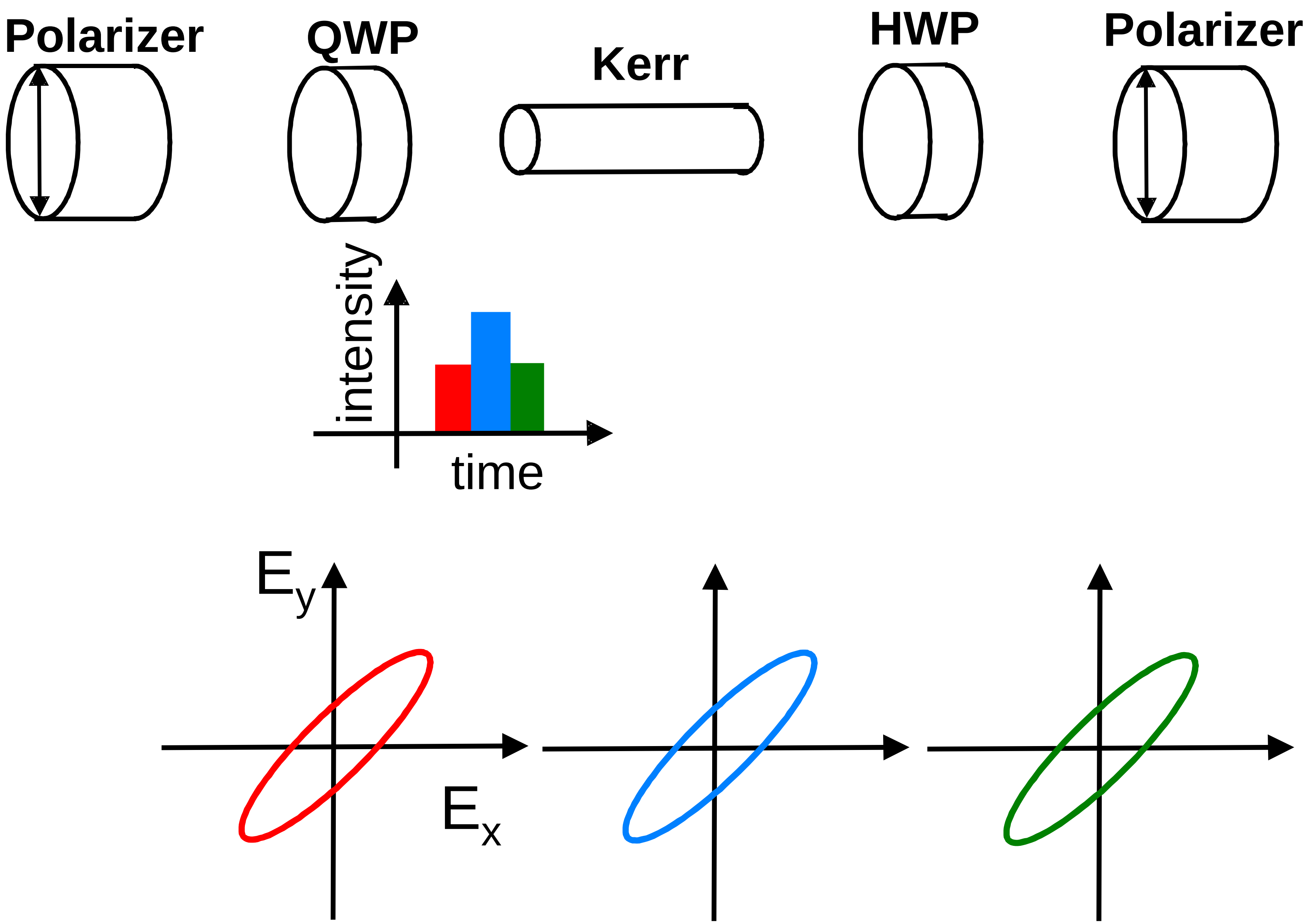
Soliton Ring Laser



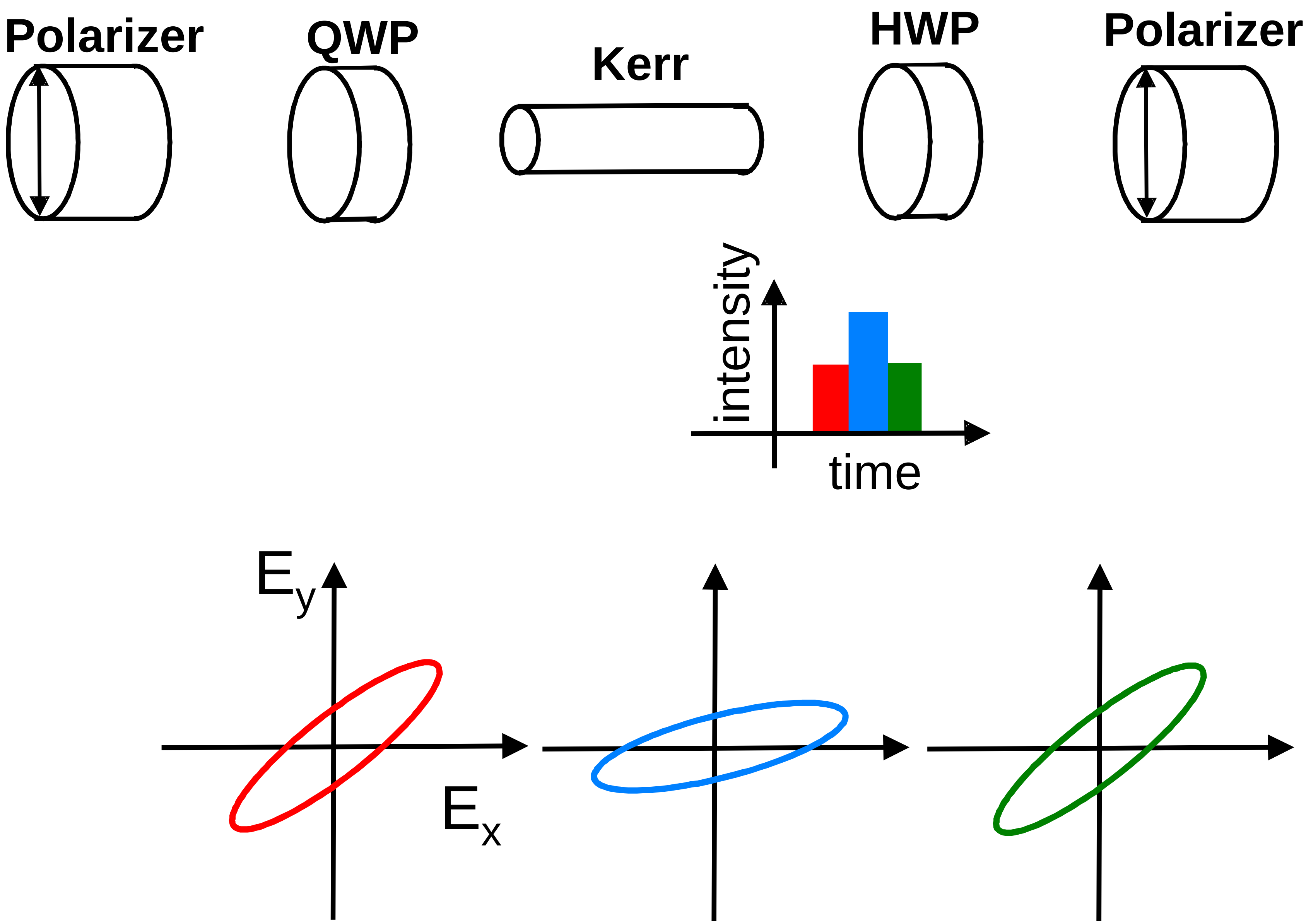
Nonlinear Polarization Rotation



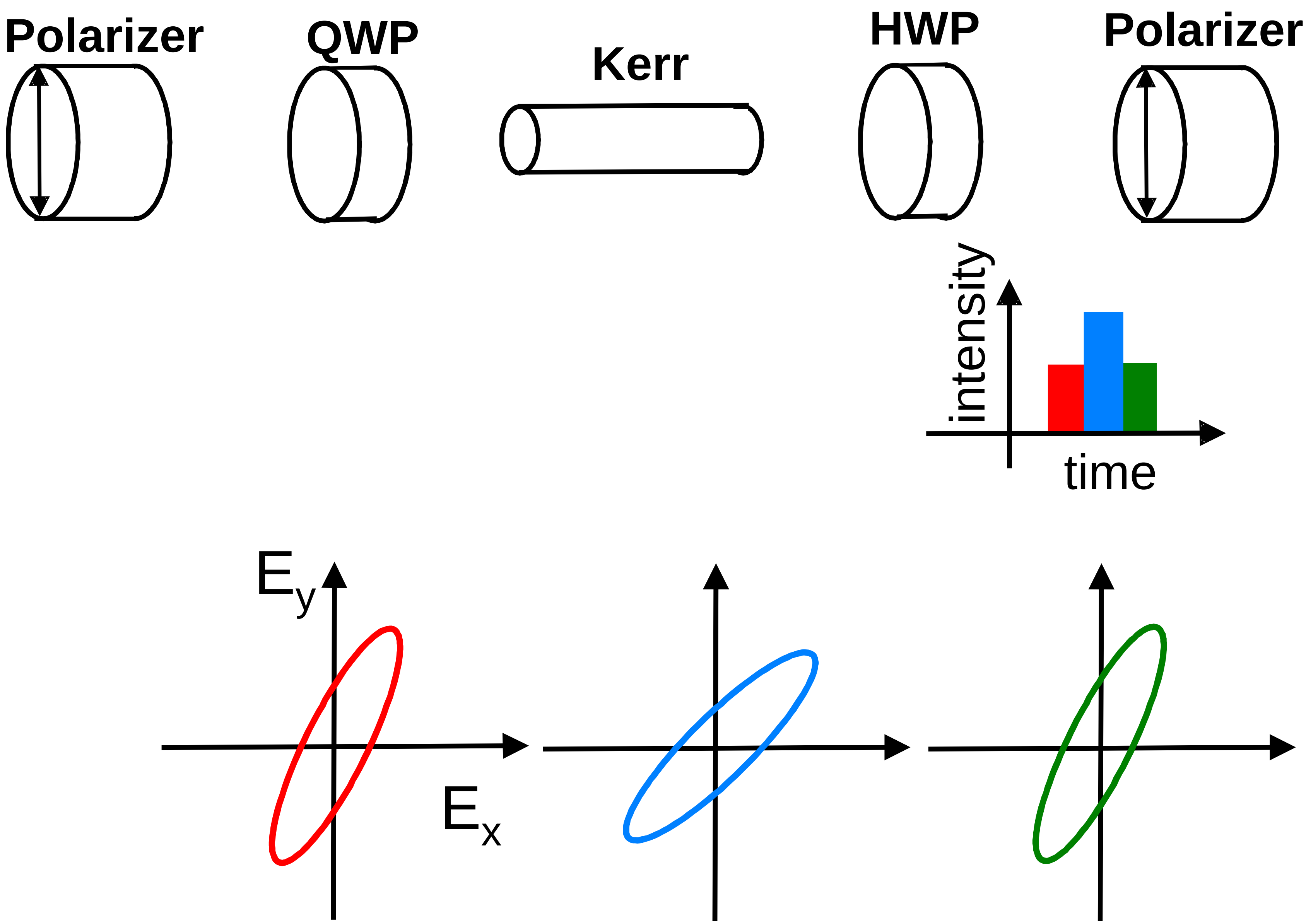
Nonlinear Polarization Rotation



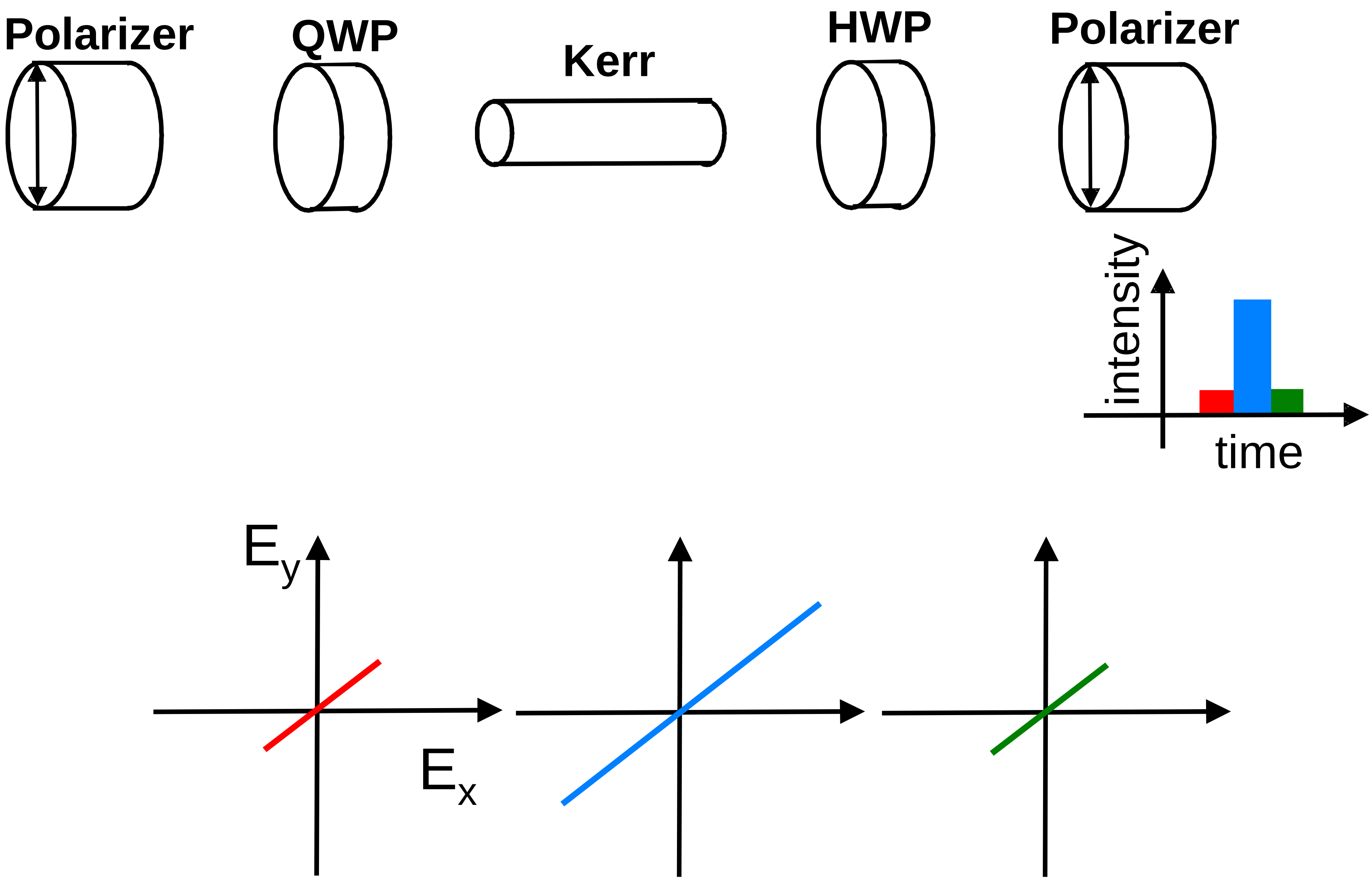
Nonlinear Polarization Rotation



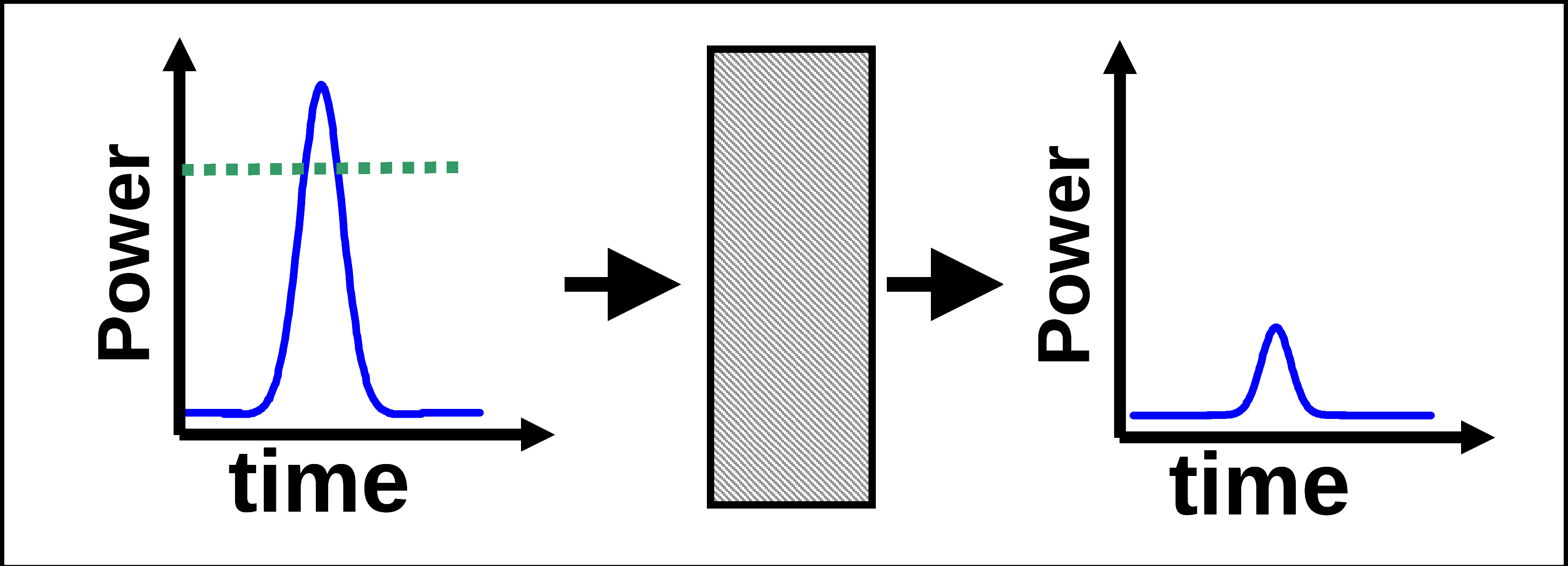
Nonlinear Polarization Rotation



Nonlinear Polarization Rotation



Saturable Absorber for Passive Mode-Locking

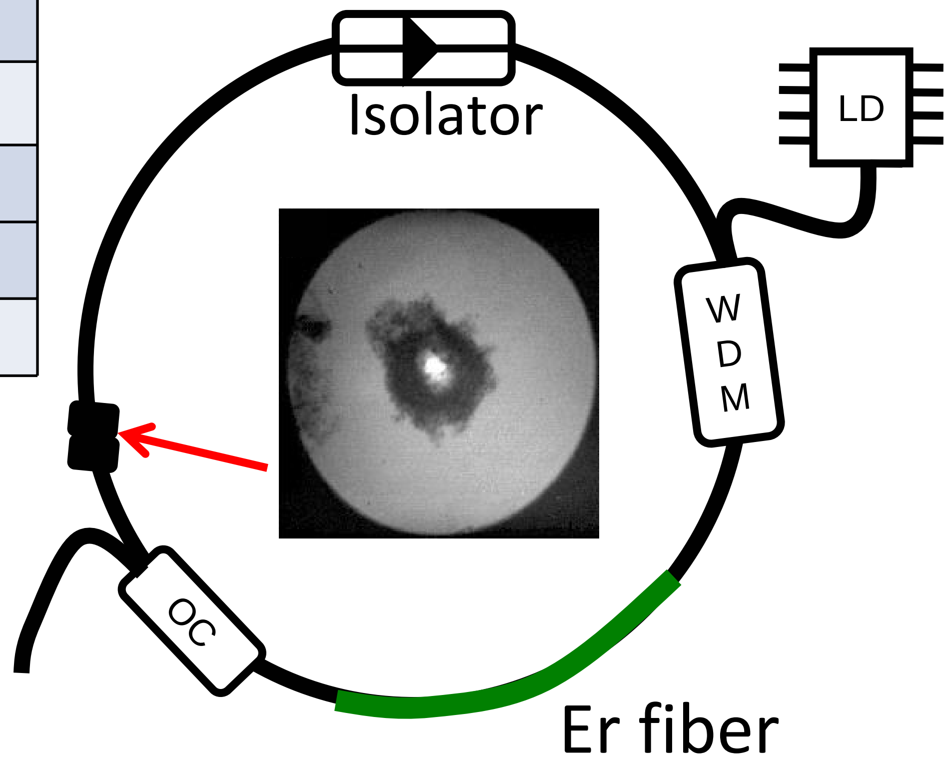


- A saturable allows the laser cavity to “favor” high peak power, ultrashort pulses
- Interferometric designs based on gain and saturable absorber section
  - Figure eight laser
  - Soliton ring laser (Additive-pulse mode-locked)

Carbon Nanotube Fiber Laser (CNFL)

First laser build by Jeff Nicholson, OFS
We have build many others laser in our laboratory

Repetition frequency	**167 MHz**
Spectral bandwidth	**10.5 nm**
Pulse duration	**250 fs (TL)**
Self starting !!	
Output power	**1 mW**



J. W. Nicholson et al. “Optically driven deposition of single-walled carbon-nanotube saturable absorbers on optical fiber end-face,” Opt. Express 15, 9176-9183 (2007)

J. W. Nicholson and D. J. DiGiovanni, "High repetition frequency, low noise, fiber ring lasers modelocked with carbon nanotubes," IEEE Photon. Technol. Lett. 20, 2123-2125 (2008).

Single Walled Carbon Nanotubes

Single wall carbon nanotubes have semiconductor, semimetal or metallic properties depending on the chiral vector of the nanotube

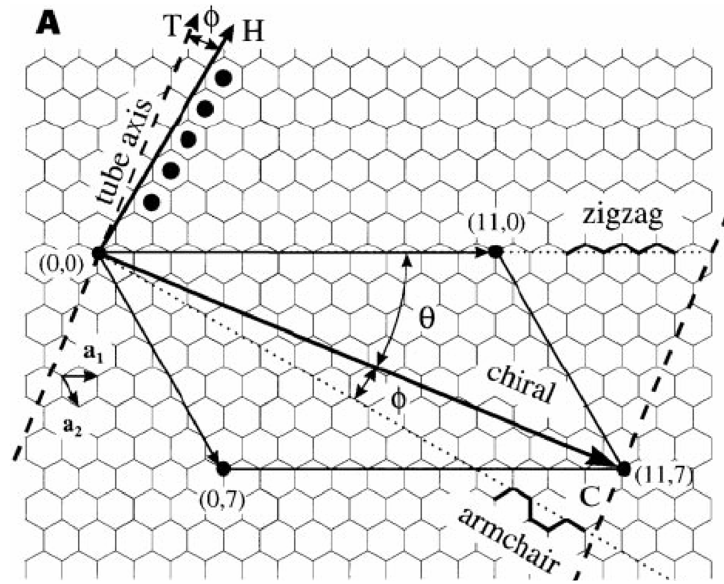
$$\mathbf{C} = n\mathbf{a}_1 + m\mathbf{a}_2$$

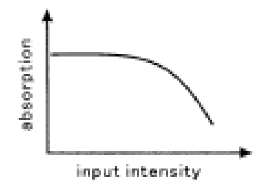
Metallic	$n - m =$ integer multiple of 3
Semiconductor	$n - m \neq$ integer multiple of 3
Semimetal	$n - m = 0$

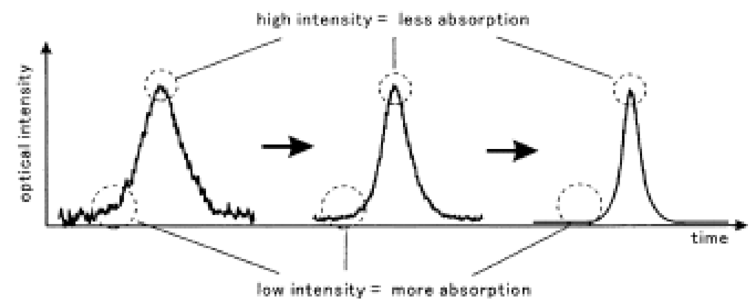
Excitonic absorption in the semiconductor nanotube is responsible for the saturable absorption property

Ultrafast recovery of the saturable absorber is due to metallic nanotubes serving a recombination centers

NATURE | VOL 391 | 1 JANUARY 1998



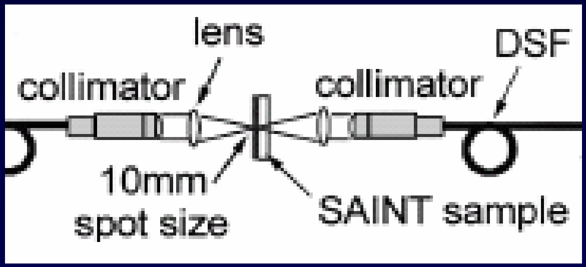




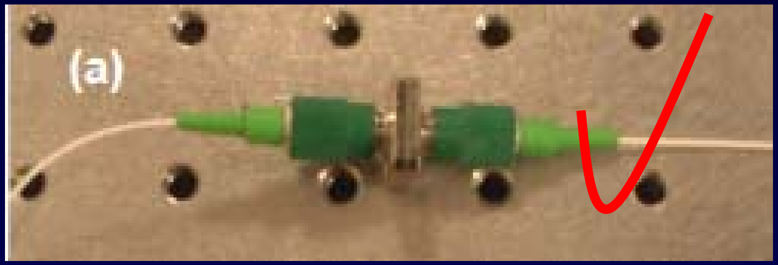
IEEE JOURNAL OF SELECTED TOPICS IN QUANTUM ELECTRONICS, VOL. 10, NO. 1, JANUARY/FEBRUARY 2004

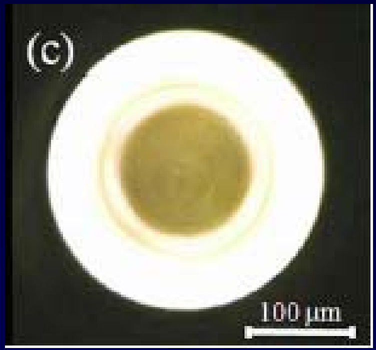
Incorporation of CNT

[1] Direct synthesis (film)

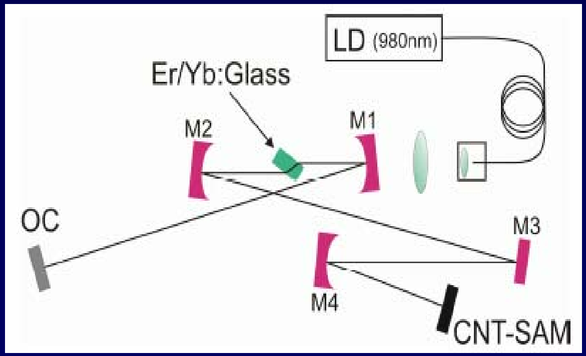


[3] Direct deposition on a connector

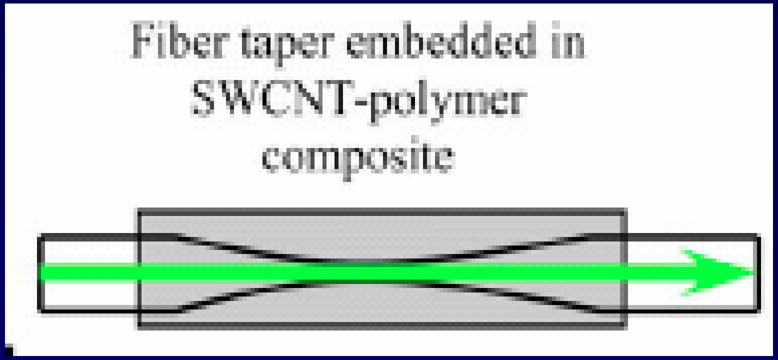




[2] Spin coating (Mirror)



[4] Taper fiber

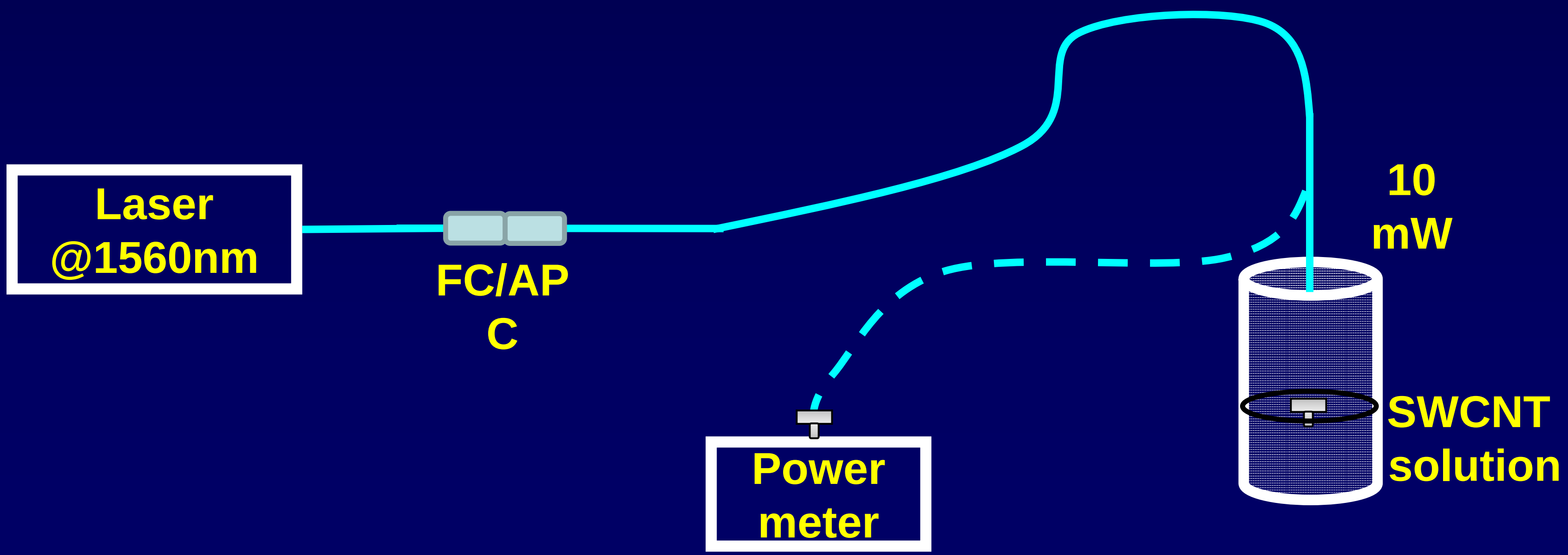


[5] CNT doped polymer optical fiber

[1] S. Yamashita et al., Optics. letters **29**, 1581-1583 (2004)
[2] T. R. Schibli et al., Optics express **13**, 8025-8031 (2005)
[3] J. W. Nicholson et al., Optics express **15** , 9176-9183 (2007)
[4] K. Kieu et al., Optics letters **32**, 2242-2244 (2007)
[5] S. Uchida et al., Optics letters **34**, 3077-3079 (2009)

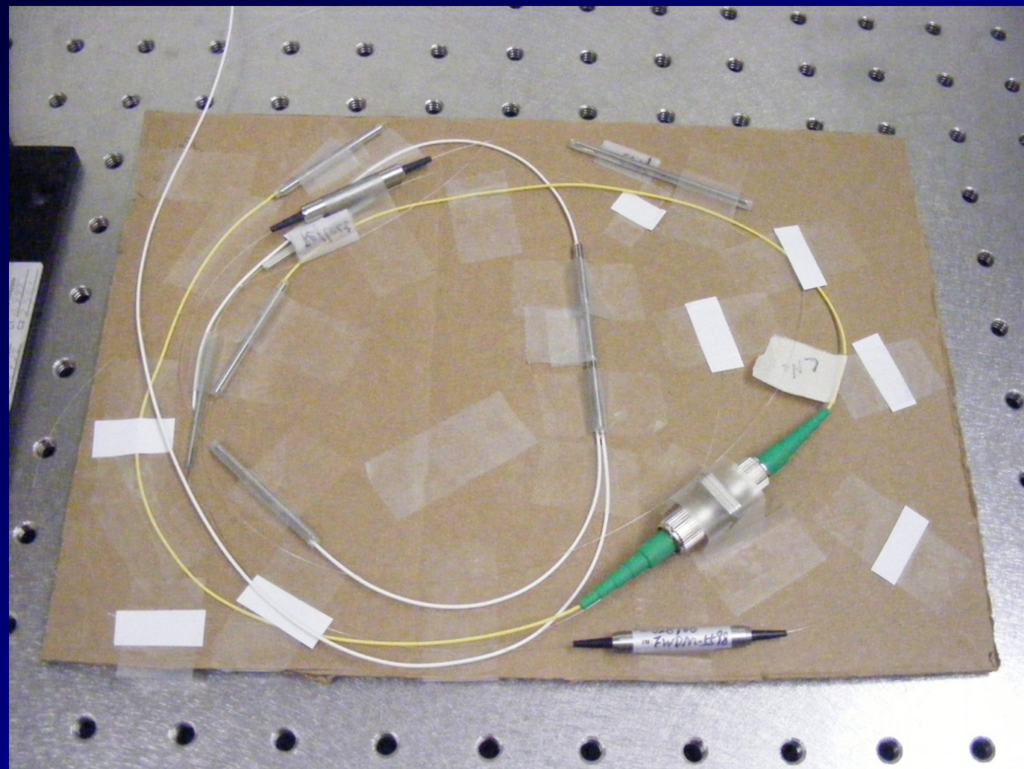
Carbon Nanotube Deposition

1) Prepare CNT solution: 0.5 mg + 12 cc ethanol and ultrasonicate for 30 minites.
2) Dip the fiber connector end with radiation for 30 secs.
3) Put it out and wait 1 min.
4) Measure the optical power.
5) Repeat the step 2)-4) until the measured loss is ~ 2 dB.



J. W. Nicholson et al., "Optically driven deposition of single-walled carbon nanotube saturable absorbers on optical fiber end-faces," Optics Express **15**, 9176–9183 (2007).

Carbon Nanotube Fiber Laser Comb

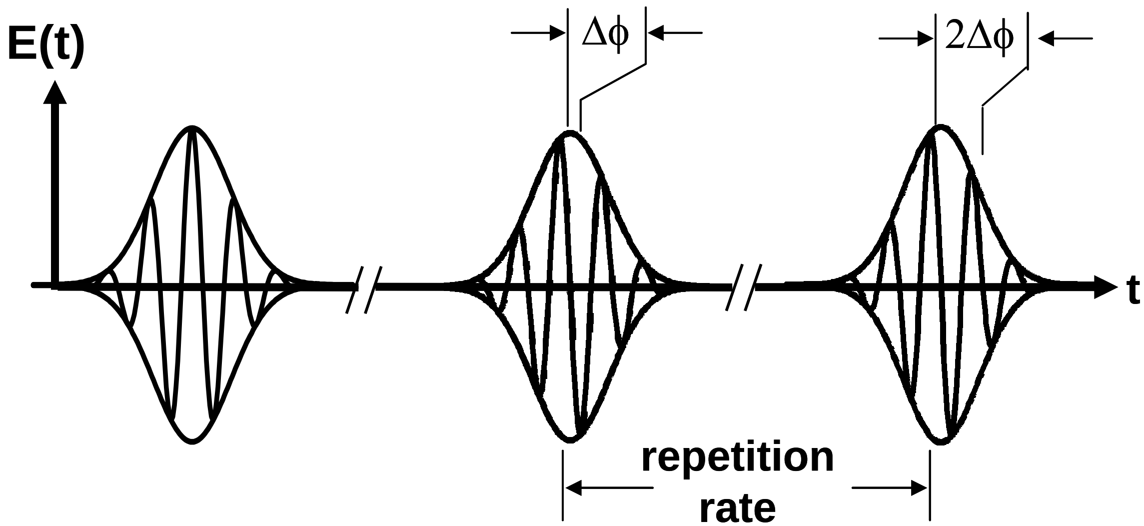


Built by JinKang and two undergraduates in half a day!
Modelocked right away by increasing the power
Cheap to build : $1000 of optical components
Great laser for an undergraduate laboratory!

Advantage: Self-starting laser, easy to mode-lock
Disadvantage: Carbon nanotube lasers are too noisy to phase stabilize
Can we phase stabilize the carbon nanotube fiber laser?

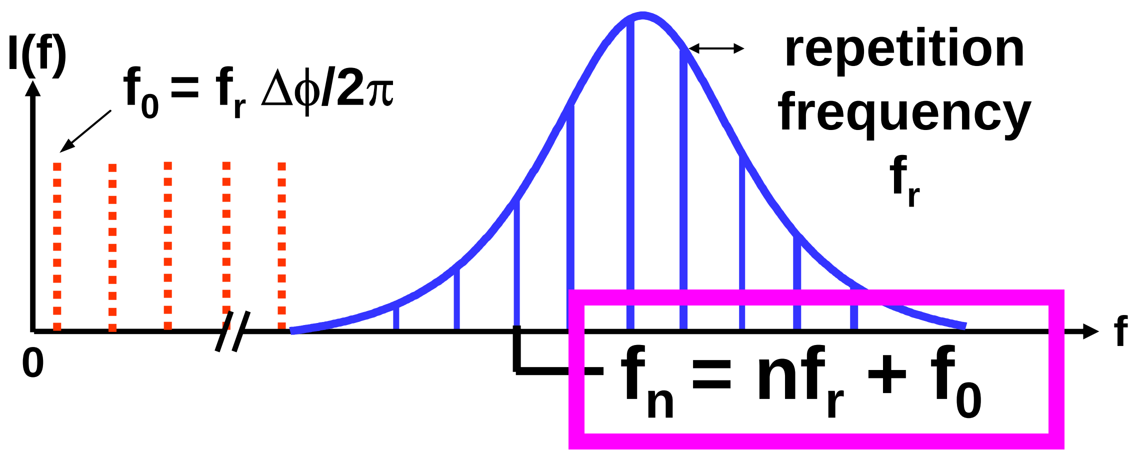
The Frequency Comb

Time domain (Pulses in time)



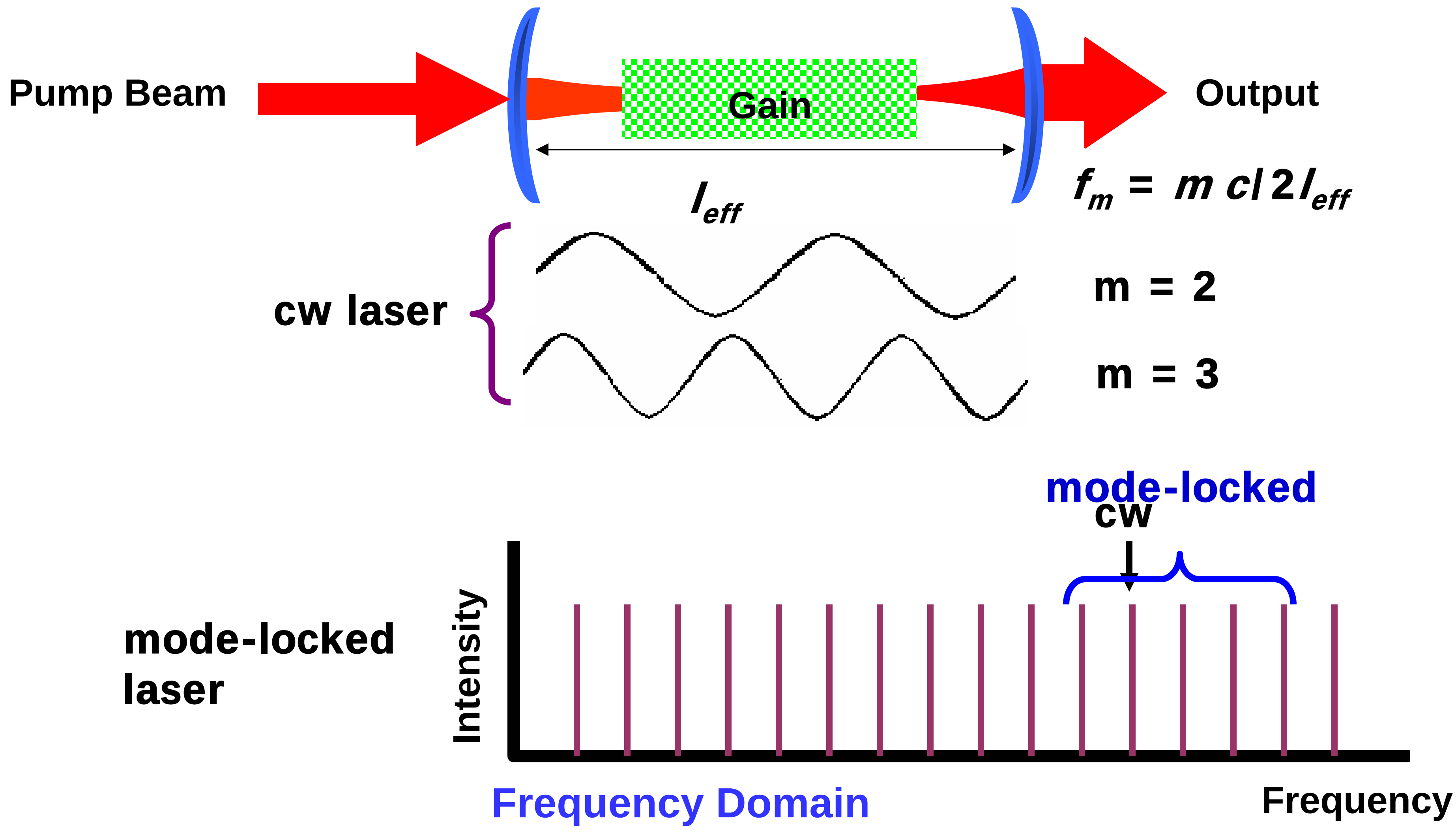
Carrier-envelope phase slip from pulse to pulse because group and phase velocities differ

Frequency domain (Comb of lines)



Stable frequency comb if
1) Repetition rate (f_r) locked
2) Offset frequency (f_0) (phase slip) locked

Pulsed lasers must give a frequency comb



Comb-like nature of ultrafast lasers

1979- mode-locked (pulsed) dye laser used as comb

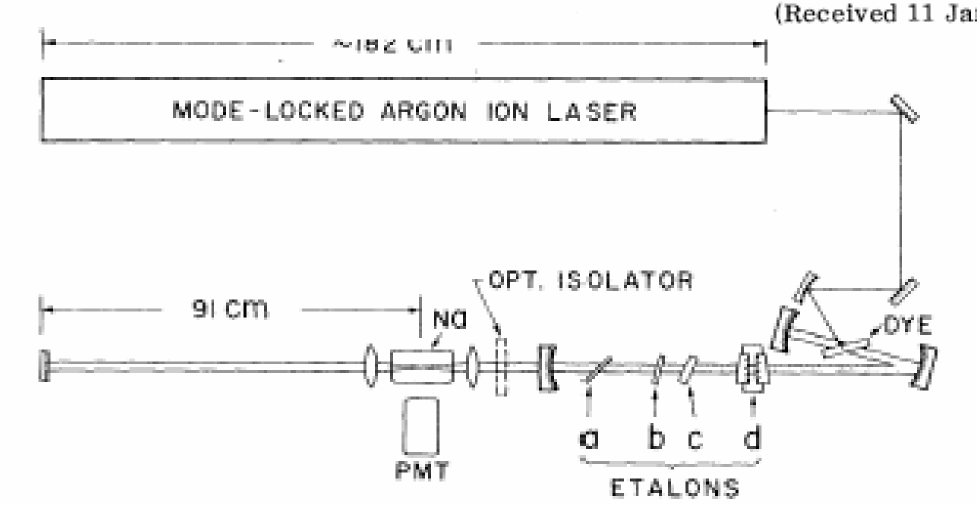
- **500 ps pulses ~0.003 nm, ~1 GHz wide**

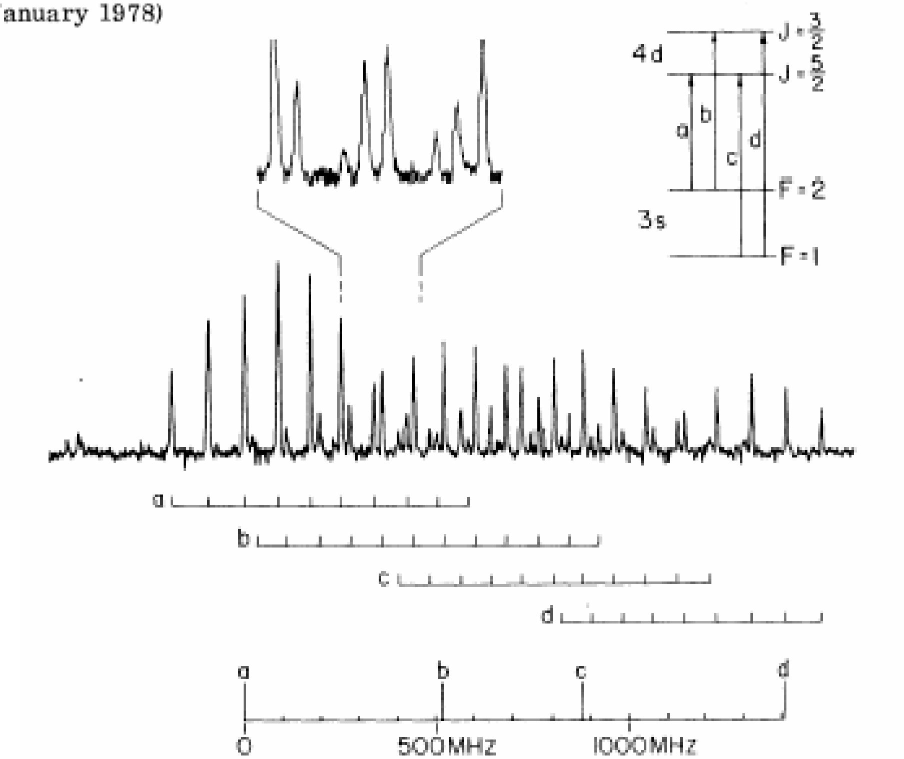
VOLUME 40, NUMBER 13 PHYSICAL REVIEW LETTERS 27 MARCH 1978

High-Resolution Two-Photon Spectroscopy with Picosecond Light Pulses

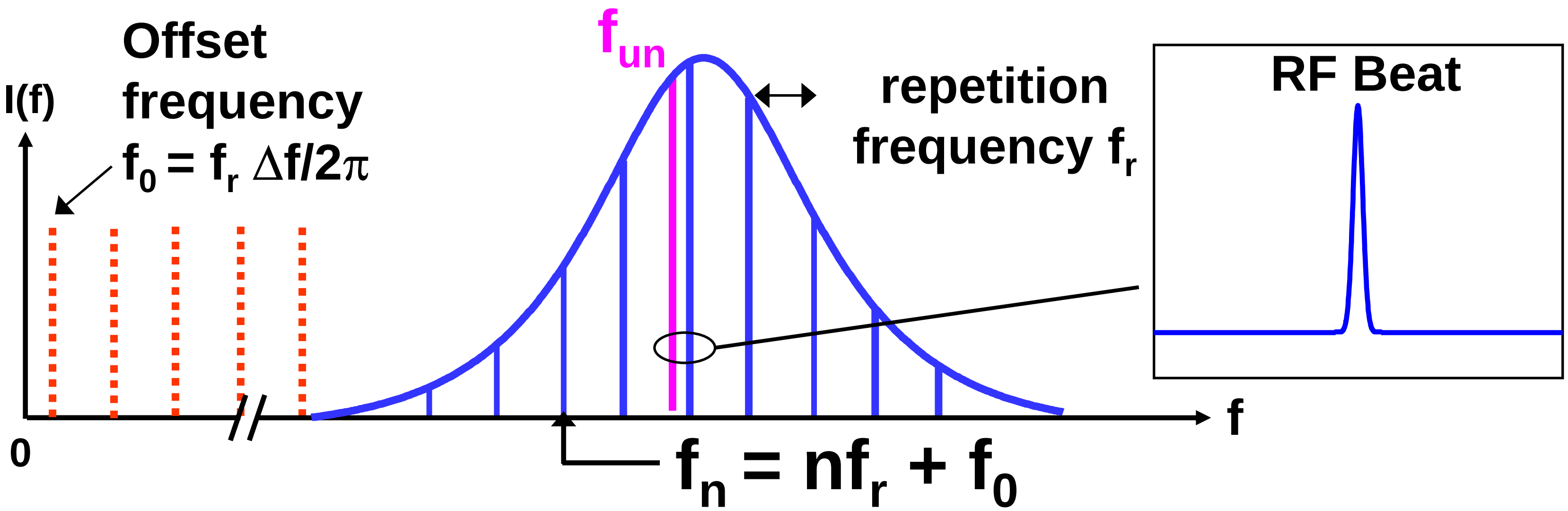
Na

J. N. Eckstein, A. I. Ferguson, and T. W. Hänsch
Department of Physics, Stanford University, Stanford, California 94305
(Received 11 January 1978)





Optical Frequency Metrology



Frequency comb as a spectral ruler

References:

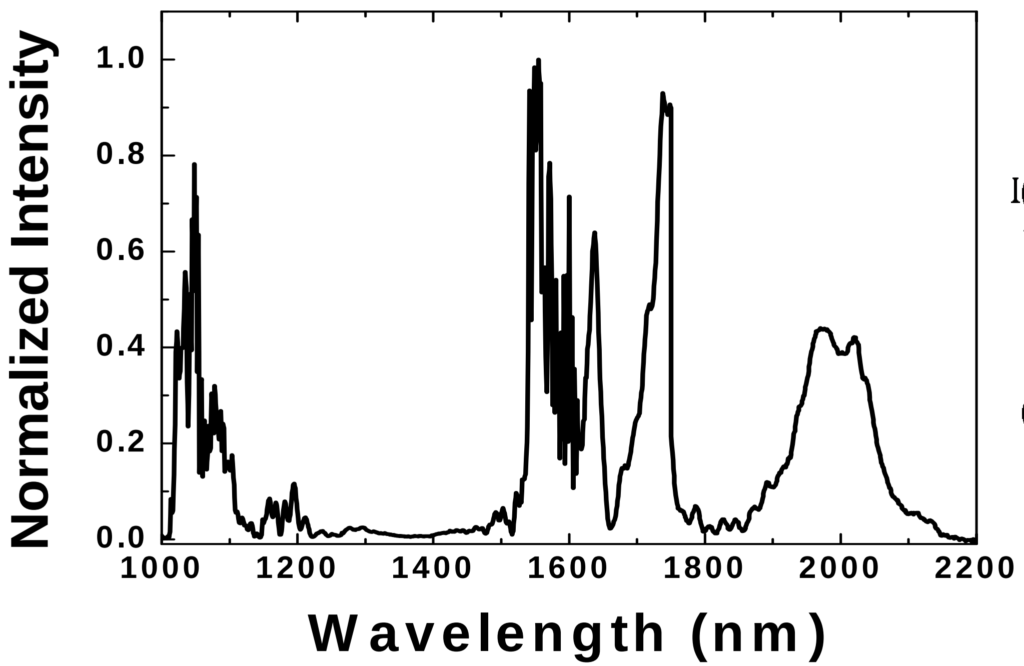
Udem, Reichert, Holwartz, Hänsch, *Phys. Rev. Lett*., vol. 82 (1999)

Jones et al., *Science*, vol. 288 (2000)

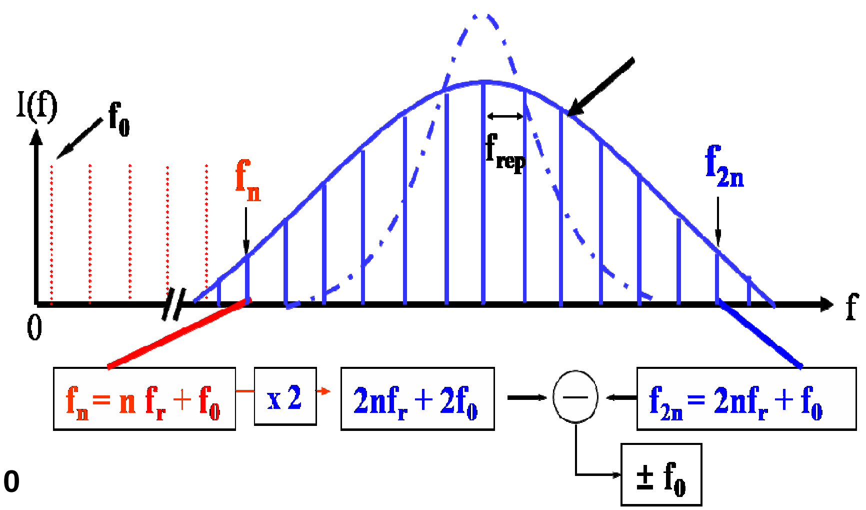
Udem, Holwartz, Hänsch, *Nature*, vol. 416 (2002)

Supercontinuum generation for self referencing f_0

Supercontinuum spectrum

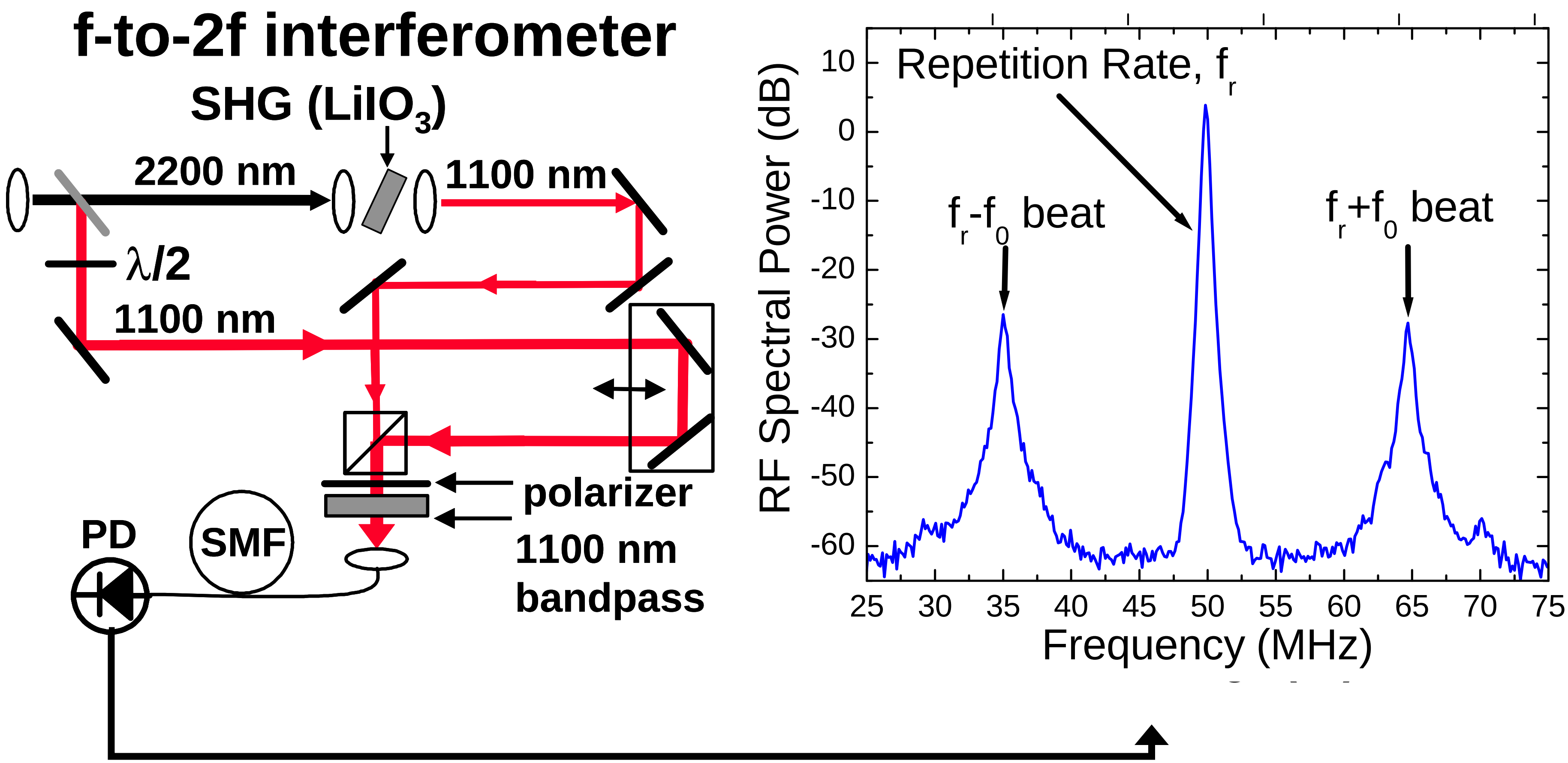


f-to-2f Interferometer



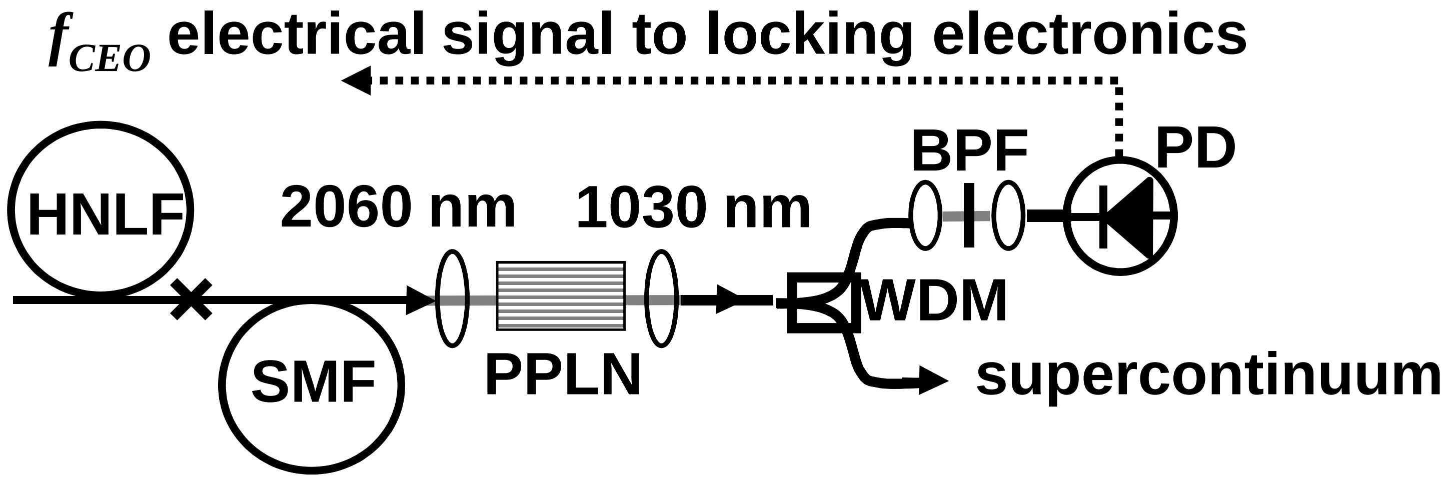
- D. J. Jones, S. A. Diddams, J. K. Ranka, A. Stentz, R. S. Windeler, J. L. Hall, and S. T. Cundiff, "Carrier-envelope phase control of femtosecond mode-locked lasers and direct optical frequency synthesis," Science 288, 635-9 (2000).

f-to-2f Interferometer

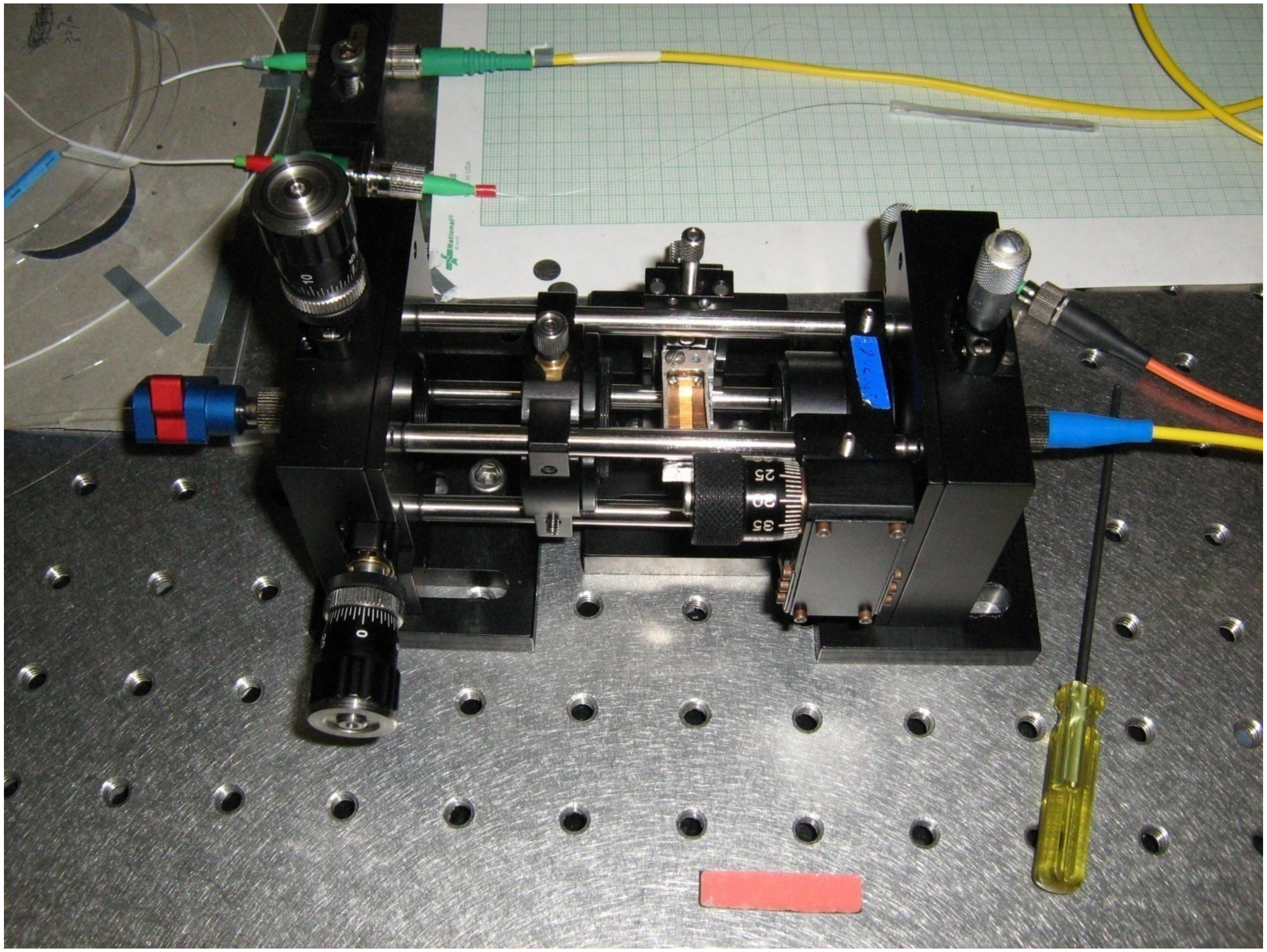


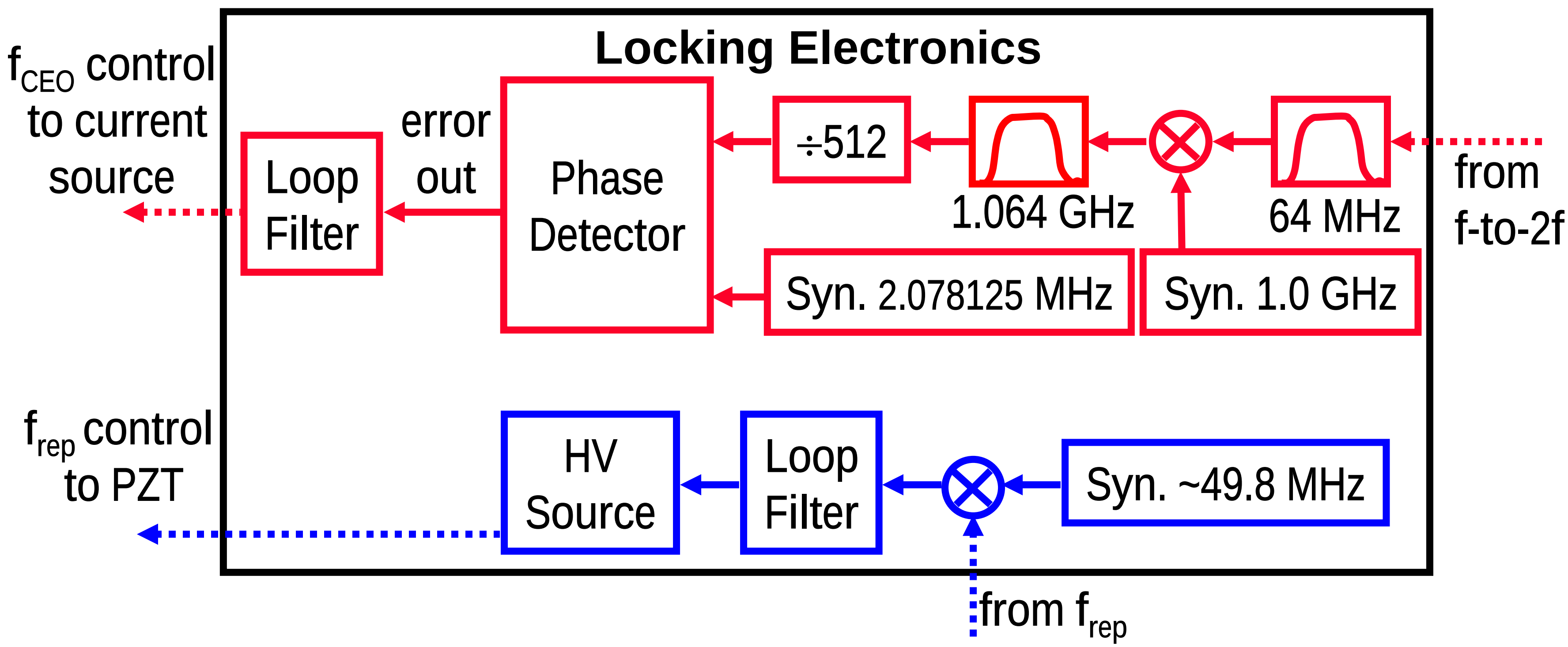
An octave of supercontinuum allow the generation of beat frequencies with a SNR of 30 dB

Co-linear all fiber geometry

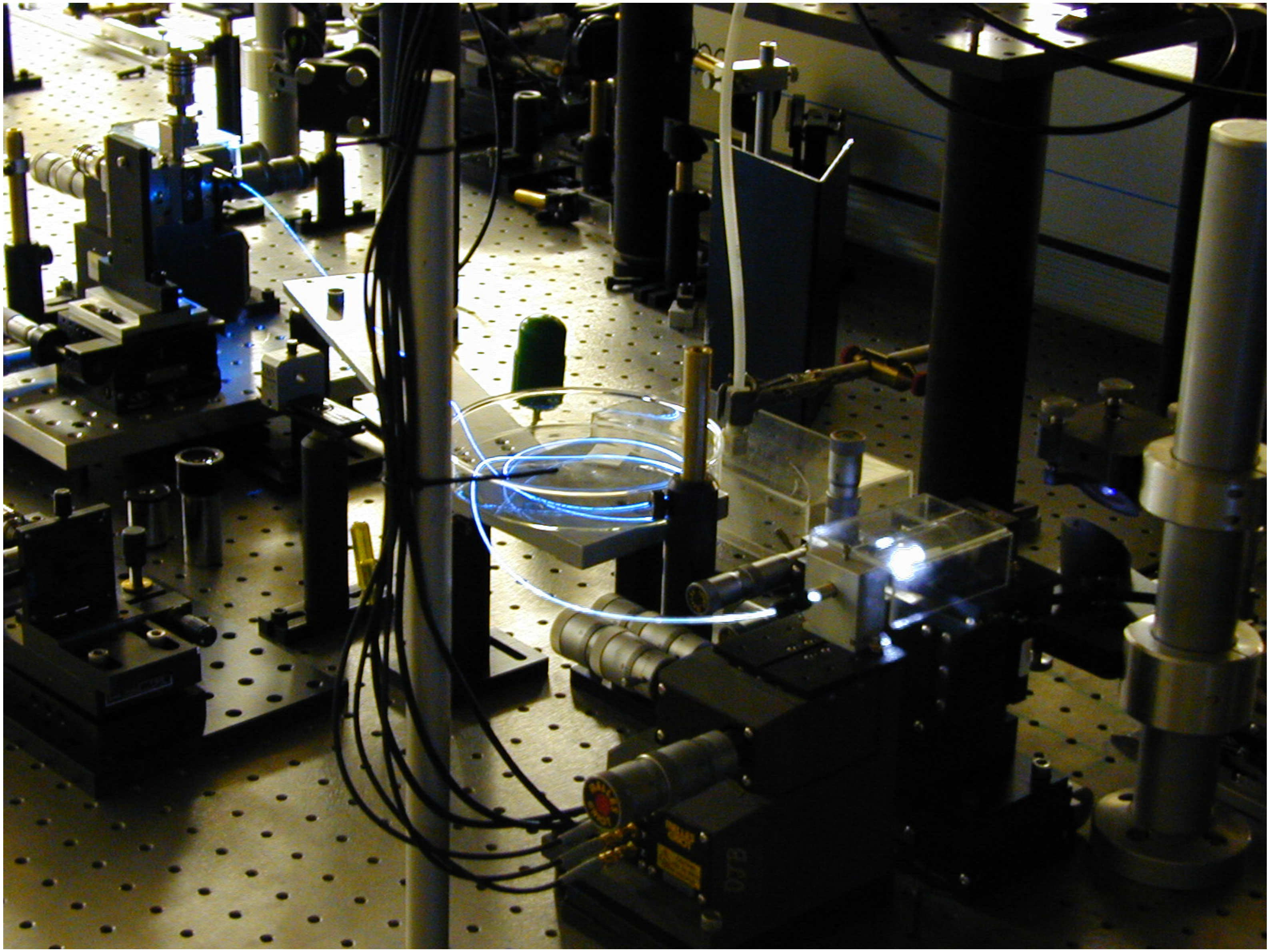


Colinear f-to-2f Inteferometer

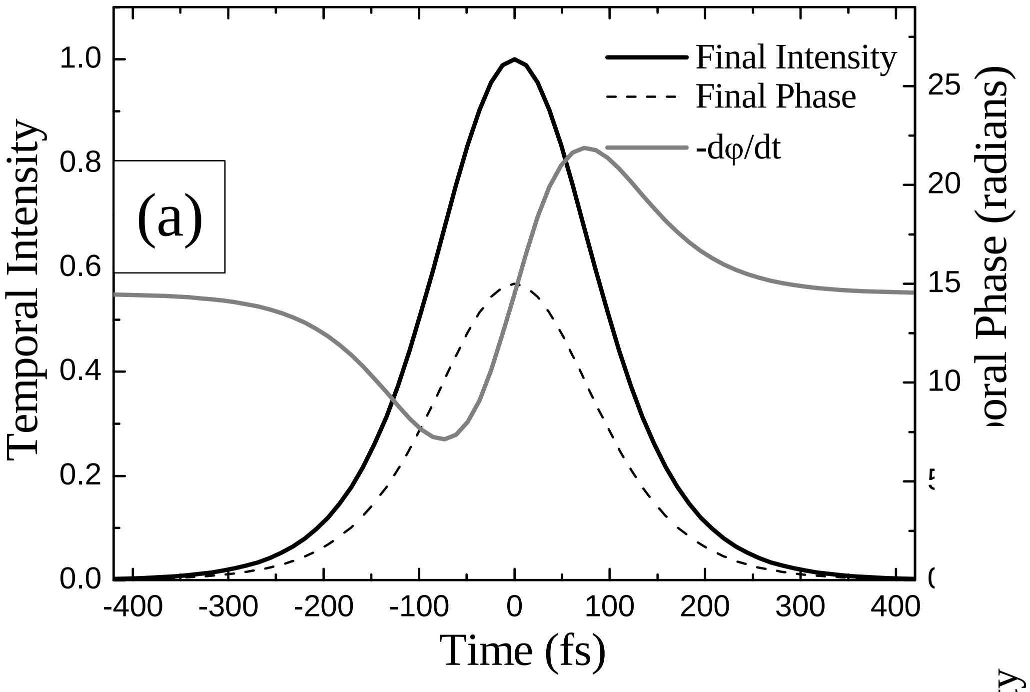


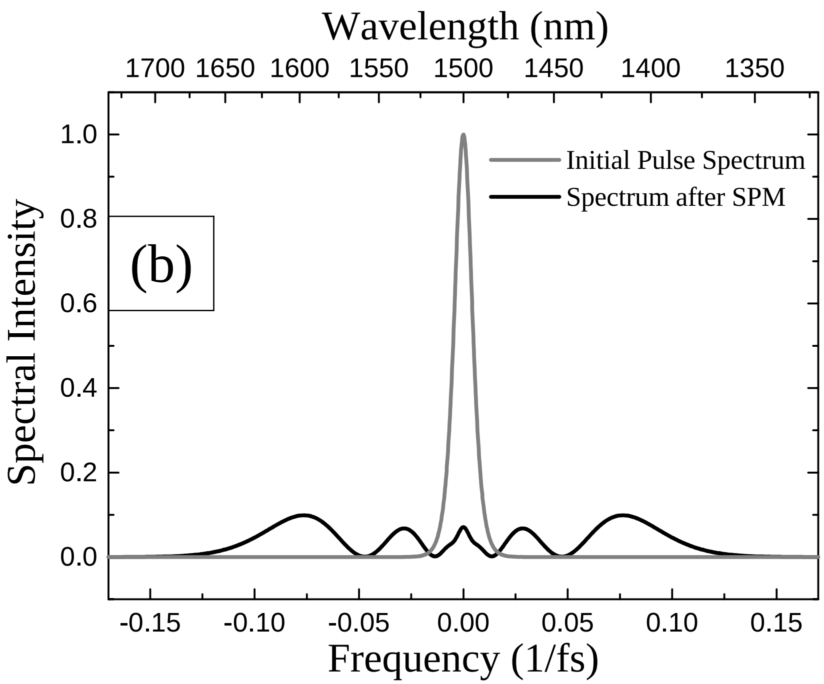


f_{CEO} control to current source
Locking Electronics
Loop Filter
error out
Phase Detector
÷512
1.064 GHz
64 MHz
from f-to-2f
Syn. 2.078125 MHz
Syn. 1.0 GHz
f_{rep} control to PZT
HV Source
Loop Filter
Syn. ~49.8 MHz
from f_{rep}



Nonlinear propagation in fiber: nonlinearity only





Generate new spectral components

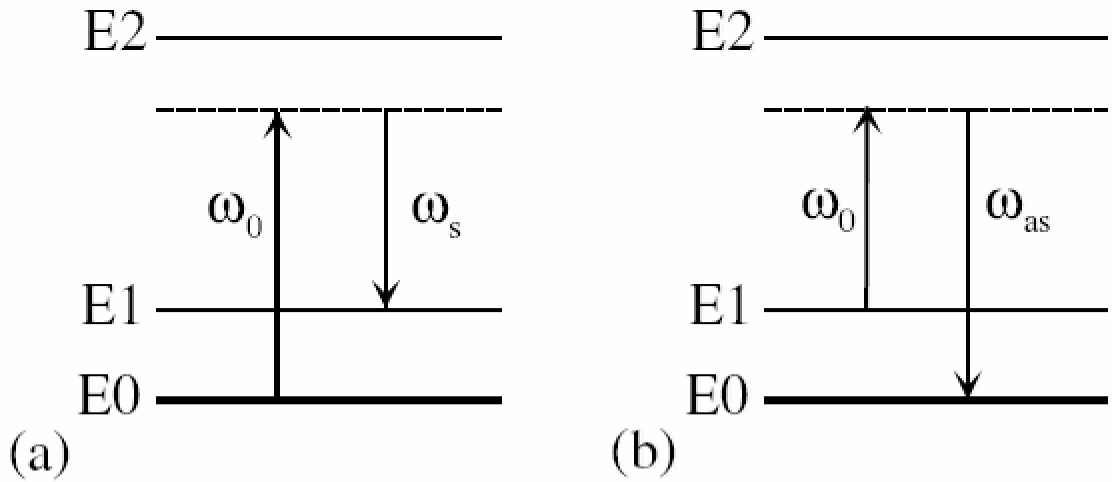
BRW

$$\frac{\partial E(z,t)}{\partial z} = \overbrace{-\frac{\alpha}{2}E}^{\text{Absorption}} - \overbrace{\left(\sum_{m=2}\beta_m \frac{i^{m-1}}{m!}\frac{\partial^m}{\partial t^m}\right)E}^{\text{Dispersion}} + (1-f_R)\left\{\overbrace{i\gamma|E|^2 E}^{\text{SPM}} - \overbrace{\frac{2\gamma}{\omega_0}\frac{\partial}{\partial t}\left(|E|^2 E\right)}^{\text{Self Steepening}}\right\}$$

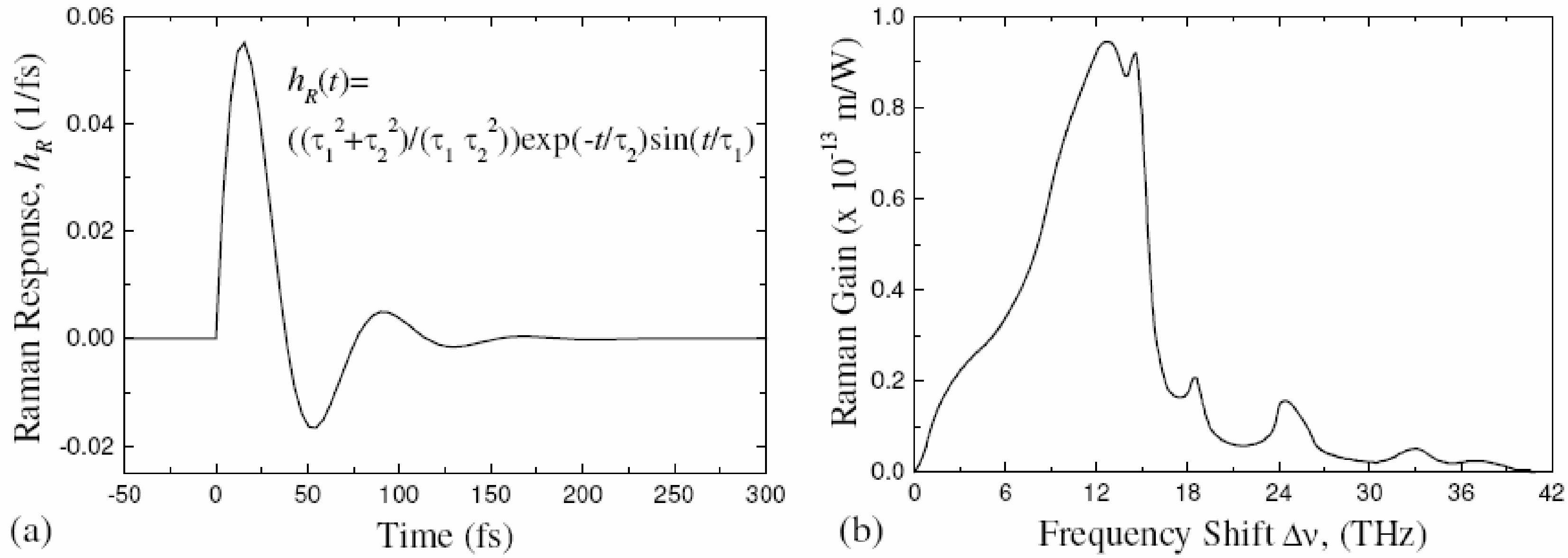
$$\overbrace{+ i\gamma f_R\left(1+\frac{i}{\omega_0}\frac{\partial}{\partial t}\right)\left(E\int_0^\infty h_R(t')|E(z,t-t')|^2\,dt'\right)}^{\text{Raman Effect}}.$$

$$h_R(t)=$$

$$((\tau_1{}^2+\tau_2{}^2)/(\tau_1\ \tau_2{}^2))\exp(-t/\tau_2)\sin(t/\tau_1)$$



$$g_R(\omega) = \frac{\omega_0}{cn_0} f_R \chi^{(3)} \,\mathrm{Im}\left[\mathcal{F}\{h_R(t)\} \right]$$

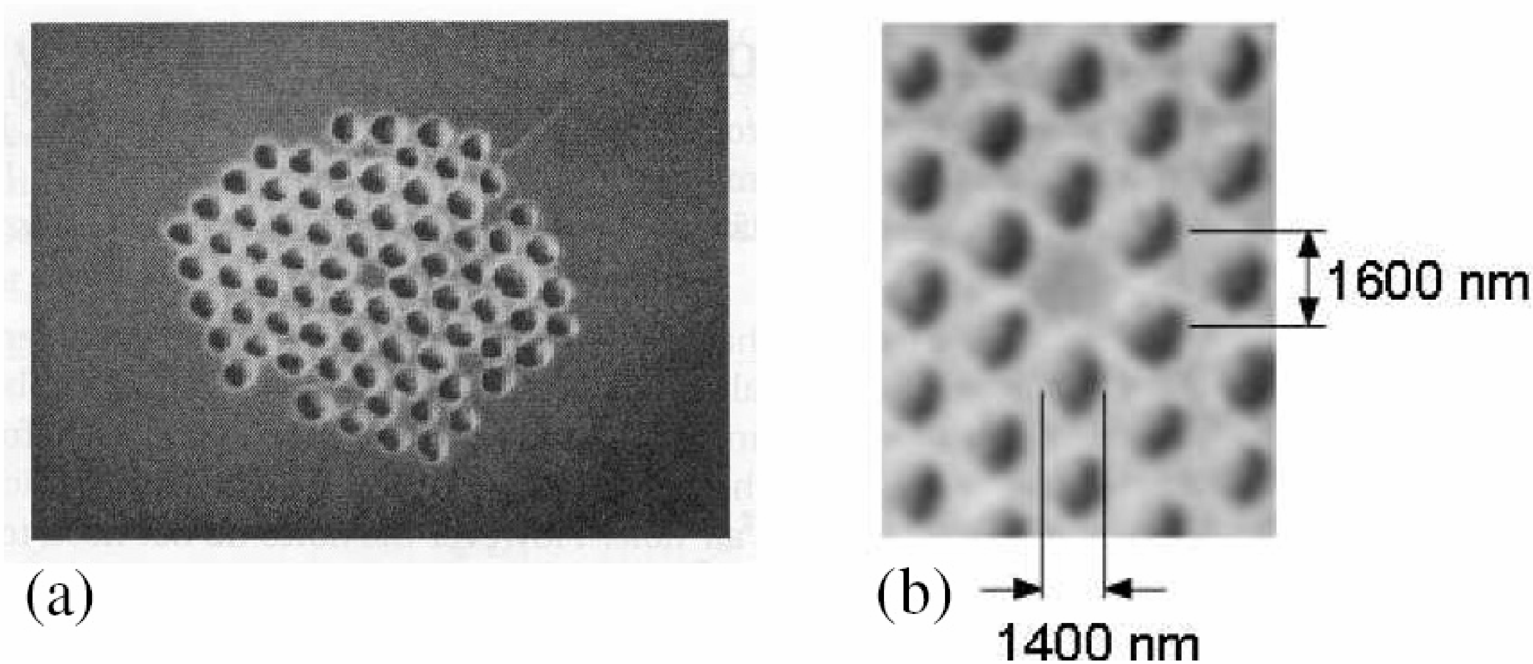

0.06
0.04
0.02
0.00
-0.02
Raman Response, h_R (1/fs)
$h_R(t)=$
$((\tau_1^2+\tau_2^2)/(\tau_1\ \tau_2^2))\exp(-t/\tau_2)\sin(t/\tau_1)$
-50
0
50
100
150
200
250
300
(a)
Time (fs)
1.0
0.8
0.6
0.4
0.2
0.0
Raman Gain (x 10^{-13} m/W)
0
6
12
18
24
30
36
42
(b)
Frequency Shift $\Delta\nu$, (THz)

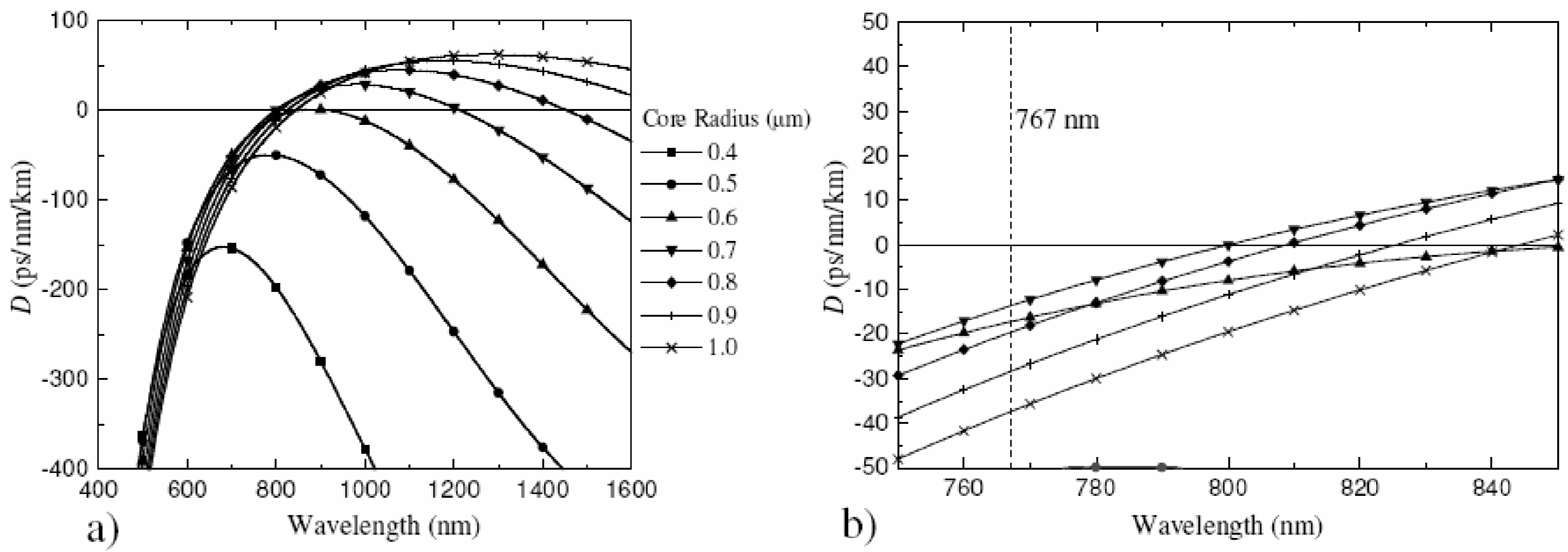
$$\frac{g_R(\Delta\omega_R)P_R}{\alpha\pi r_0^2}\left[1-\exp(-\alpha L)\right]\approx 16$$

$$g_R(\omega)=\frac{\omega_0}{cn_0}f_R\chi^{(3)}\,\mathrm{Im}\left[\boldsymbol{\mathcal{F}}\left\{h_R(t)\right\}\right]$$

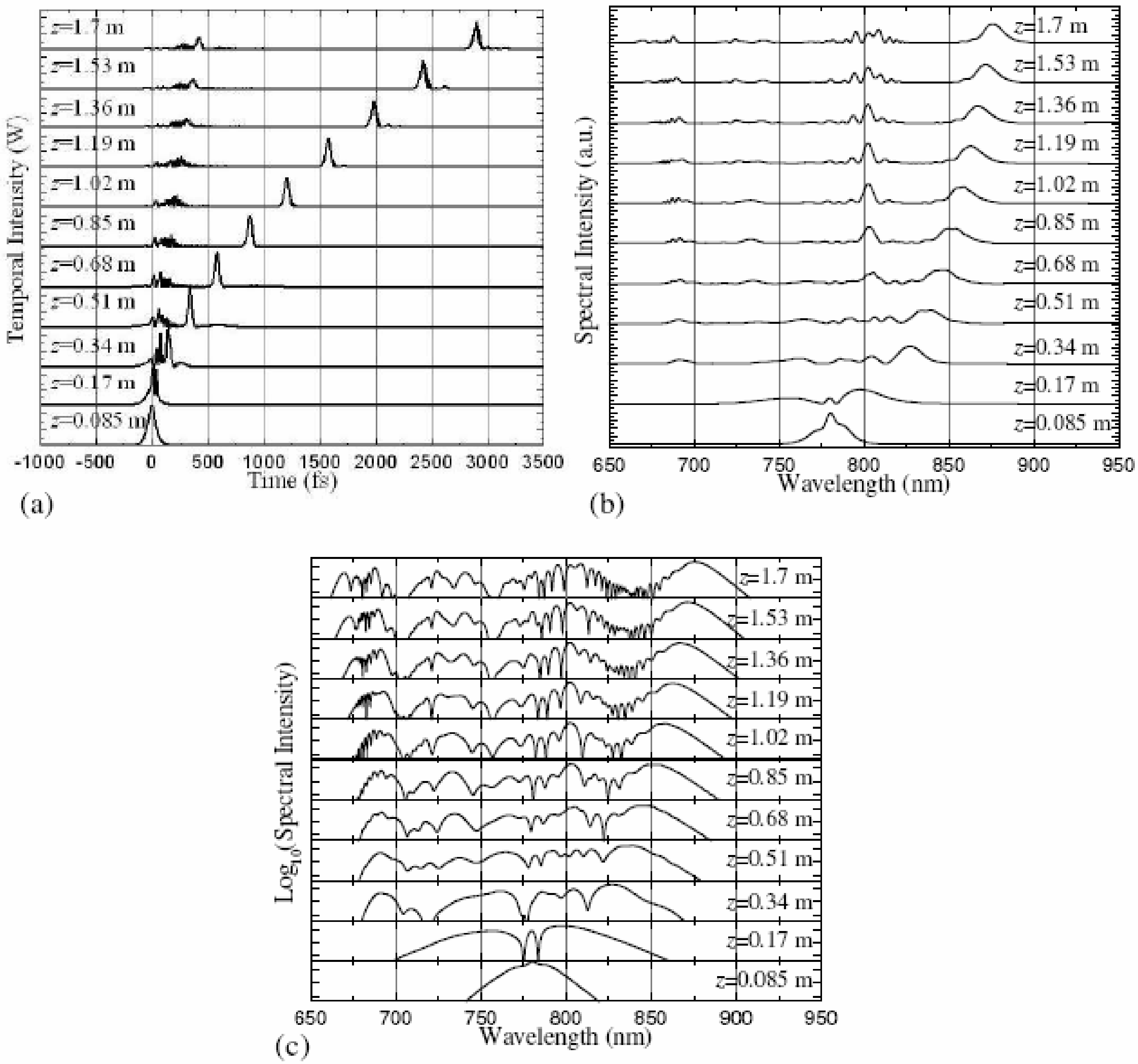
$$\frac{d\omega_{SSFS}}{dz}=-\frac{\lambda_0}{16n_2}\int\Omega^3\frac{g_R\left(-\Omega/2\pi T_0\right)}{\sinh^2(\pi\Omega/2)}d\Omega, \tag{3.57}$$

where $\Omega\equiv\left(\omega-\omega_0\right)T_0$.


(a)
(b)
1600 nm
1400 nm



a)
D (ps/nm/km)
Wavelength (nm)
Core Radius (μm)
0.4
0.5
0.6
0.7
0.8
0.9
1.0
b)
767 nm
D (ps/nm/km)
Wavelength (nm)



Temporal Intensity (W)
Time (fs)
-1000 -500 0 500 1000 1500 2000 2500 3000 3500
z=1.7 m
z=1.53 m
z=1.36 m
z=1.19 m
z=1.02 m
z=0.85 m
z=0.68 m
z=0.51 m
z=0.34 m
z=0.17 m
z=0.085 m
(a)
Spectral Intensity (a.u.)
Wavelength (nm)
650 700 750 800 850 900 950
(b)
Log10(Spectral Intensity)
Wavelength (nm)
650 700 750 800 850 900 950
(c)

Introduction to Quantum Optics: Seminal Paper

Please read Papers 1,2, and 4 (3 if you have time) in the ‘Hanbury-Brown and Twiss’ folder on Kstate Online **in order in which they were published** (as ordered in folder). For in-class discussion on Oct. 21, be able to answer the following questions:

1.In *Hanbury-Brown and Twiss* (Paper 1), why did they measure coincidences in “cathodes aligned” positions and no coincidences in “cathodes not aligned position”?

2.Why did *Brannen and Ferguson* (Paper 2) not measure any coincidences?

3.At the end of *Brannen and Ferguson*, they stated that “if such a correlation did exist, it would call for a major revision of some fundamental concepts of quantum mechanics”. How did they come to that conclusion?

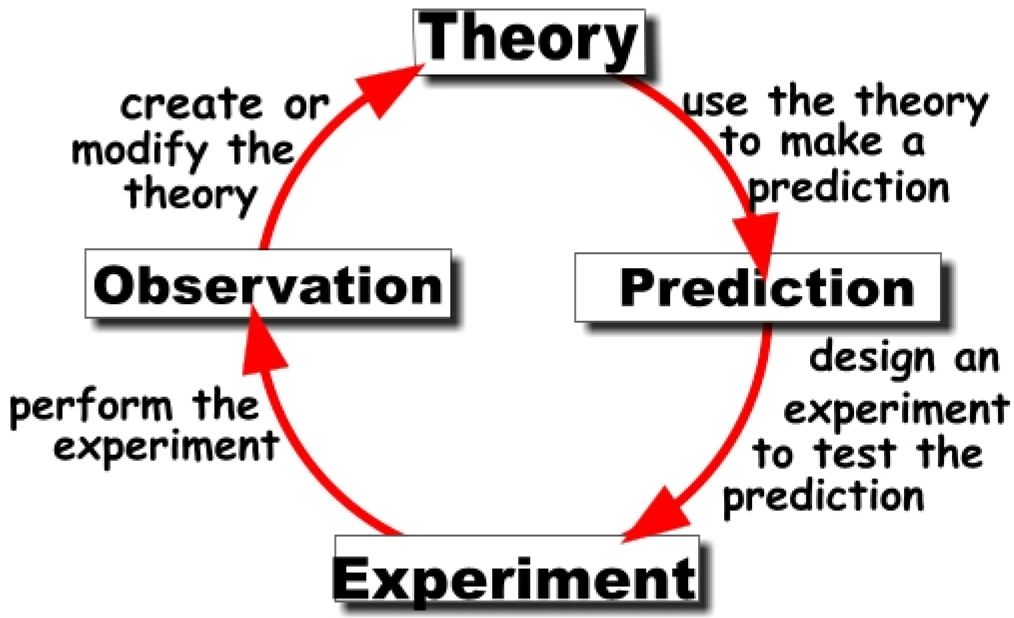
4.Why did Hanbury-Brown and Twiss do their first experiment? What was their overall goal?

5.Why did *Purcell* (Paper 4) state that “the Brown-Twiss effect, far from requiring a revision of quantum mechanics, is an instructive illustration of its elementary principles.”?

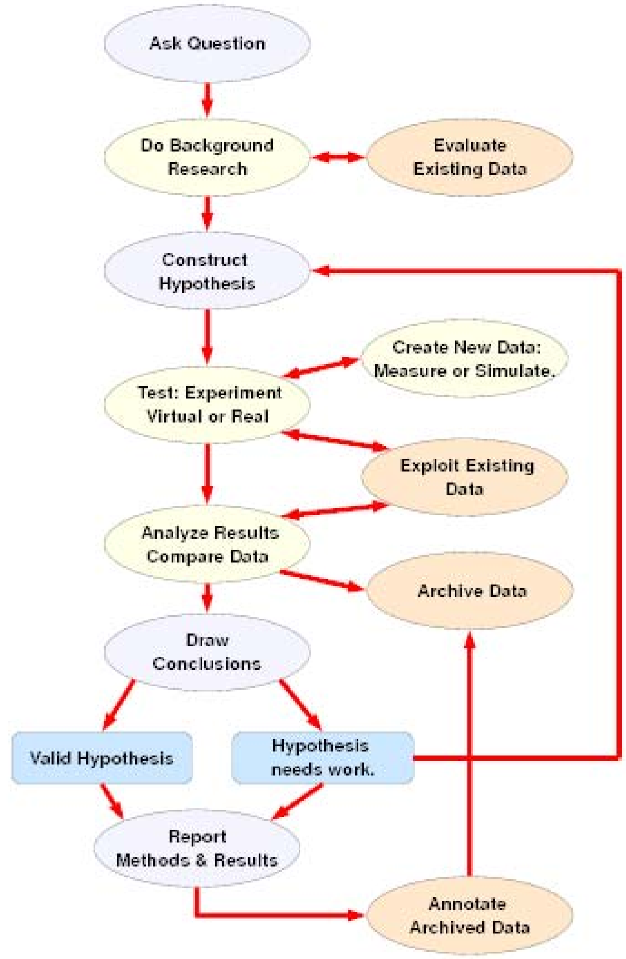
6.Given the experiment in *Brannen and Ferguson*, what ‘apparatus’ would be required for them to use in their experiment in order to observe coincidences? How would this ‘apparatus’ solve their problems?

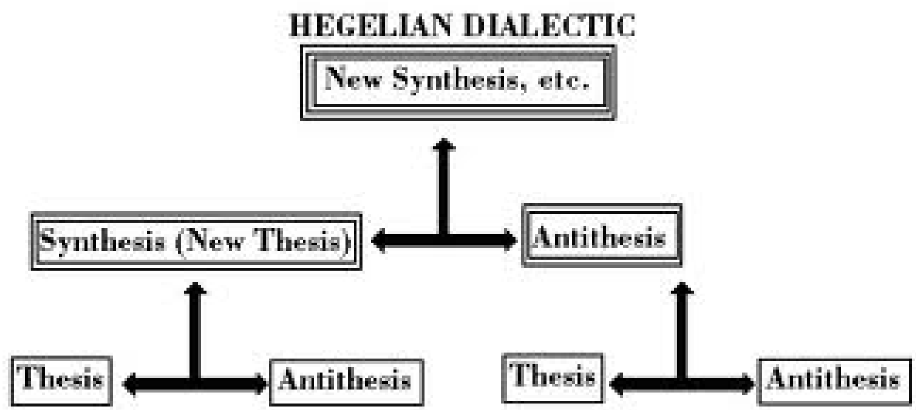
Introduction to Quantum Mechanics: Preliminaries

The Scientific Method



vs. The Hegelian Dielectic


Ask Question
Do Background Research
Evaluate Existing Data
Construct Hypothesis
Create New Data: Measure or Simulate.
Test: Experiment Virtual or Real
Exploit Existing Data
Analyze Results Compare Data
Archive Data
Draw Conclusions
Valid Hypothesis
Hypothesis needs work.
Report Methods & Results
Annotate Archived Data


HEGELIAN DIALECTIC
New Synthesis, etc.
Synthesis (New Thesis)
Antithesis
Thesis
Antithesis
Thesis
Antithesis

Frontiers of Nonlinear Optics
Higher Harmonic Generation and Attosecond Pulses

References

Trebino lecture notes

Intense few-cycle laser fields: Frontiers of nonlinear optics

Thomas Brabec and Ferenc Krausz*

Reviews of Modern Physics, Vol. 72, No. 2, April 2000

Attosecond science

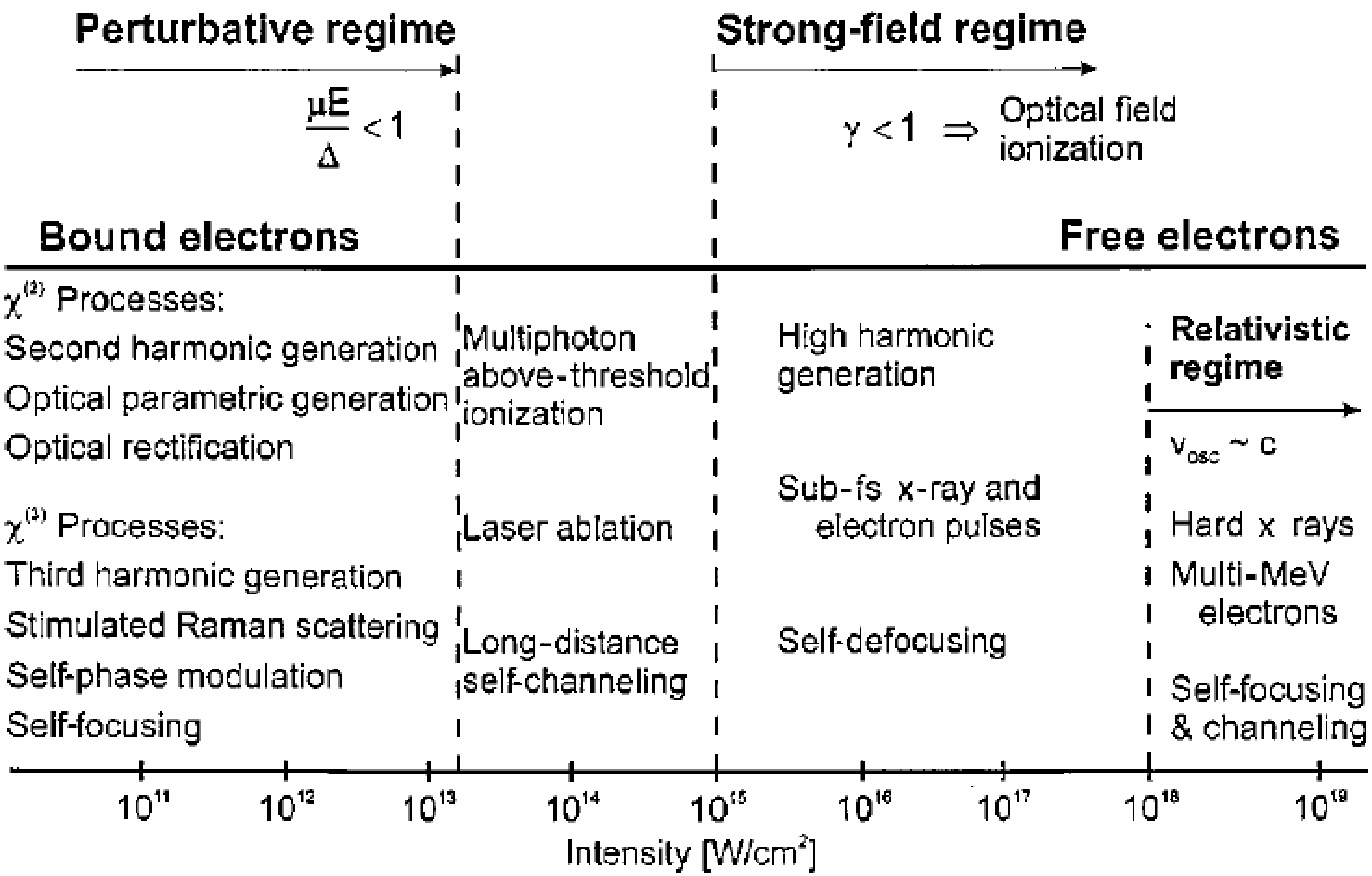
P. B. CORKUM[1] AND FERENC KRAUSZ[2,3]

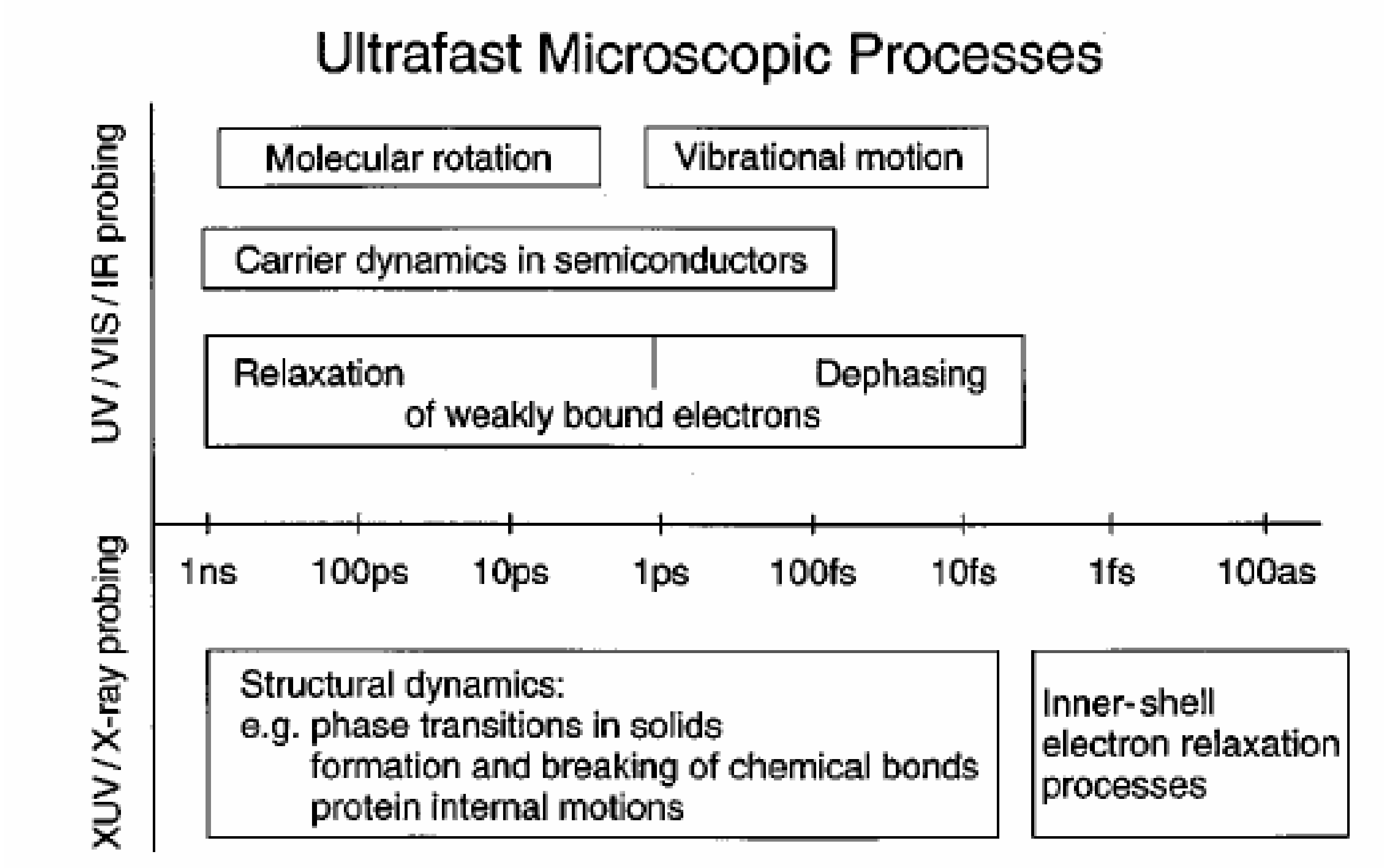
nature physics | VOL 3 | JUNE 2007 | www.nature.com/naturephysics

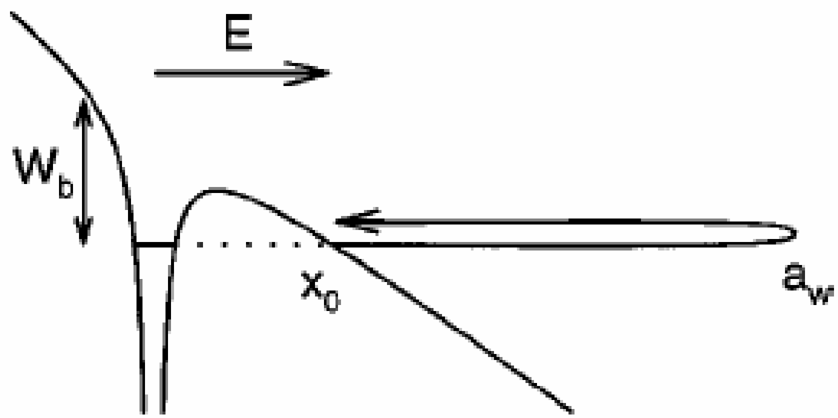
The Attosecond Revolution

P.B. Corkum and Zenghu Chang   OPN October 2008

Regimes of Nonlinear Optics

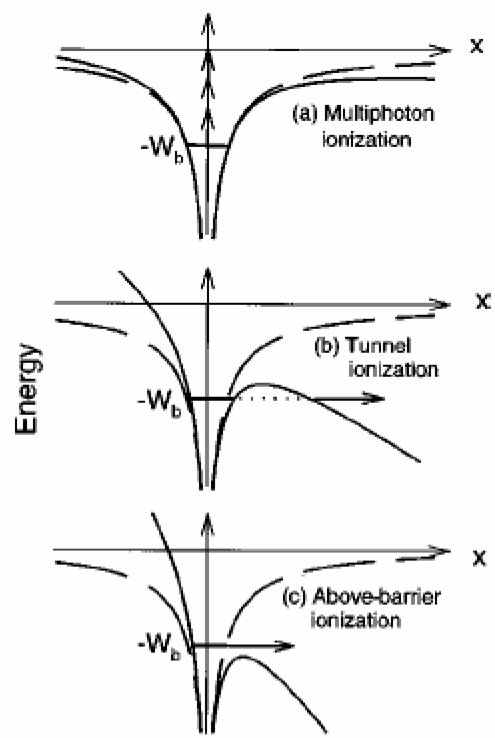



Ultrafast Microscopic Processes
UV / VIS / IR probing
Molecular rotation
Vibrational motion
Carrier dynamics in semiconductors
Relaxation
Dephasing
of weakly bound electrons
1ns
100ps
10ps
1ps
100fs
10fs
1fs
100as
XUV / X-ray probing
Structural dynamics:
e.g. phase transitions in solids
formation and breaking of chemical bonds
protein internal motions
Inner-shell
electron relaxation
processes

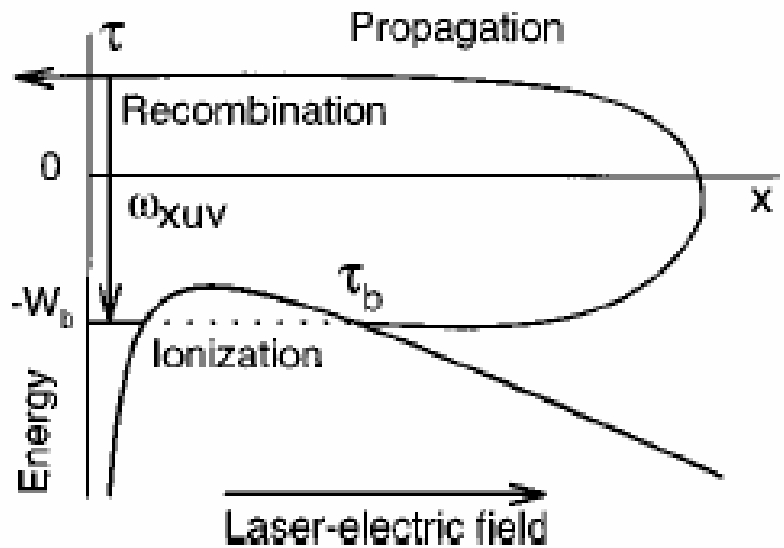

E
W_b
x_0
a_w

$$\frac{\chi^{(k+1)} E^{k+1}}{\chi^{(k)} E^{k}} \approx \frac{\mu_{ik} E_a}{\hbar \Delta} \approx \frac{e E_a a_B}{\hbar \Delta} = \alpha_{bb}$$

$$\frac{1}{\gamma} = \frac{e E_a}{\omega_0 \sqrt{2 m W_b}} = \frac{e E_a a_B}{\hbar \omega_0} = \alpha_{bf},$$



x
(a) Multiphoton ionization
-W_b
Energy
x
(b) Tunnel ionization
-W_b
x
(c) Above-barrier ionization
-W_b


Propagation
τ
Recombination
0
x
ω_xuv
-W_b
τ_b
Ionization
Energy
Laser-electric field

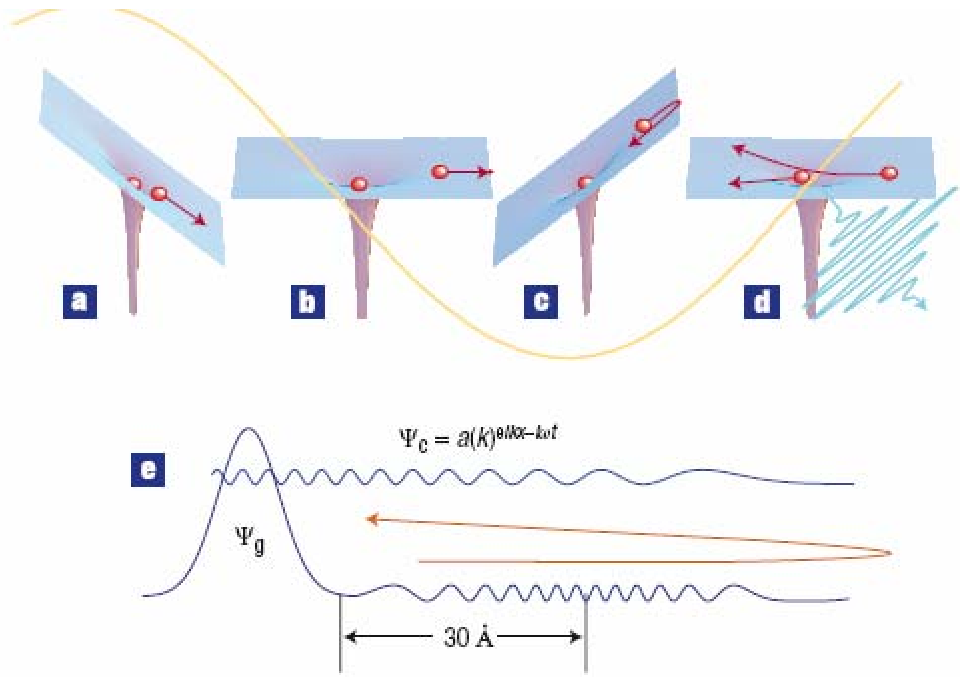
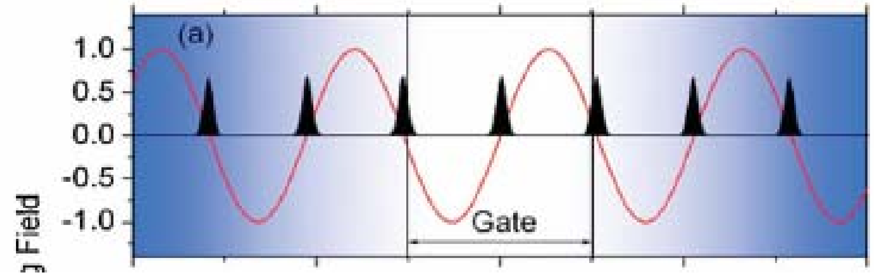


Figure 2 Creating an attosecond pulse. **a–d**, An intense femtosecond near-infrared or visible (henceforth: optical) pulse (shown in yellow) extracts an electron wavepacket from an atom or molecule. For ionization in such a strong field (**a**), Newton's equations of motion give a relatively good description of the response of the electron. Initially, the electron is pulled away from the atom (**a**, **b**), but after the field reverses, the electron is driven back (**c**) where it can 'recollide' during a small fraction of the laser oscillation cycle (**d**). The parent ion sees an attosecond electron pulse. This electron can be used directly, or its kinetic energy, amplitude and phase can be converted to an optical pulse on recollision[12]. **e**, The quantum mechanical perspective. Ionization splits the wavefunction: one portion remains in the original orbital, the other portion becomes a wave packet moving in the continuum. The laser field moves the wavepacket much as described in **a–d**, but when it returns the two portions of the wavefunction overlap. The resulting dynamic interference pattern transfers the kinetic energy, amplitude and phase from the recollision electron to the photon.



PRL **100**, 103906 (2008)

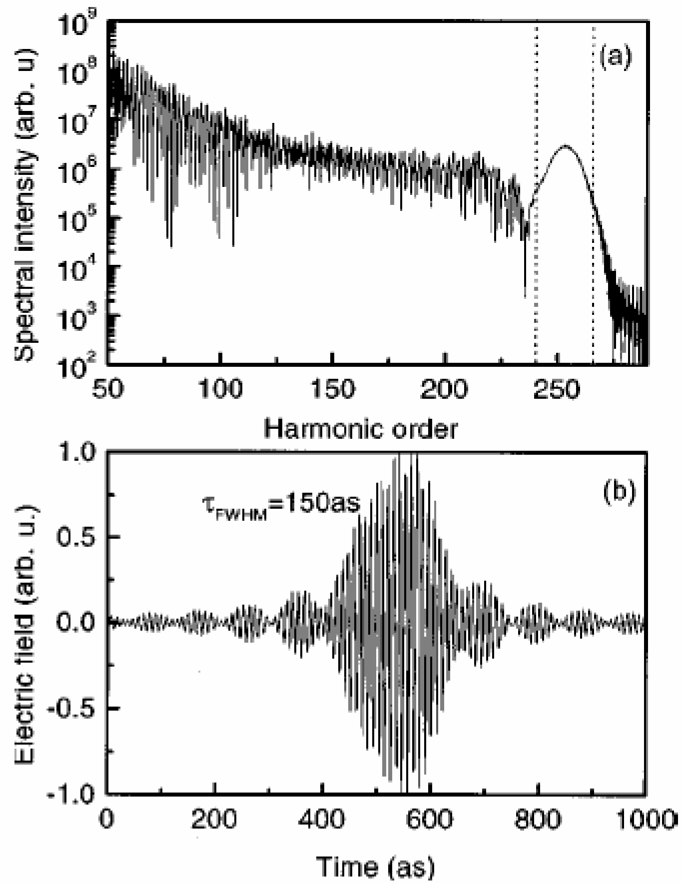
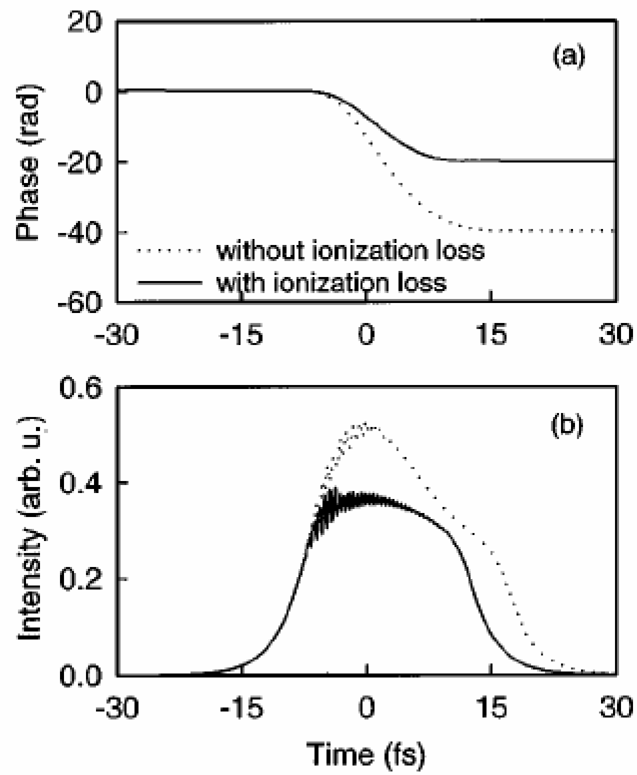
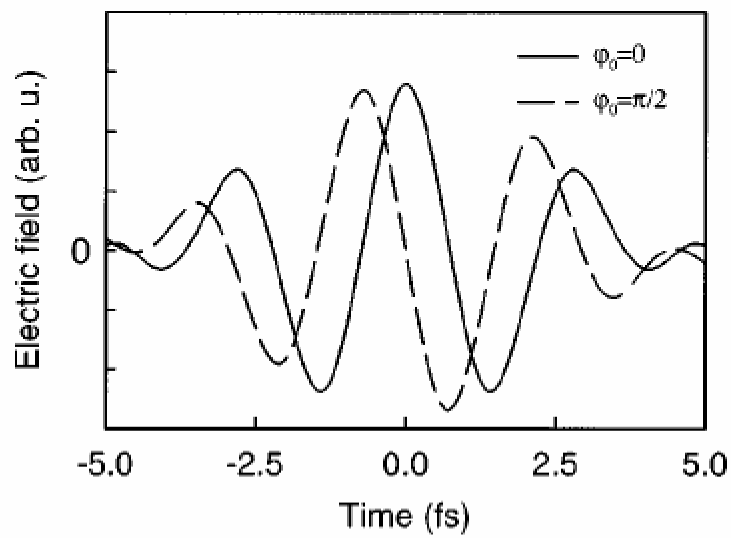


FIG. 45. Harmonic spectrum in helium (500 Torr) after a propagation distance of 9 μm for $\lambda_0 = 0.8\,\mu$m, $\tau_p = 5$ fs, $I_0 = 2 \times 10^{15}$ W/cm^2, and $\varphi_0 = 0$. (a) Computed harmonic spectrum; dotted lines, harmonic orders $N = 240$ and $N = 265$; (b) Fourier transform of the spectral band between $N = 240$ and $N = 265$.



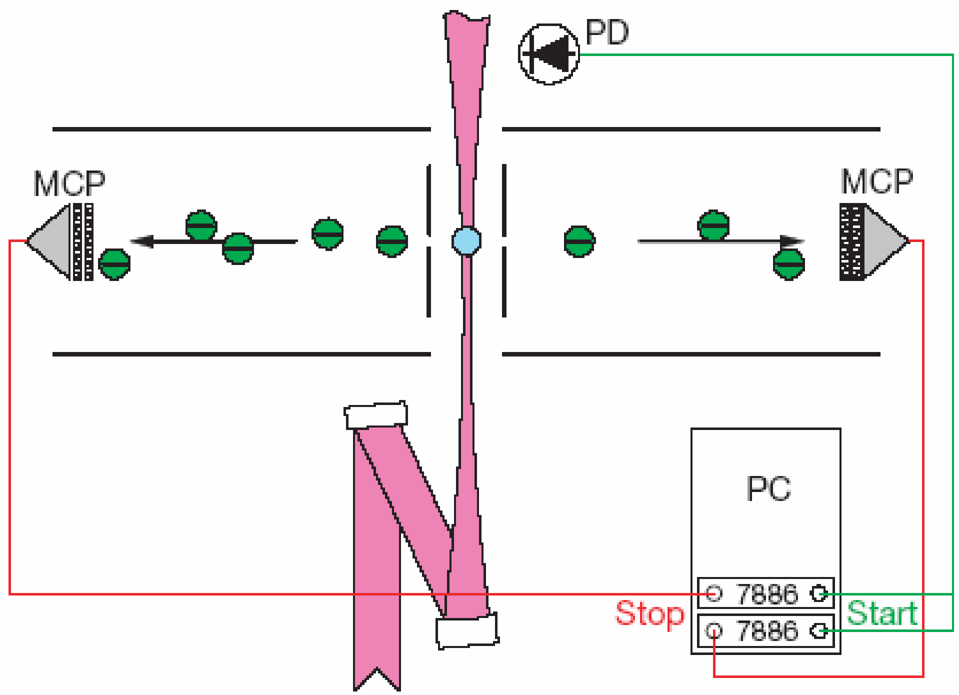
(a)
Phase (rad)
20
0
-20
-40
-60
without ionization loss
with ionization loss
-30
-15
0
15
30
(b)
Intensity (arb. u.)
0.6
0.4
0.2
0.0
-30
-15
0
15
30
Time (fs)



$\varphi_0=0$
$\varphi_0=\pi/2$
Electric field (arb. u.)
0
-5.0
-2.5
0.0
2.5
5.0
Time (fs)

CE Phase is Important for Atomic Physics

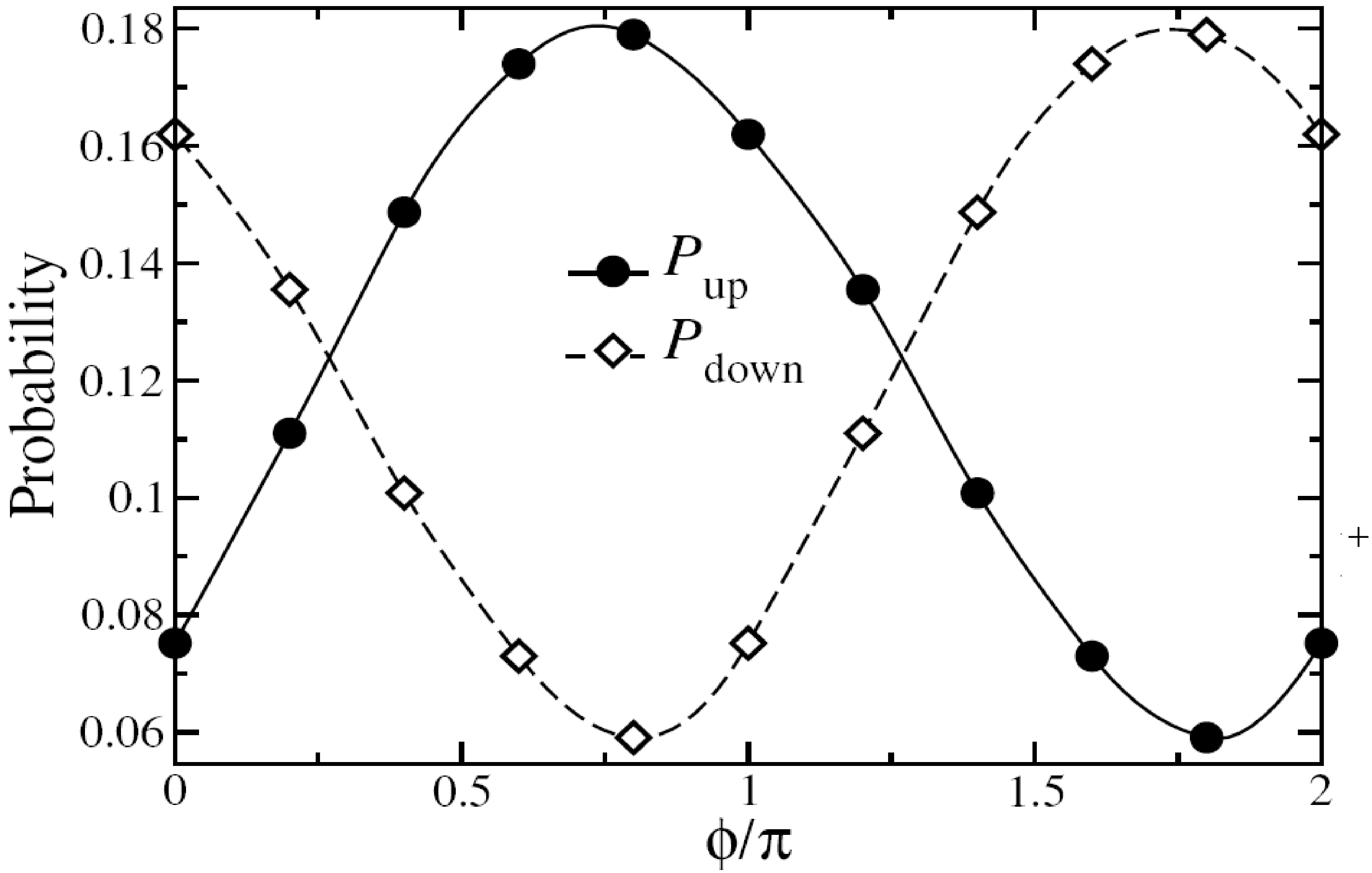
- **The carrier envelope phase has a strong influence on laser-atomic interactions**
  - **Intense-field photoionization**
  - **Above threshold ionization**
  - **Higher harmonic / attosecond pulse generation**



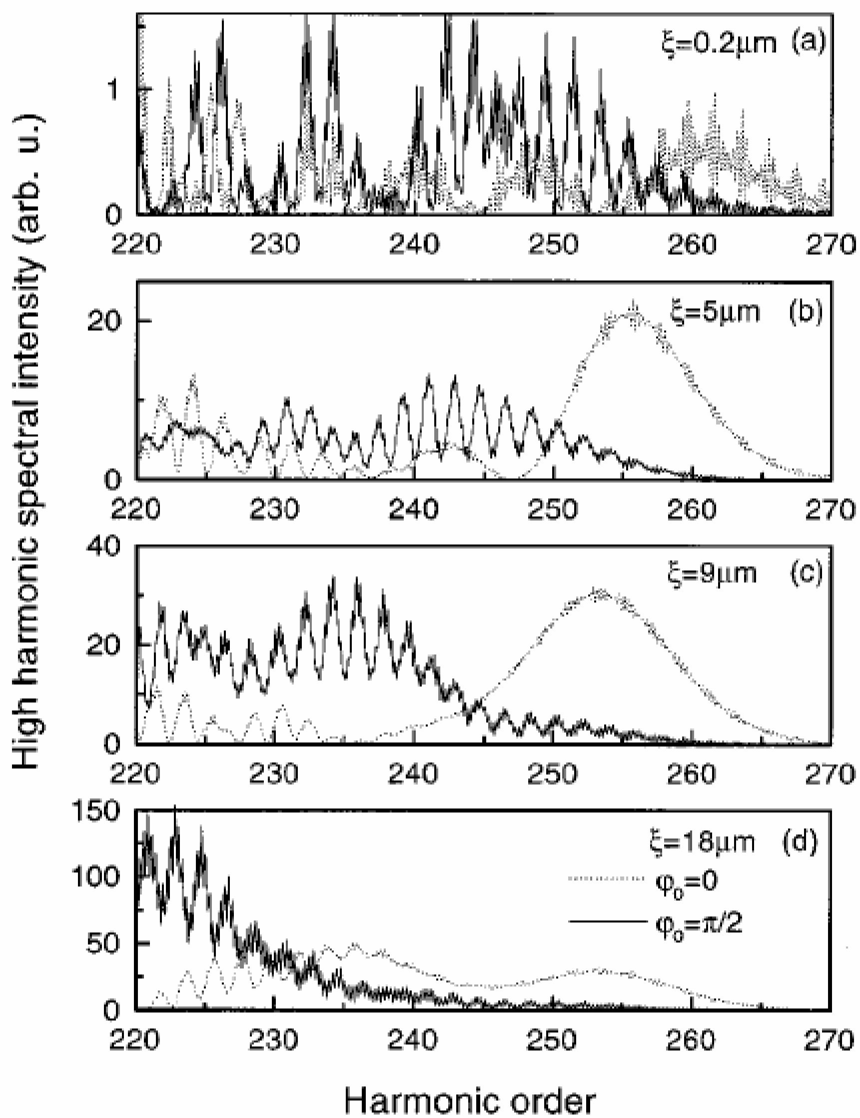
From Paulus et al, Nature 414, 8, November 2001, p 182

Phase measurement from study of ions

- **Theory by Brett Esry Group**



V. Roudnev et al., Phys. Rev. Lett. 93, 163601 (2004)



ξ=0.2μm (a)
ξ=5μm (b)
ξ=9μm (c)
ξ=18μm (d)
φo=0
φo=π/2
High harmonic spectral intensity (arb. u.)
Harmonic order

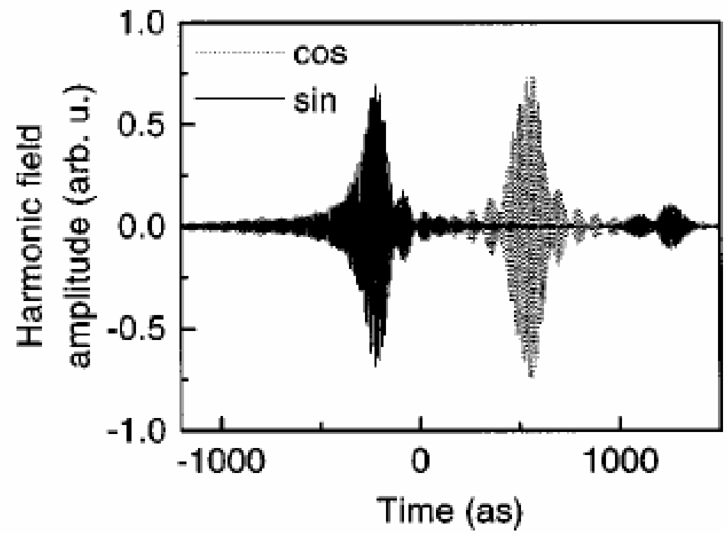
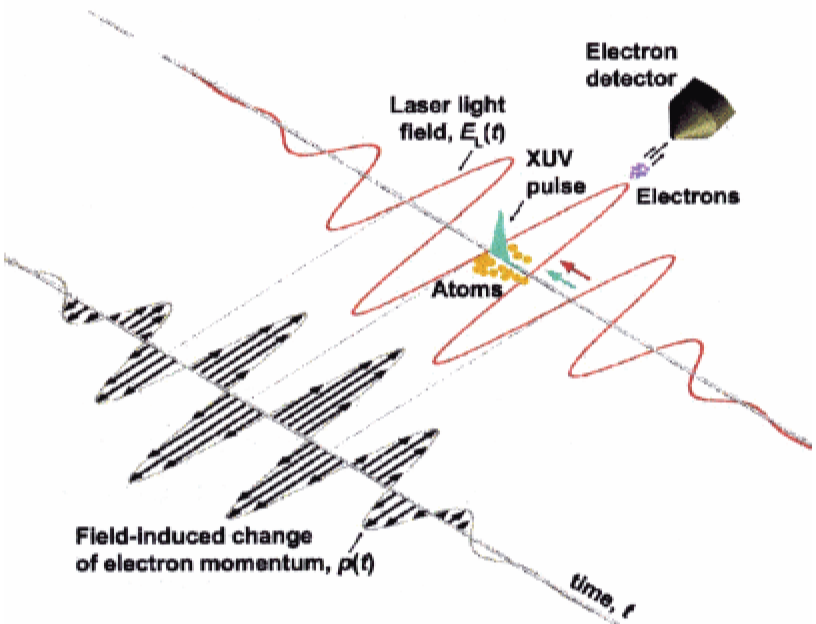
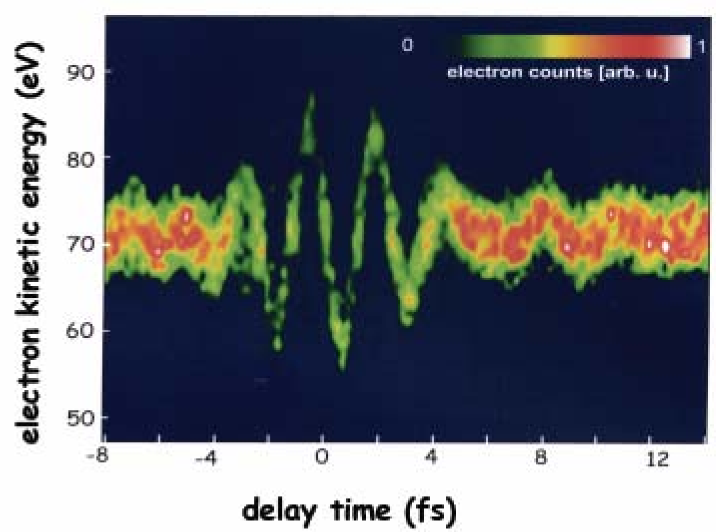
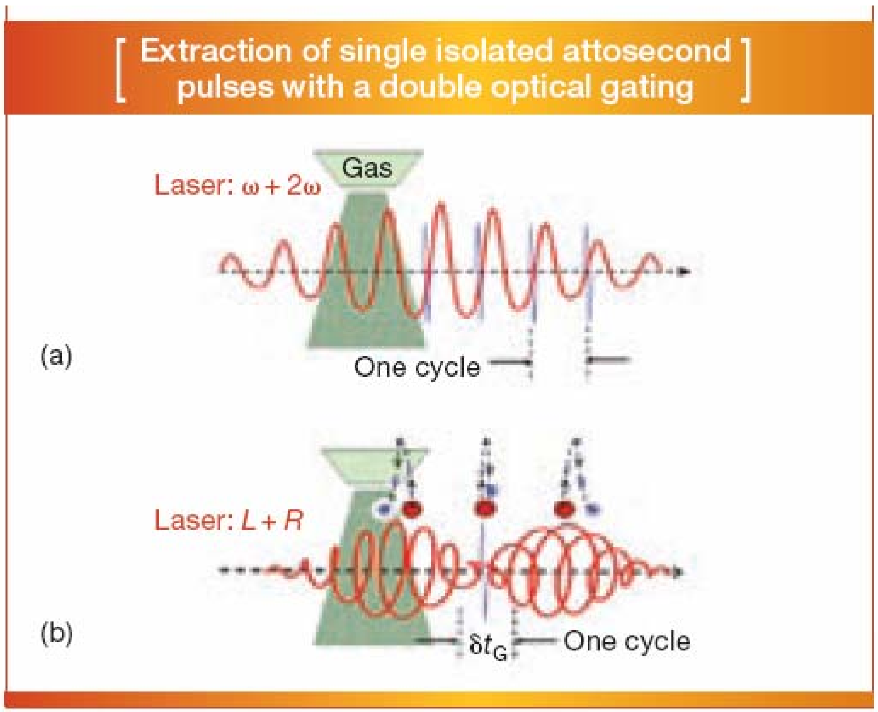


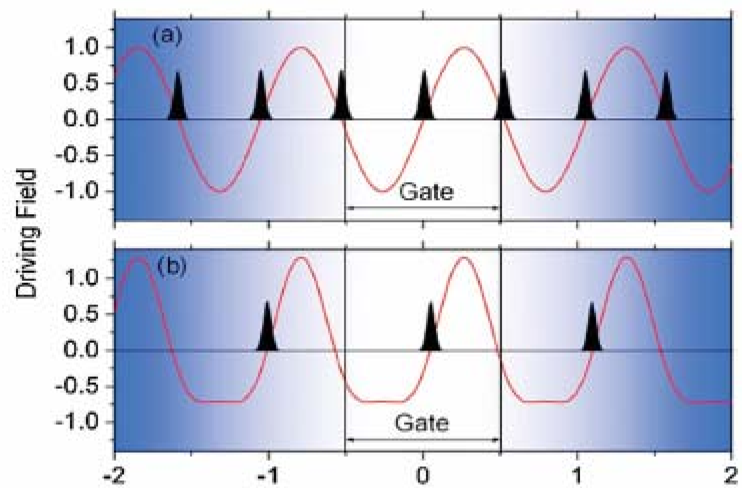
FIG. 49. Fourier transform of the harmonic amplitude spectrum between the orders 230 and 260 for the parameters of Fig. 48(c). Pulse phases: dotted line, $\varphi_0 = 0$; solid line, $\varphi_0 = \pi/2$.


Electron detector
Laser light field, $E_L(t)$
XUV pulse
Electrons
Atoms
Field-induced change of electron momentum, $p(t)$
time, t


0
1
electron counts [arb. u.]
electron kinetic energy (eV)
90
80
70
60
50
-8
-4
0
4
8
12
delay time (fs)

Isolated attosecond pulses: Double optical gating

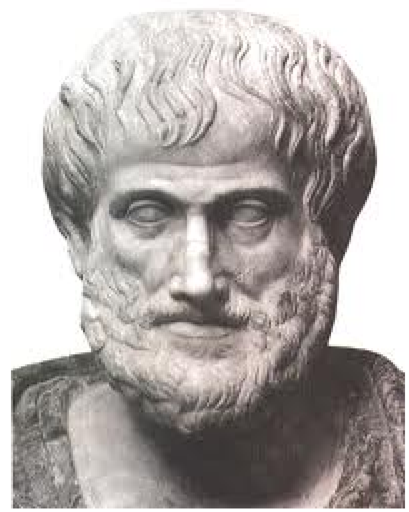




PRL **100**, 103906 (2008)

Newton

Huygens

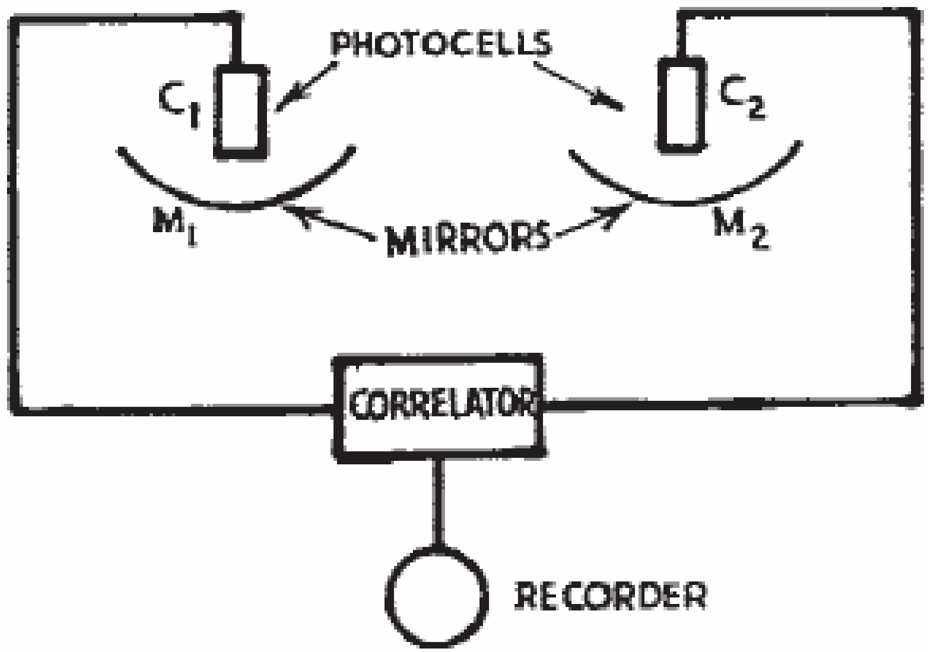


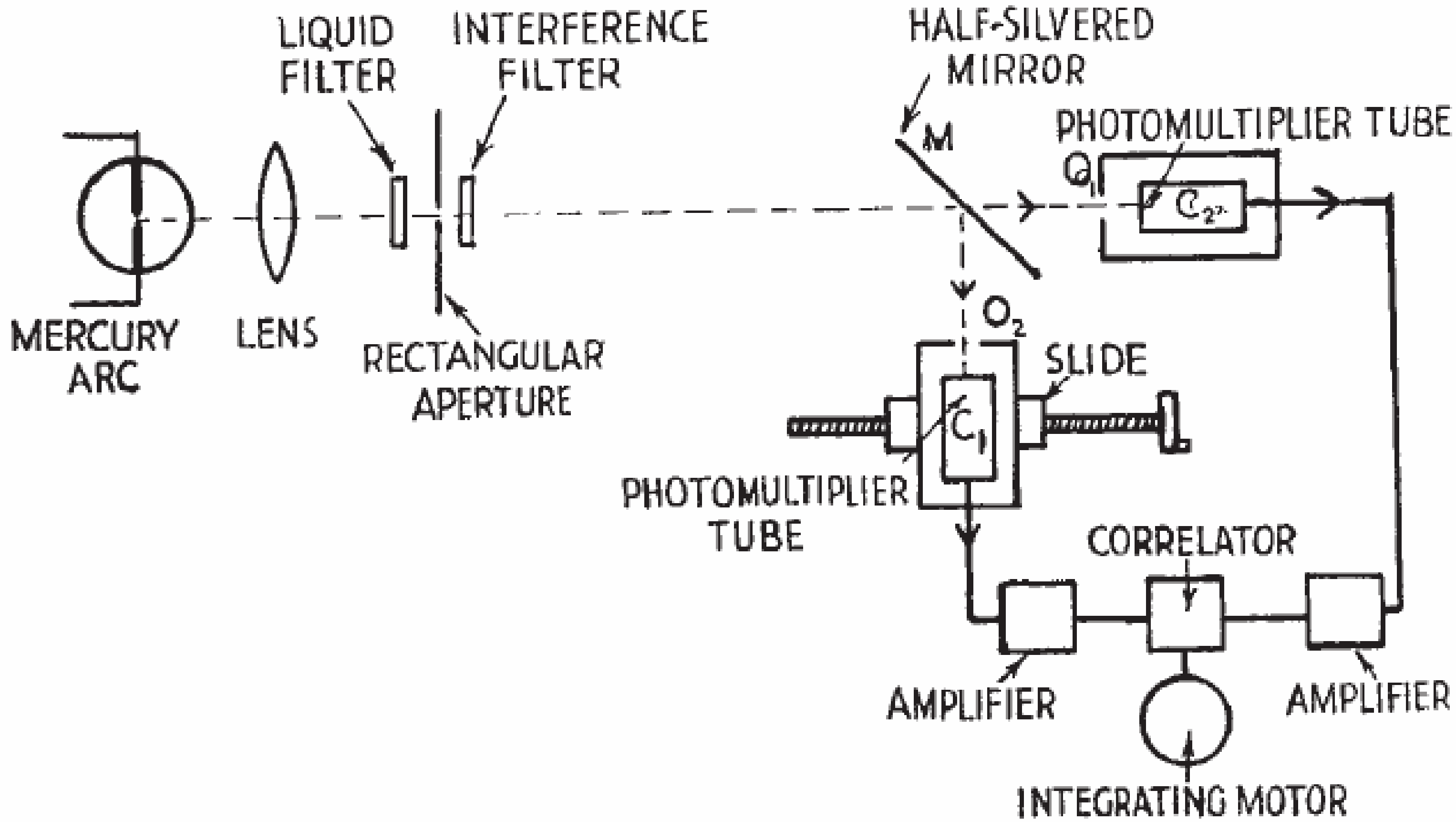
Aristotle

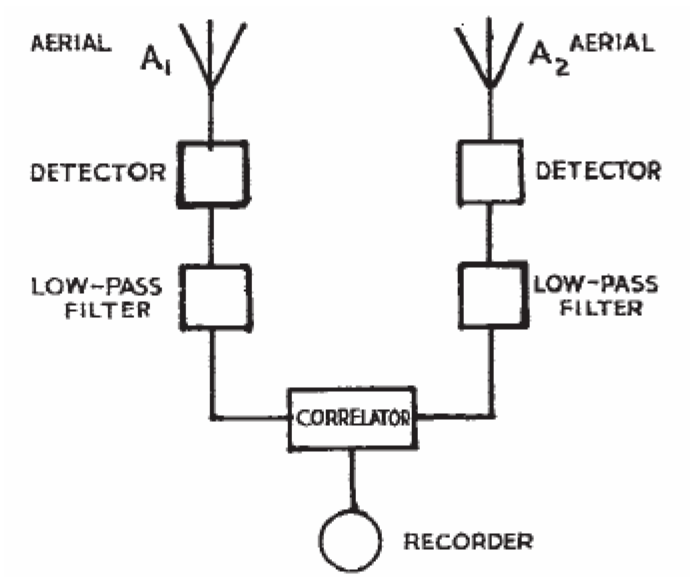
Laozi 老子

The Scientific Method

Richard Feynman Lectures:
http://www.youtube.com/watch?v=b240PGCMwV0
http://www.youtube.com/watch?v=wLaRXYai19A
http://www.youtube.com/watch?v=_MmpUWEW6Is&feature=related


PHOTOCELLS
C_1
C_2
M_1
MIRRORS
M_2
CORRELATOR
RECORDER


LIQUID FILTER
INTERFERENCE FILTER
HALF-SILVERED MIRROR
M
PHOTOMULTIPLIER TUBE
O_1
C_2
MERCURY ARC
LENS
RECTANGULAR APERTURE
O_2
SLIDE
C_1
PHOTOMULTIPLIER TUBE
CORRELATOR
AMPLIFIER
AMPLIFIER
INTEGRATING MOTOR



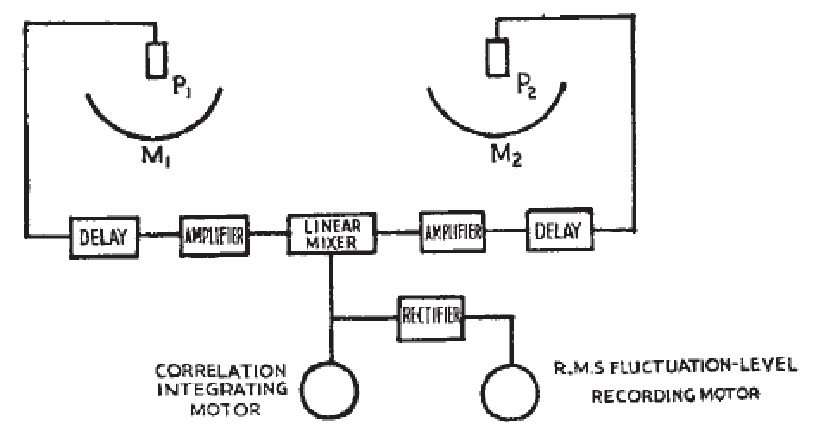
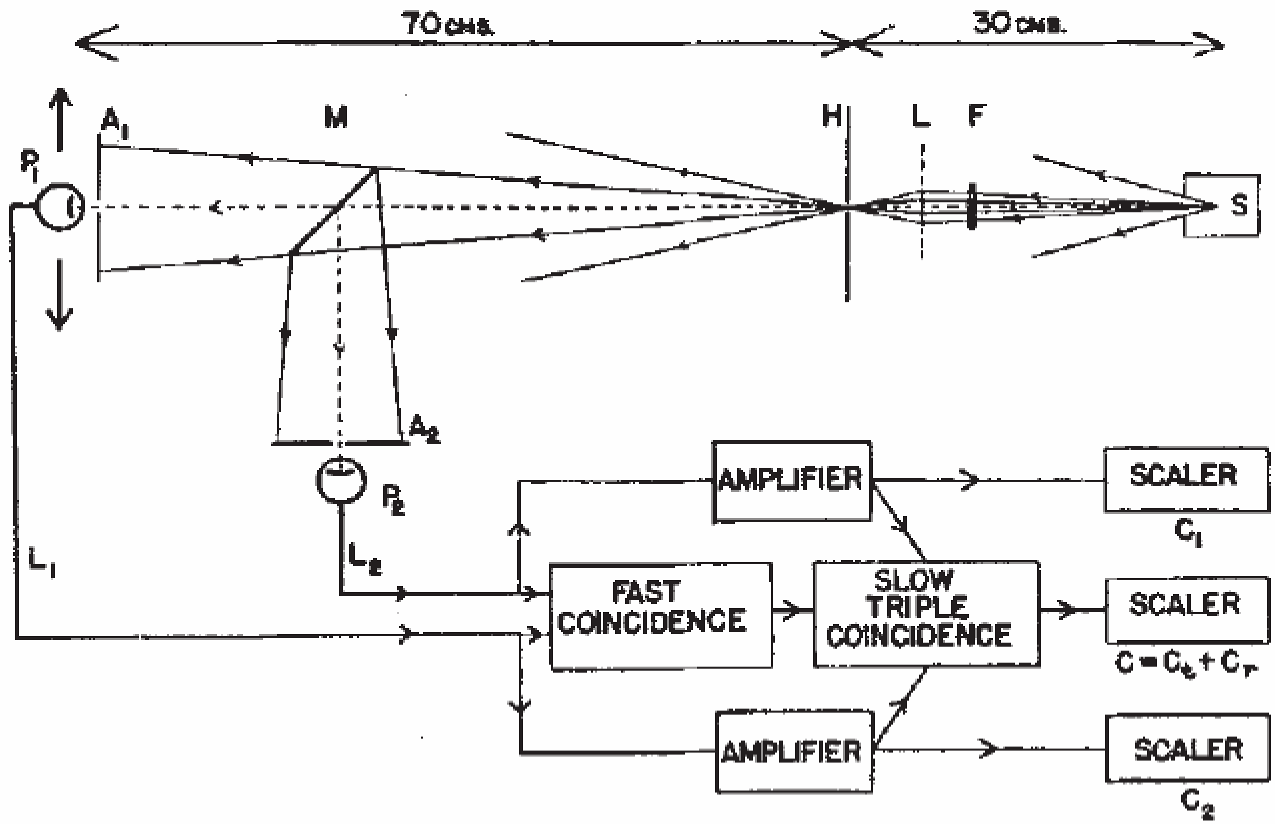


Fig. 1. Simplified diagram of the apparatus



70 CMS.
30 CMS.
A1
M
H
L
F
P1
S
A2
P2
L1
L2
AMPLIFIER
FAST COINCIDENCE
SLOW TRIPLE COINCIDENCE
AMPLIFIER
SCALER
C1
SCALER
C = Ct + Cr
SCALER
C2

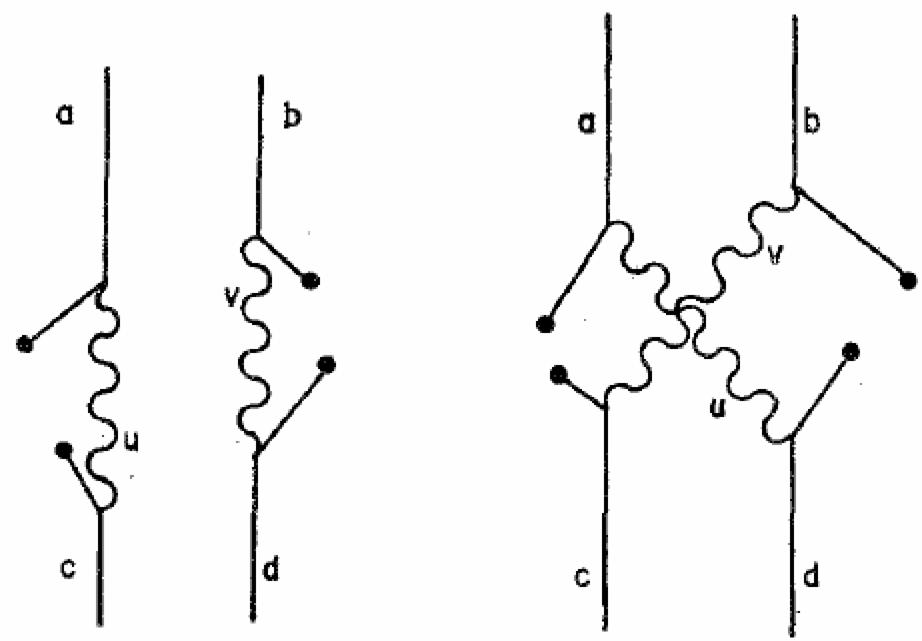
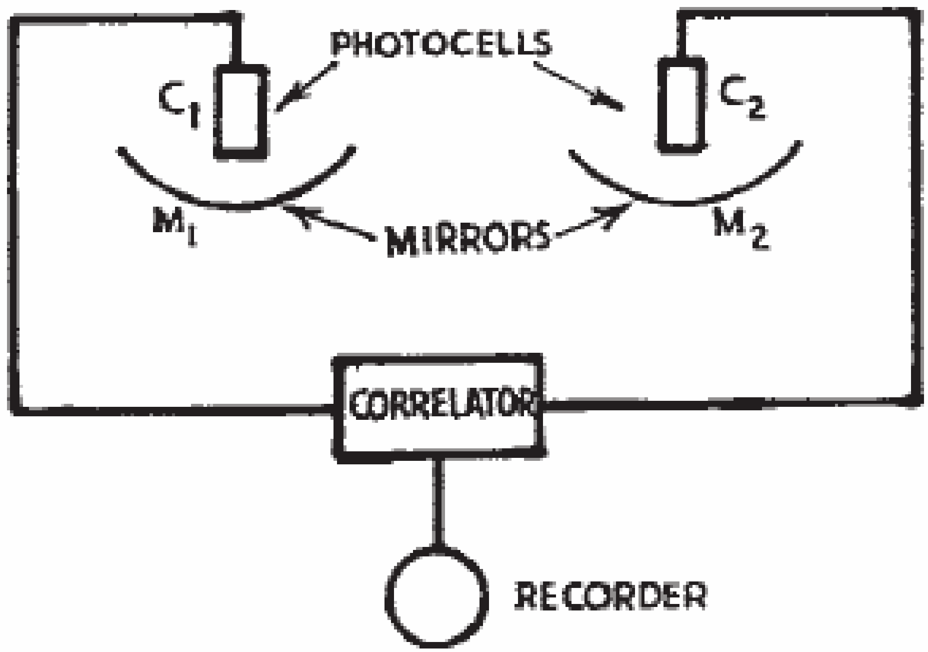
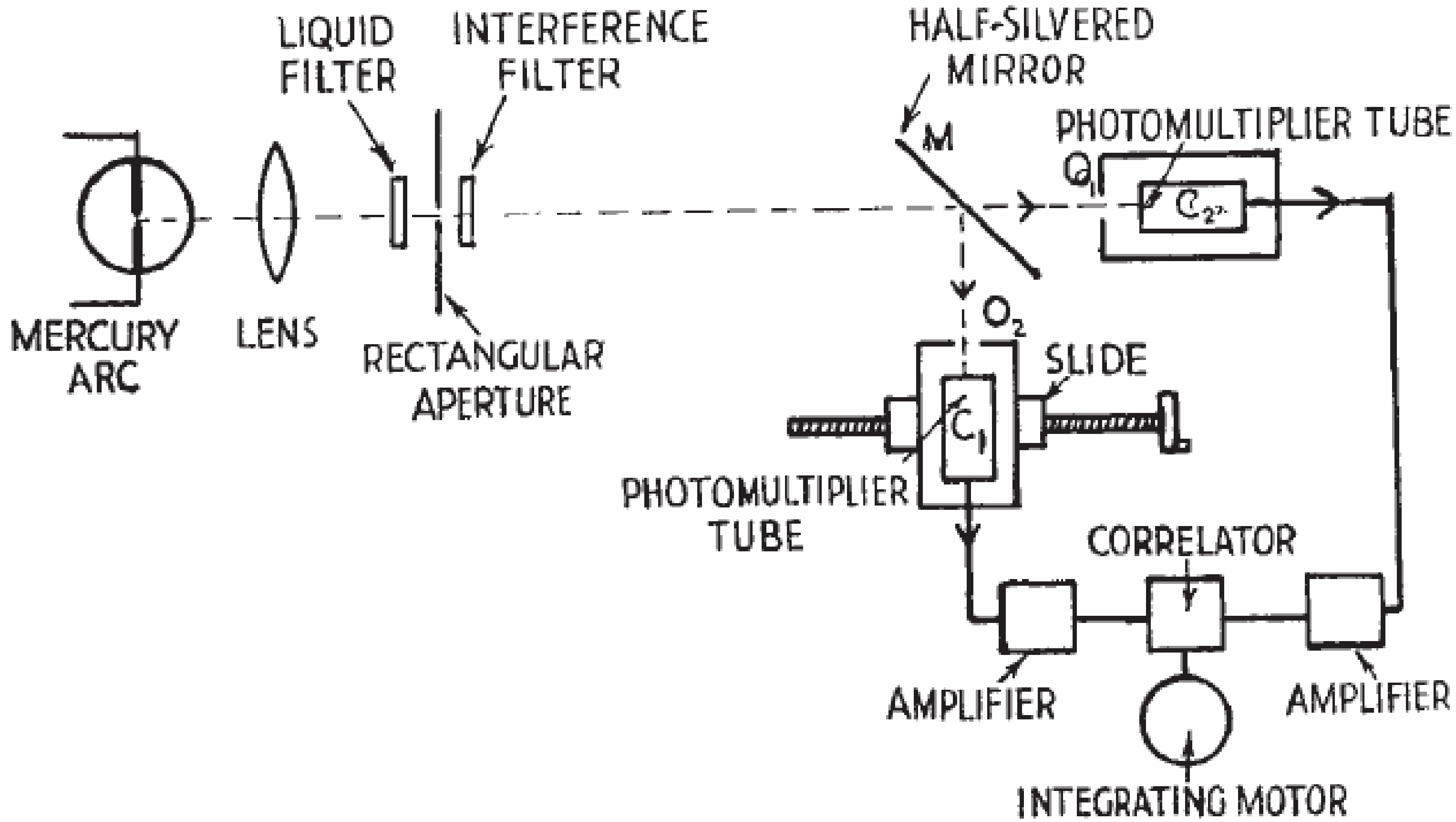


FIG. 1. Lowest-order diagrams representing light emission by a pair of atoms and its absorption by another pair of atoms. The heavy dots indicate ground-state lines.


PHOTOCELLS
C1
C2
M1
MIRRORS
M2
CORRELATOR
RECORDER



LIQUID
FILTER
INTERFERENCE
FILTER
HALF-SILVERED
MIRROR
M
PHOTOMULTIPLIER TUBE
O_1
C_2
O_2
MERCURY
ARC
LENS
RECTANGULAR
APERTURE
SLIDE
C_1
PHOTOMULTIPLIER
TUBE
CORRELATOR
AMPLIFIER
AMPLIFIER
INTEGRATING MOTOR

1. In *Hanbury-Brown and Twiss* (Paper 1), why did they measure coincidences in "cathodes aligned" positions and no coincidences in "cathodes not aligned position"?

2. Why did *Brannen and Ferguson* (Paper 2) not measure any coincidences?

3. At the end of *Brannen and Ferguson*, they stated that "if such a correlation did exist, it would call for a major revision of some fundamental concepts of quantum mechanics". How did they come to that conclusion?

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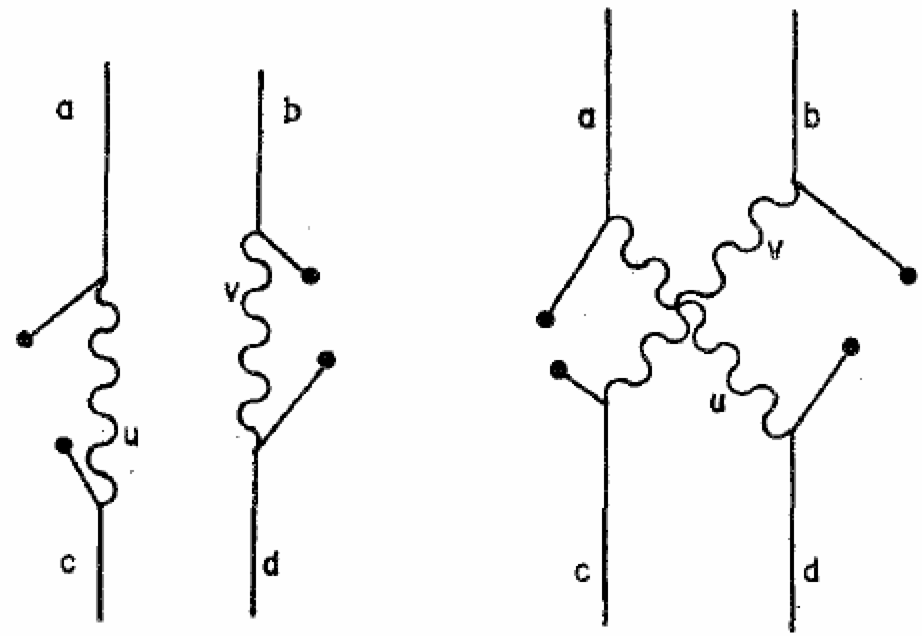

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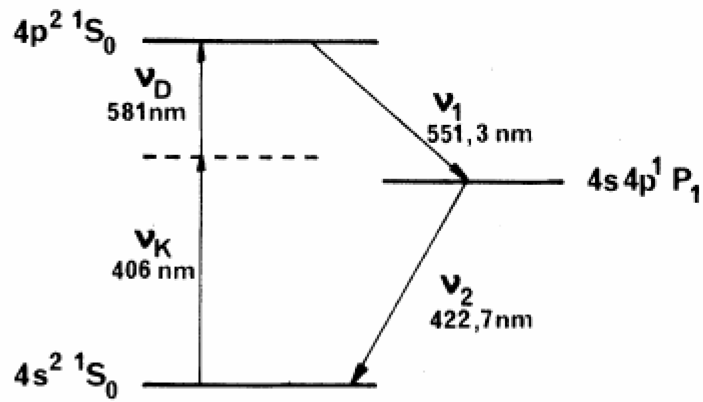


FIG. 1. Relevant levels of calcium. The atoms, selectively pumped to the upper level by the nonlinear absorption of ν_K and ν_L, emits the photons ν_1 and ν_2 correlated in polarization.

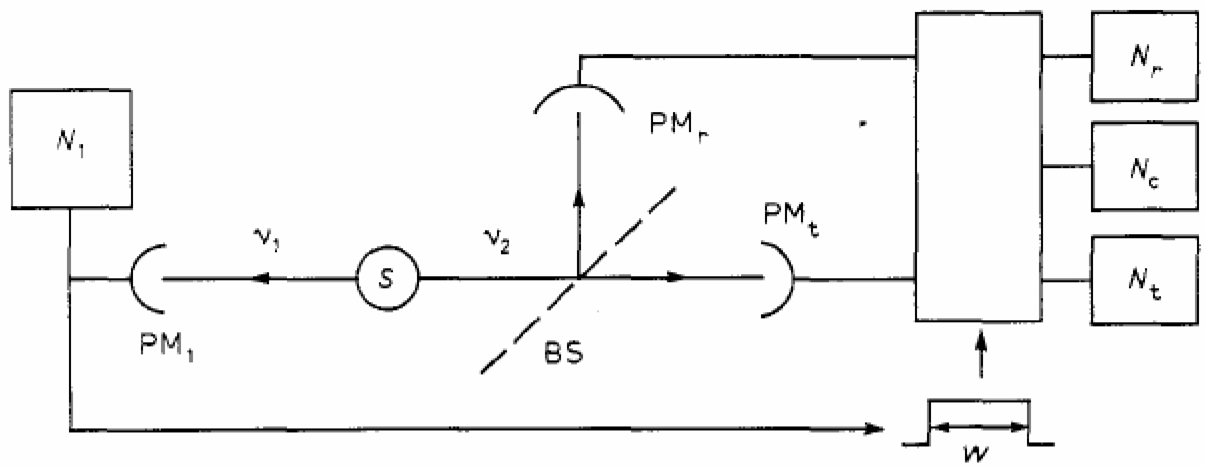


Fig. 1. – Triggered experiment. The detection of the first photon of the cascade produces a gate w, during which the photomultipliers PM$_t$ and PM$_r$ are active. The probabilities of detection during the gate are $p_t = N_t/N_1$, $p_r = N_r/N_1$ for singles, and $p_c = N_c/N_1$ for coincidences.

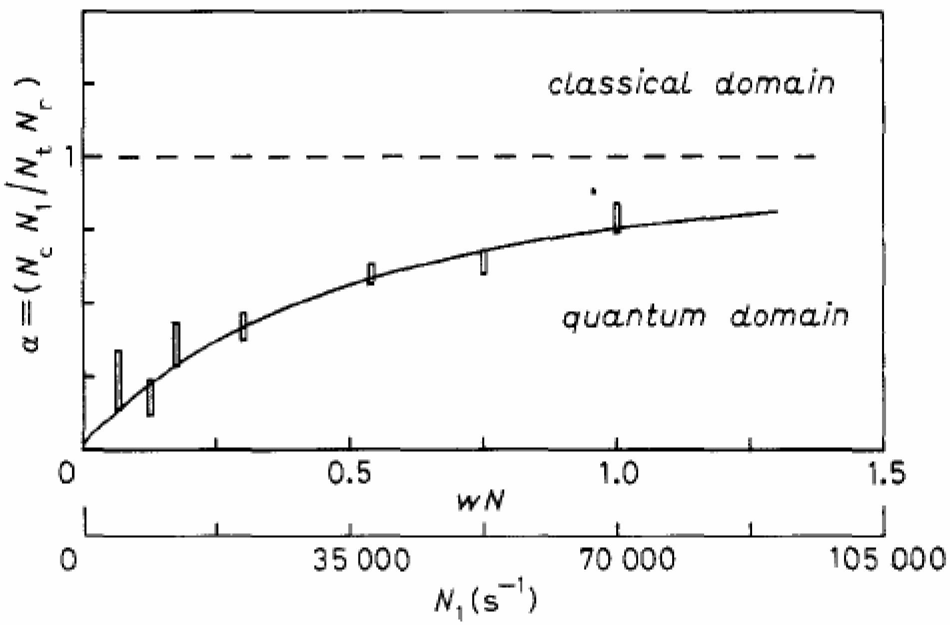


Fig. 2. – Anticorrelation parameter α as a function of wN (number of cascades emitted during the gate) and of N_1 (trigger rate). The indicated error bars are $\pm$ one standard deviation. The full-line curve is the theoretical prediction from eq. (8). The inequality $\alpha \geqslant 1$ characterizes the classical domain.

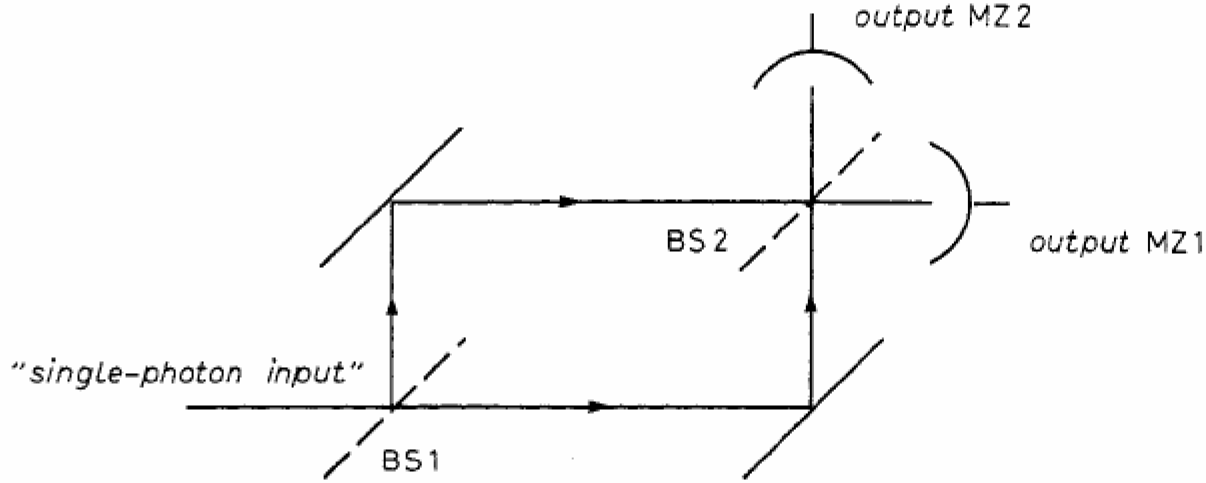


Fig. 3. – Mach-Zehnder interferometer. The detection probabilities in outputs MZ1 and MZ2 are oppositely modulated as a function of the path difference between the arms of the interferometer.

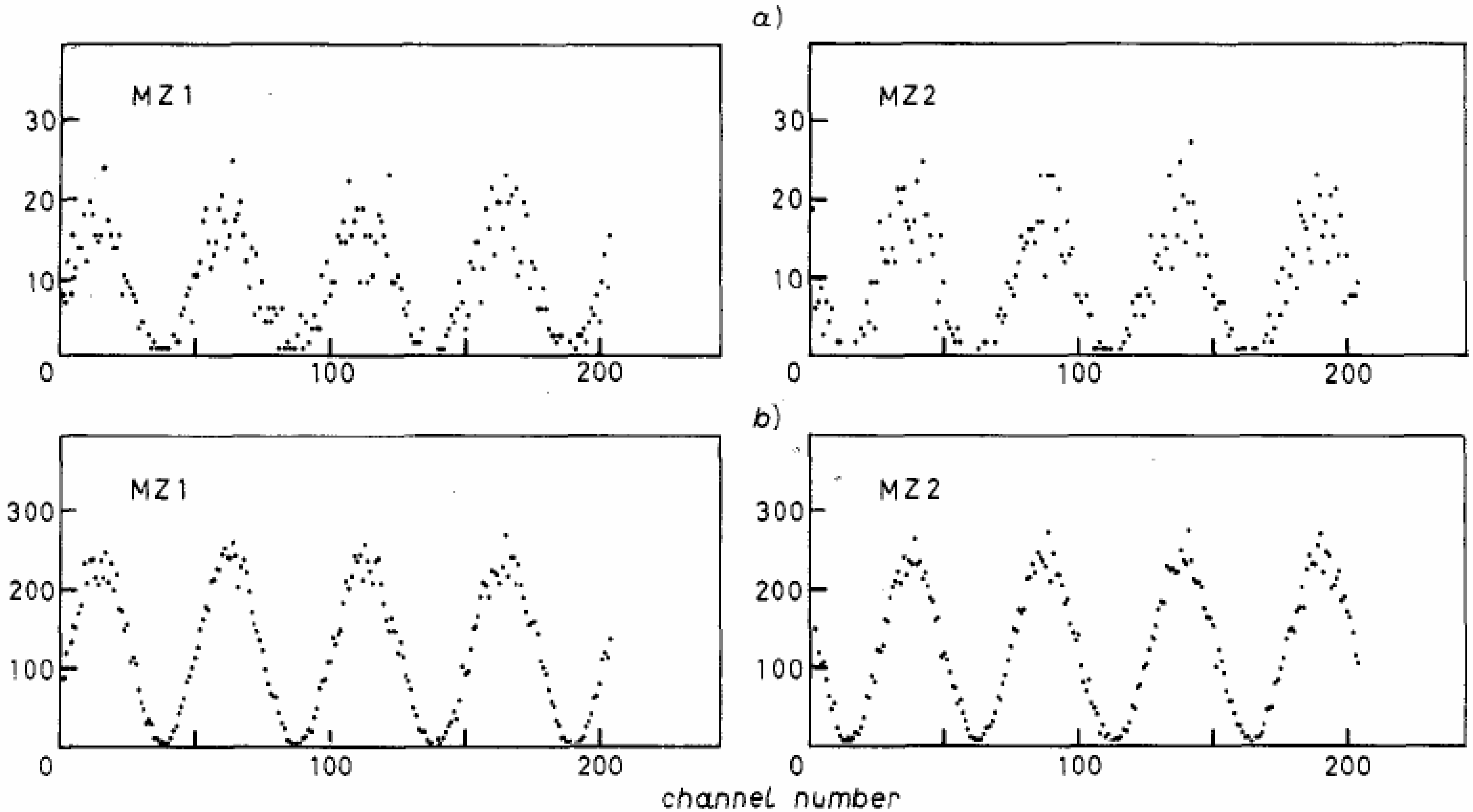
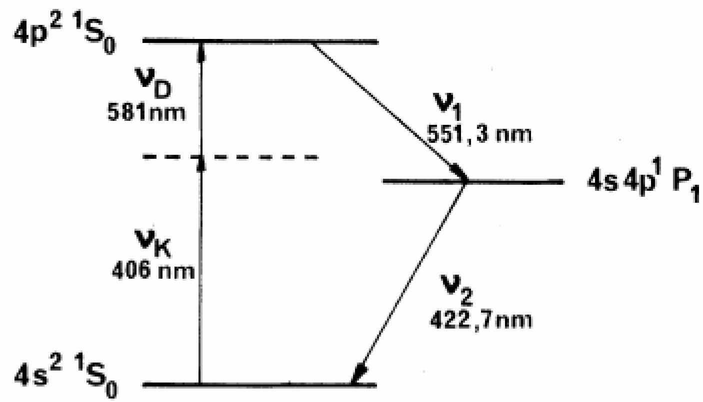


Fig. 4. – Number of counts in outputs MZ1 and MZ2 as a function of the path difference δ (one channel corresponds to a $\lambda/50$ variation of δ). *a*) 1 s counting time per channel *b*) 15 s counting time per channel (compilation of 15 elementary sweeps (like (*a*)). This experiment corresponds to an anticorrelation parameter $\alpha = 0.18$.

Sun Wu (Sun Tzu)

孙**武**

Experiment of Aspect *et al,* Europhys. Lett. 1 173 1986



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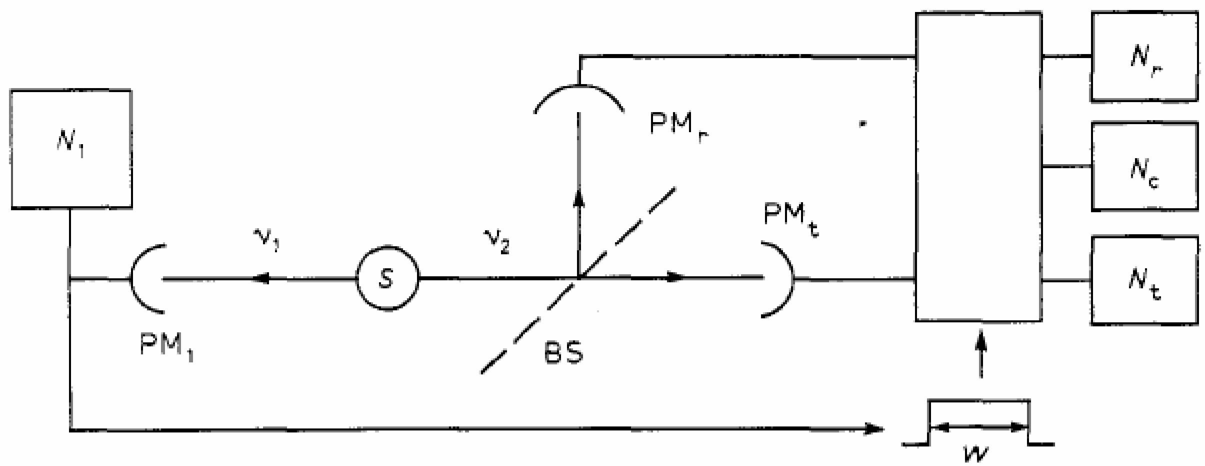


Fig. 1. – Triggered experiment. The detection of the first photon of the cascade produces a gate w, during which the photomultipliers PM$_t$ and PM$_r$ are active. The probabilities of detection during the gate are $p_t = N_t/N_1$, $p_r = N_r/N_1$ for singles, and $p_c = N_c/N_1$ for coincidences.

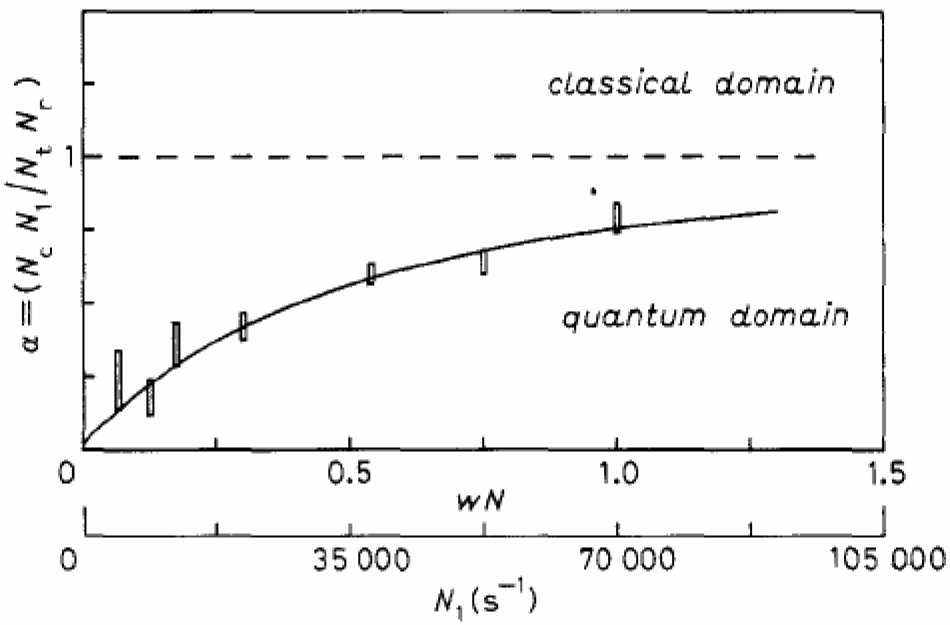


Fig. 2. – Anticorrelation parameter α as a function of wN (number of cascades emitted during the gate) and of N_1 (trigger rate). The indicated error bars are $\pm$ one standard deviation. The full-line curve is the theoretical prediction from eq. (8). The inequality $\alpha \geqslant 1$ characterizes the classical domain.

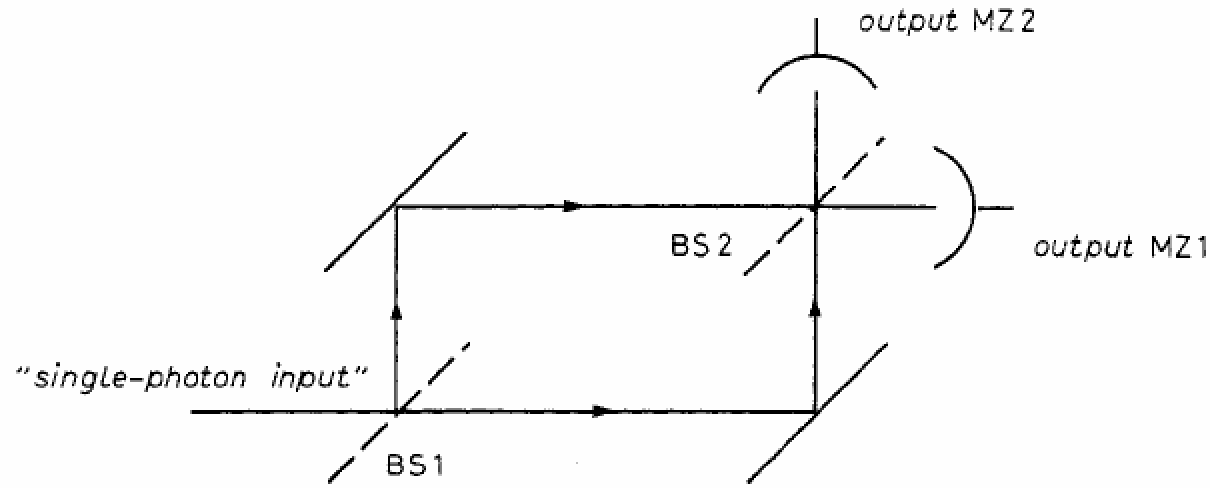


Fig. 3. – Mach-Zehnder interferometer. The detection probabilities in outputs MZ1 and MZ2 are oppositely modulated as a function of the path difference between the arms of the interferometer.

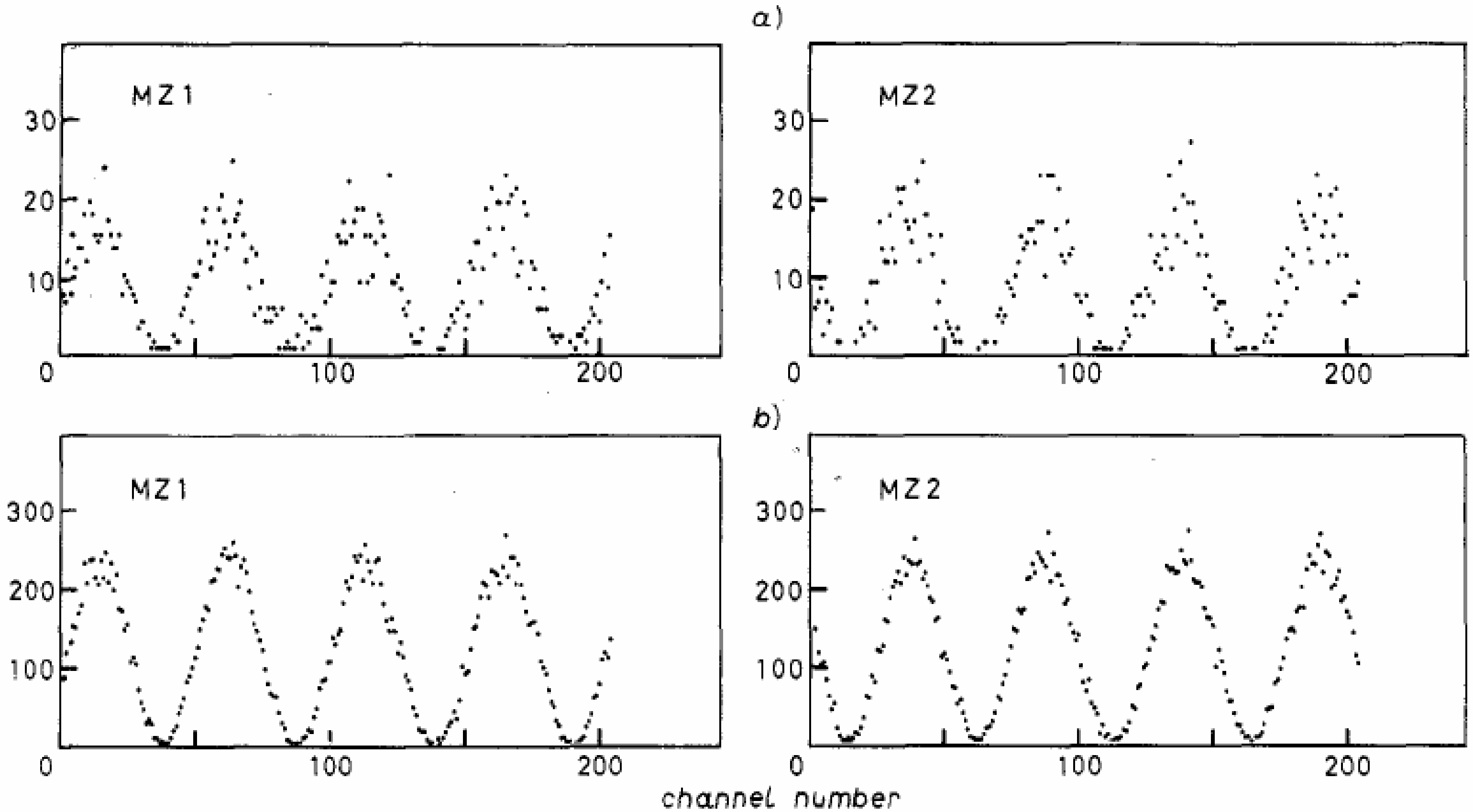
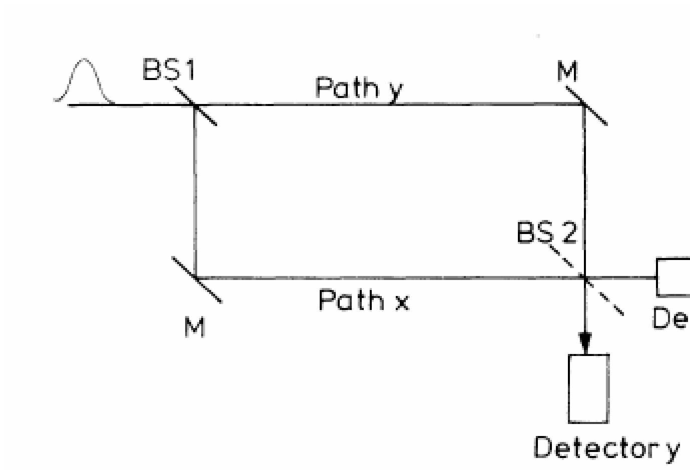


Fig. 4. – Number of counts in outputs MZ1 and MZ2 as a function of the path difference δ (one channel corresponds to a $\lambda/50$ variation of δ). *a*) 1 s counting time per channel *b*) 15 s counting time per channel (compilation of 15 elementary sweeps (like (*a*)). This experiment corresponds to an anticorrelation parameter $\alpha = 0.18$.

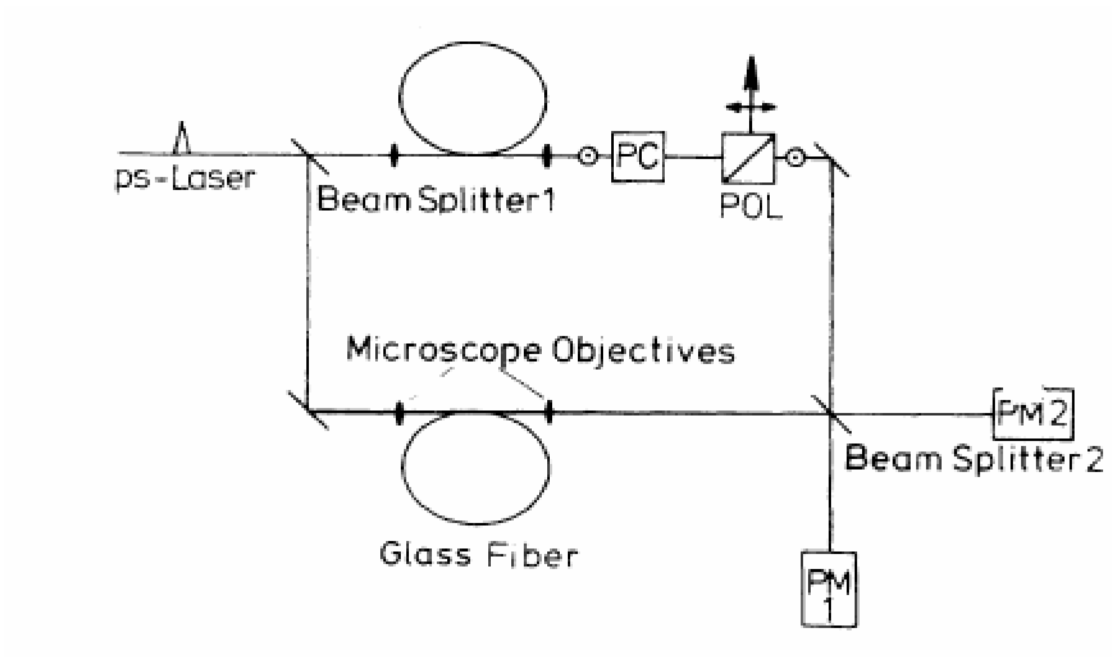
J. Wheeler's Delayed Choice Experiment



Move BS 2 in or out
after light is past BS 1

Delayed Choice Experiment

Experiment of Walther *et al.* PRA vol 35, 6 1987



Comparison of normal and delayed choice modes

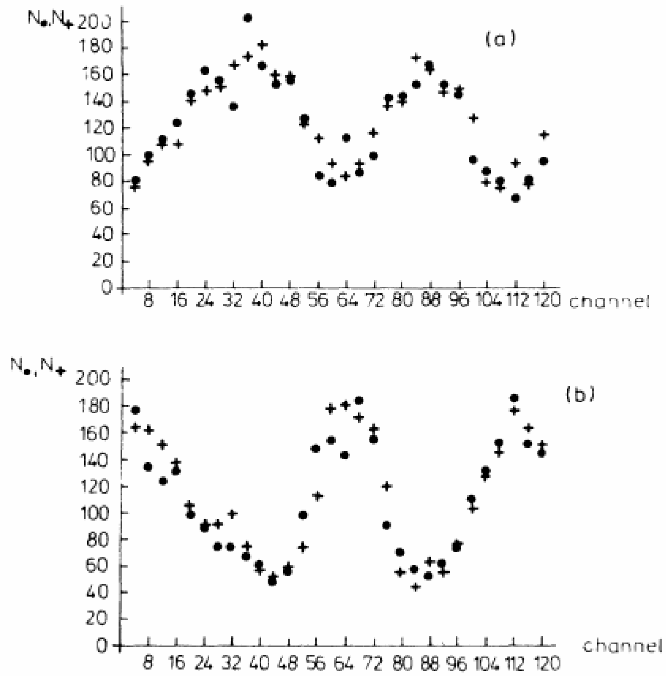


FIG. 6. Comparison of interference patterns for normal and delayed-choice configurations. Dots represent the data taken with the interferometer in its normal configuration, and crosses are data for delayed-choice operation. (a) is for photomultiplier 1, while the phase-inverted signal detected by photomultiplier 2 is shown in (b). The points are four-channel averages of the raw data. The horizontal axis is equivalent to time with 30 s/channel.

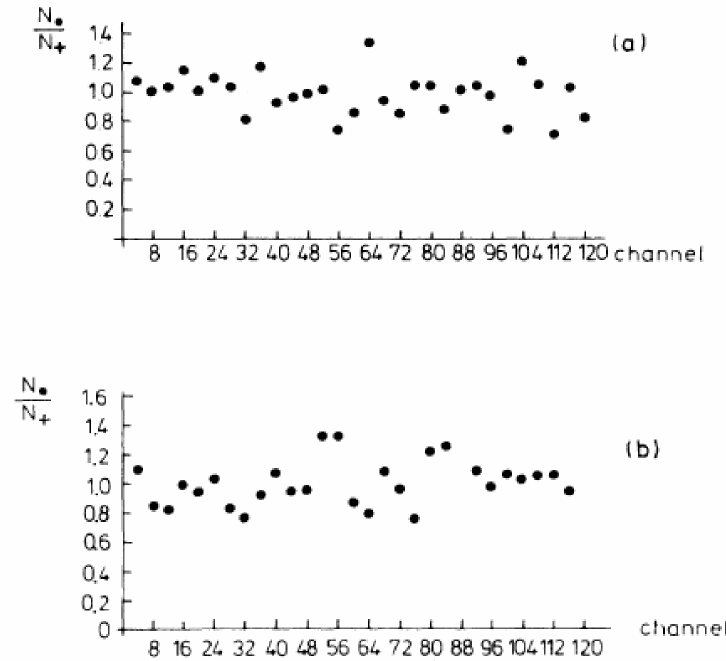


FIG. 7. Ratio $N_\bullet/N_+$ using the results from Fig. 6. Again the horizontal axis is equivalent to time. The Copenhagen interpretation of quantum mechanics predicts $N_\bullet/N_+=1$.

$N_\bullet$: Normal Mode

N_+ : Delayed Choice Mode

$N_\bullet / N_+ = 1.00 \pm 0.02$

Wheeler’s Smoky Dragon

Quantum Beat Experiment

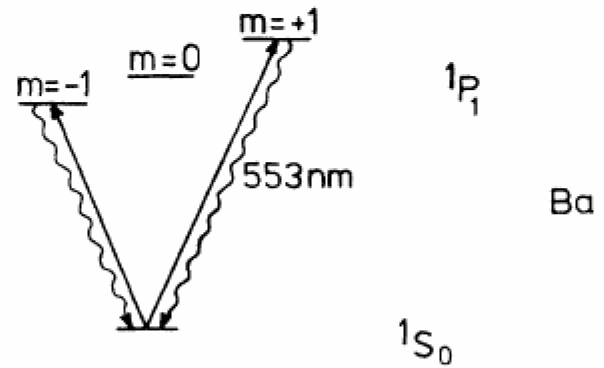


FIG. 5. Excitation scheme of barium used in the quantum-beat experiment.

$$\sigma^+ \ |0\rangle \to |+1\rangle \to |0\rangle$$
$$\sigma^- \ |0\rangle \to |-1\rangle \to |0\rangle$$

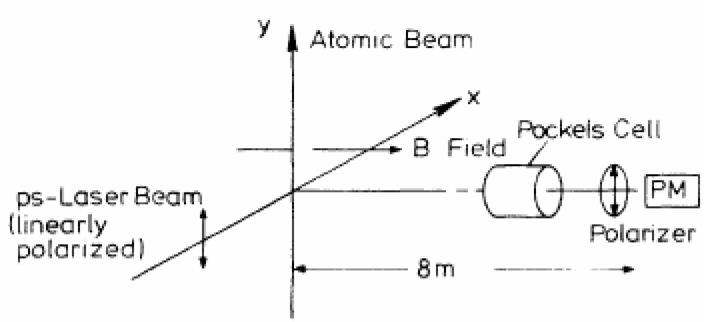


FIG. 4. Schematic arrangement of the quantum-beat experiment.

Pockels Cell ON
σ+ transmitted
σ- blocked

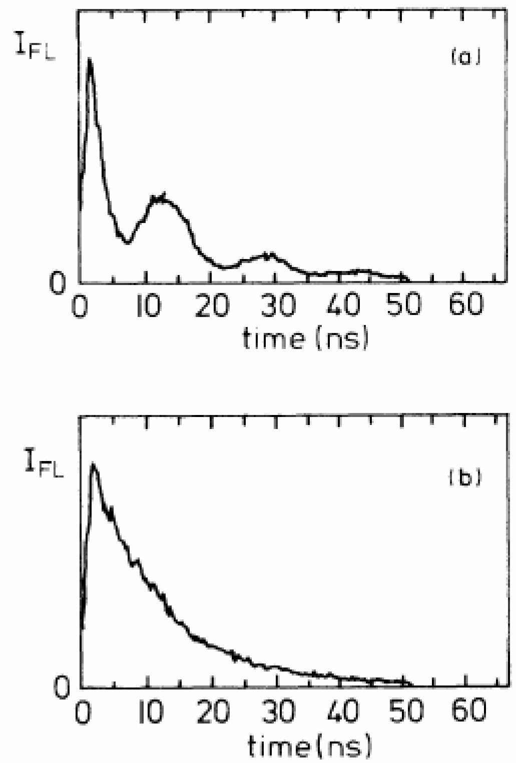


FIG. 8. Time-resolved fluorescence from pulse-excited barium in normal quantum-beat configuration when (a) Pockels cell voltage is zero, (b) a quarter-wave voltage is applied to the Pockels cell.

Pockels cell switched OFF at S (both paths after time S)

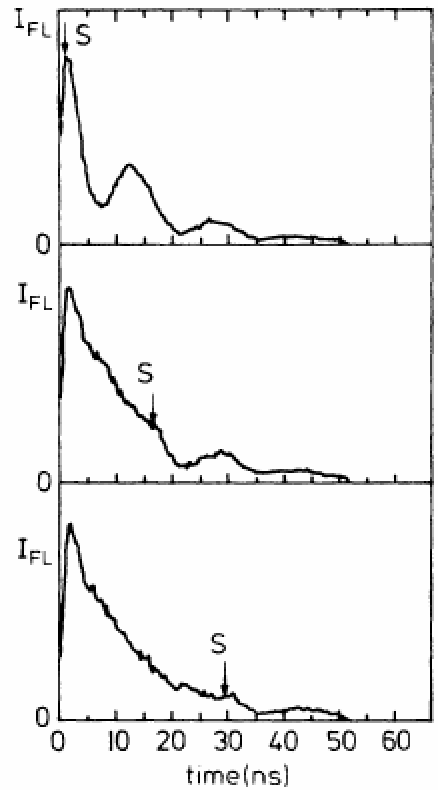


FIG. 9. Time-resolved fluorescence intensity with the Pockels cell switched off at different times (marked by S).

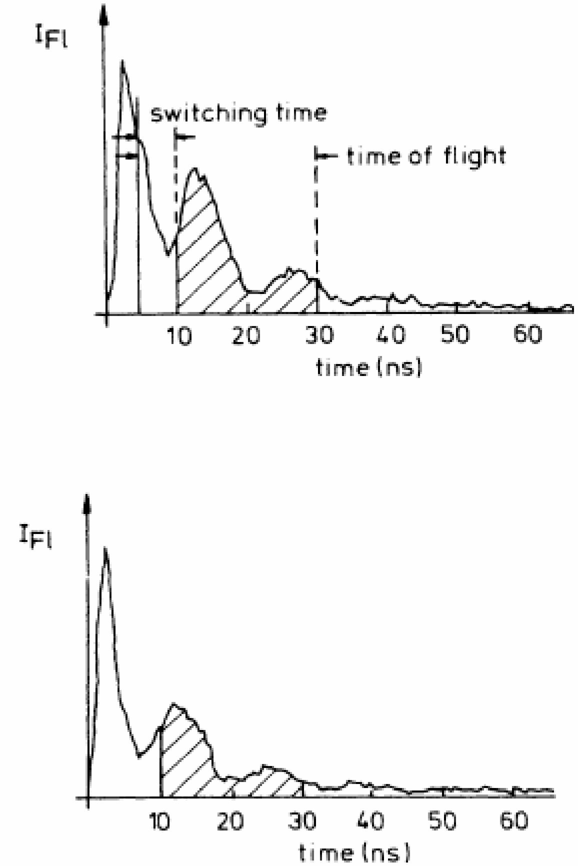


FIG. 10. Comparison of time-resolved fluorescence intensities for the normal (below) and the delayed-choice (above) modes of operation. Only the hatched area is used in evaluation (see text).

Lower time limit: rise time of Pockels cell (4 ns)
Upper time limit: time of flight between atomic beam and detection system

Comparison of normal and delayed choice modes

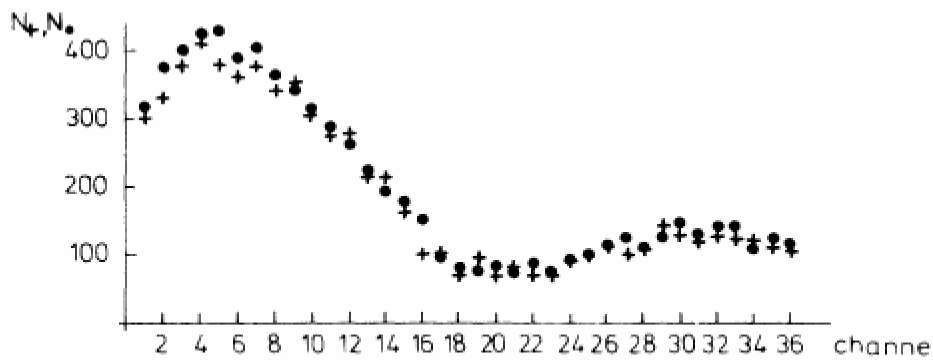


FIG. 11. Quantitative comparison of the hatched area from Fig. 10. Dots represent the delayed-choice and crosses the normal mode of operation. The vertical axis gives the number of counts per channel, and the horizontal corresponds to time with 0.56 ns/channel.

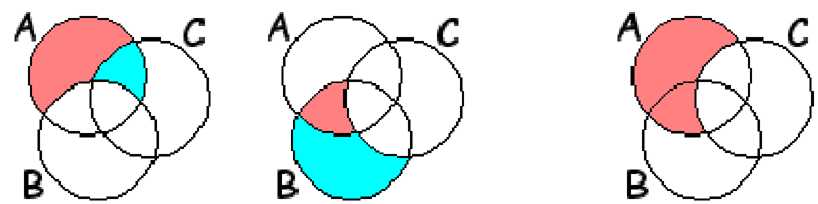
$N_{\bullet}$: Normal Mode

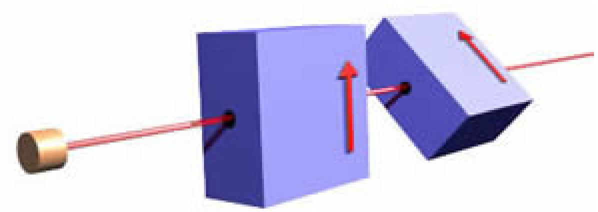
N_{+} : Delayed Choice Mode

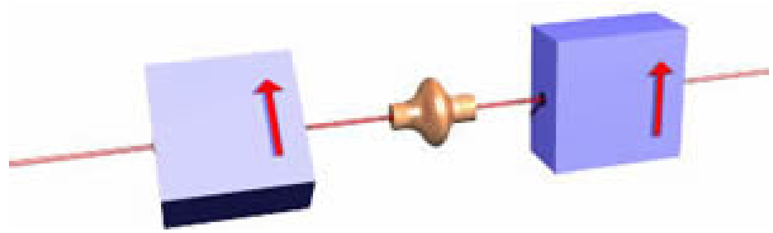
$N_{\bullet}/N_{+} = 1.03 \pm 0.02$

Example of extreme nonlocality : Voodoo Doll

$$E(A,\bar{B}) + E(B,\bar{C}) \geq E(A,\bar{C})$$







Notes on Plagiarism

http://www.aps.org/policy/statements/02_2.cfm#supplementary_guidelines1

http://www.scribd.com/doc/18773744/How-to-Publish-a-Scientific-Comment-in-1-2-3-Easy-Steps

From Trebino's "How to publish a scientific comment in 1 2 3 easy steps"

"Reviewers (of any paper) should themselves be reviewable. Currently, reviewers can say whatever they like, and there is no check on them. Authors should be allowed to single out potentially irresponsible reviewers, such as Reviewer #2 in the above scenario, whose review would be reviewed by another reviewer. Confirmed irresponsible reviewers should then be identified and removed from reviewer databases, which would be shared with other journals. Writing an irresponsible review should be considered a form of scientific misconduct. "

“While removing unethical reviewers would help, improving reviews of ethical ones is also important. Currently there is no compensation of any sort for reviewers and hence no encouragement to do a good job. I believe that reviewers should be paid for their services. People take paid jobs much more seriously than volunteer efforts. Knowing this, social psychologists pay their subjects simply to fill out questionnaires because it yields much higher-quality results. And what could be more important than the accuracy of the archival scientific literature? “

“Require scientific ethics courses in grad school. Problems like those that I encountered are a proverbial ticking time bomb for science. What if those opposed to taking action against global warming were to make the claim that science shouldn’t be believed in this matter because its process is so rife with poor ethics that it can’t be trusted?”

Three Aspect et al. Experiments

VOLUME 47, NUMBER 7 PHYSICAL REVIEW LETTERS 17 AUGUST 1981

Experimental Tests of Realistic Local Theories via Bell's Theorem

Alain Aspect, Philippe Grangier, and Gérard Roger
Institut d'Optique Théorique et Appliquée, Université Paris-Sud, F-91406 Orsay, France
(Received 30 March 1981)

VOLUME 49, NUMBER 2 PHYSICAL REVIEW LETTERS 12 JULY 1982

Experimental Realization of Einstein-Podolsky-Rosen-Bohm *Gedankenexperiment*: A New Violation of Bell's Inequalities

Alain Aspect, Philippe Grangier, and Gérard Roger
Institut d'Optique Théorique et Appliquée, Laboratoire associé au Centre National de la Recherche Scientifique, Université Paris-Sud, F-91406 Orsay, France
(Received 30 December 1981)

VOLUME 49, NUMBER 25 PHYSICAL REVIEW LETTERS 20 DECEMBER 1982

Experimental Test of Bell's Inequalities Using Time-Varying Analyzers

Alain Aspect, Jean Dalibard,[a] and Gérard Roger
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(Received 27 September 1982)

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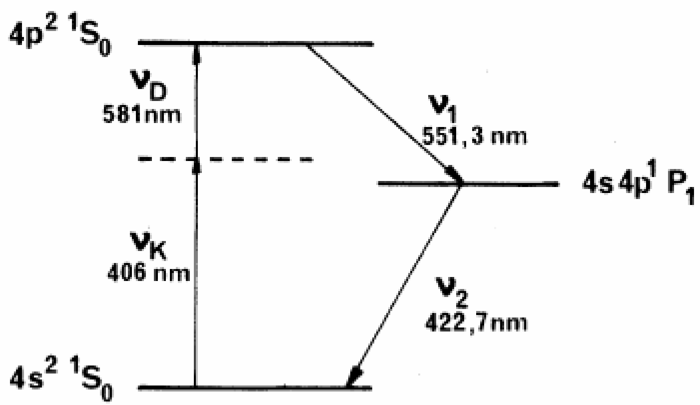

FIG. 1. Relevant levels of calcium. The atoms, selectively pumped to the upper level by the nonlinear absorption of ν_K and ν_L, emits the photons ν_1 and ν_2 correlated in polarization.

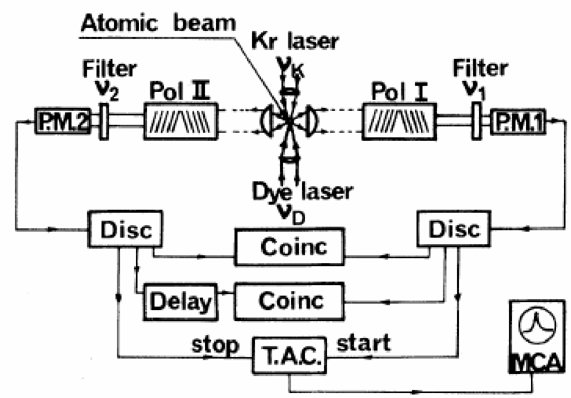

FIG. 2. Schematic diagram of apparatus and electronics. The laser beams are focused onto the atomic beam perpendicular to the figure. Feedback loops from the fluorescence signal control the krypton laser power and the dye-laser wavelength. The output of discriminators feed counters (not shown) and coincidence circuits. The multichannel analyzer (MCA) displays the time-delay spectrum.

Experimental Realization of Einstein-Podolsky-Rosen-Bohm *Gedankenexperiment*: A New Violation of Bell's Inequalities

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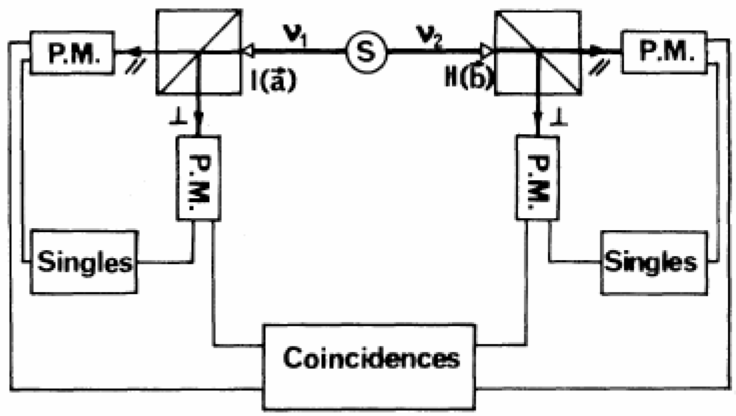

FIG. 2. Experimental setup. Two polarimeters I and II, in orientations $\vec{a}$ and $\vec{b}$, perform true dichotomic measurements of linear polarization on photons ν_1 and ν_2. Each polarimeter is rotatable around the axis of the incident beam. The counting electronics monitors the singles and the coincidences.

$$E(\vec{a},\vec{b}) = P_{++}(\vec{a},\vec{b}) + P_{--}(\vec{a},\vec{b}) - P_{+-}(\vec{a},\vec{b}) - P_{-+}(\vec{a},\vec{b})$$

Local Hidden Variable Theory Gives

$$-2 \leqslant S \leqslant 2,$$

where

$$S = E(\vec{a},\vec{b}) - E(\vec{a},\vec{b}') + E(\vec{a}',\vec{b}) + E(\vec{a}',\vec{b}')$$

Quantum Mechanics Gives

$$E(\vec{a},\vec{b}) = F\,\frac{(T_1^{\parallel} - T_1^{\perp})(T_2^{\parallel} - T_2^{\perp})}{(T_1^{\parallel} + T_1^{\perp})(T_2^{\parallel} + T_2^{\perp})}\cos 2(\vec{a},\vec{b})$$

$$S_{QM} = 2.70 \pm 0.05.$$

For no losses and perfect detection $S_{QM} = \pm 2\sqrt{2}$

Experiment Measures

$$E(\vec{a},\vec{b}) = \frac{R_{++}(\vec{a},\vec{b}) + R_{--}(\vec{a},\vec{b}) - R_{+-}(\vec{a},\vec{b}) - R_{-+}(\vec{a},\vec{b})}{R_{++}(a,b) + R_{--}(a,b) + R_{+-}(a,b) + R_{-+}(a,b)}.$$

$$S_{expt} = 2.697 \pm 0.015.$$

Quantum Mechanics is consistent with measurement

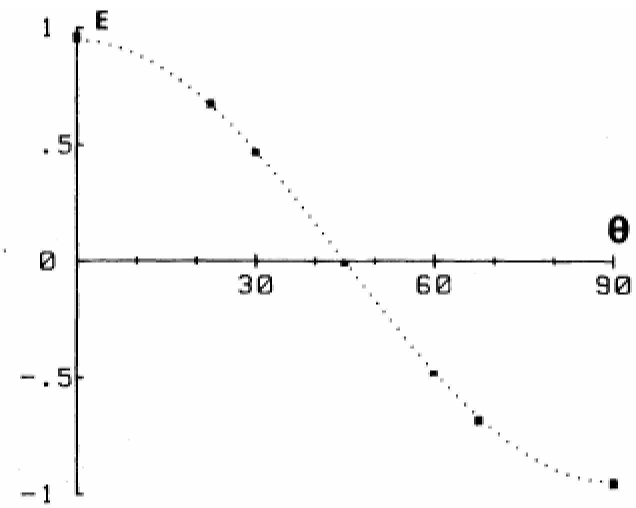

FIG. 3. Correlation of polarizations as a function of the relative angle of the polarimeters. The indicated errors are ±2 standard deviations. The dotted curve is not a fit to the data, but quantum mechanical predictions for the actual experiment. For ideal polarizers, the curve would reach the values ±1.

Experimental Test of Bell's Inequalities Using Time-Varying Analyzers

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Institut d'Optique Théorique et Appliquée, F-91406 Orsay Cédex, France

(Received 27 September 1982)

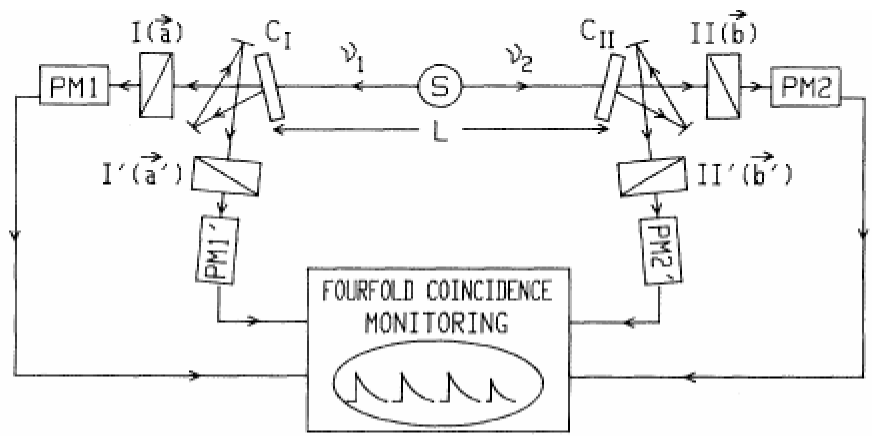


FIG. 2. Timing experiment with optical switches. Each switching device ($C_{\rm I}$, $C_{\rm II}$) is followed by two polarizers in two different orientations. Each combination is equivalent to a polarizer switched fast between two orientations.

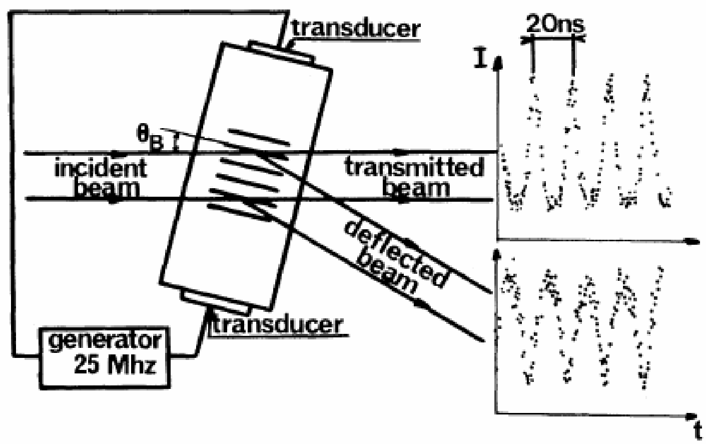


FIG. 3. Optical switch. The incident light is switched at a frequency around 50 MHz by diffraction at the Bragg angle on an ultrasonic standing wave. The intensities of the transmitted and deflected beams as a function of time have been measured with the actual source. The fraction of light directed towards other diffraction orders is negligible.

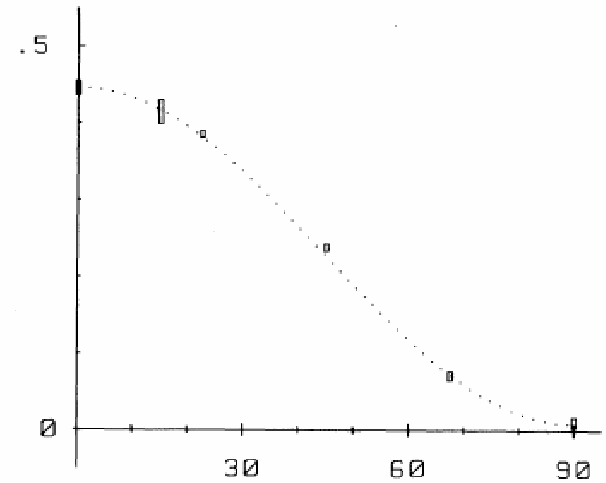
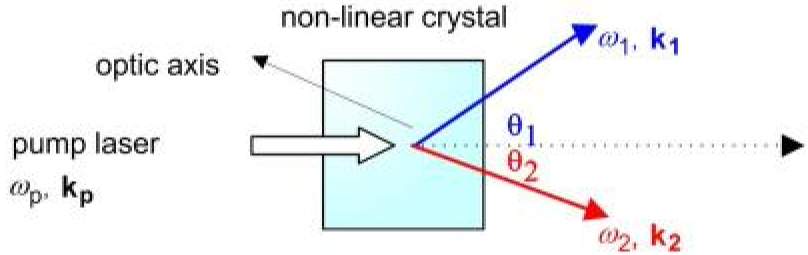
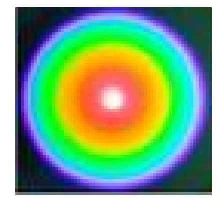


FIG. 4. Average normalized coincidence rate as a function of the relative orientation of the polarizers. Indicated errors are ±1 standard deviation. The dashed curve is not a fit to the data but the predictions by quantum mechanics for the actual experiment.

Type I Spontaneous Down Conversion



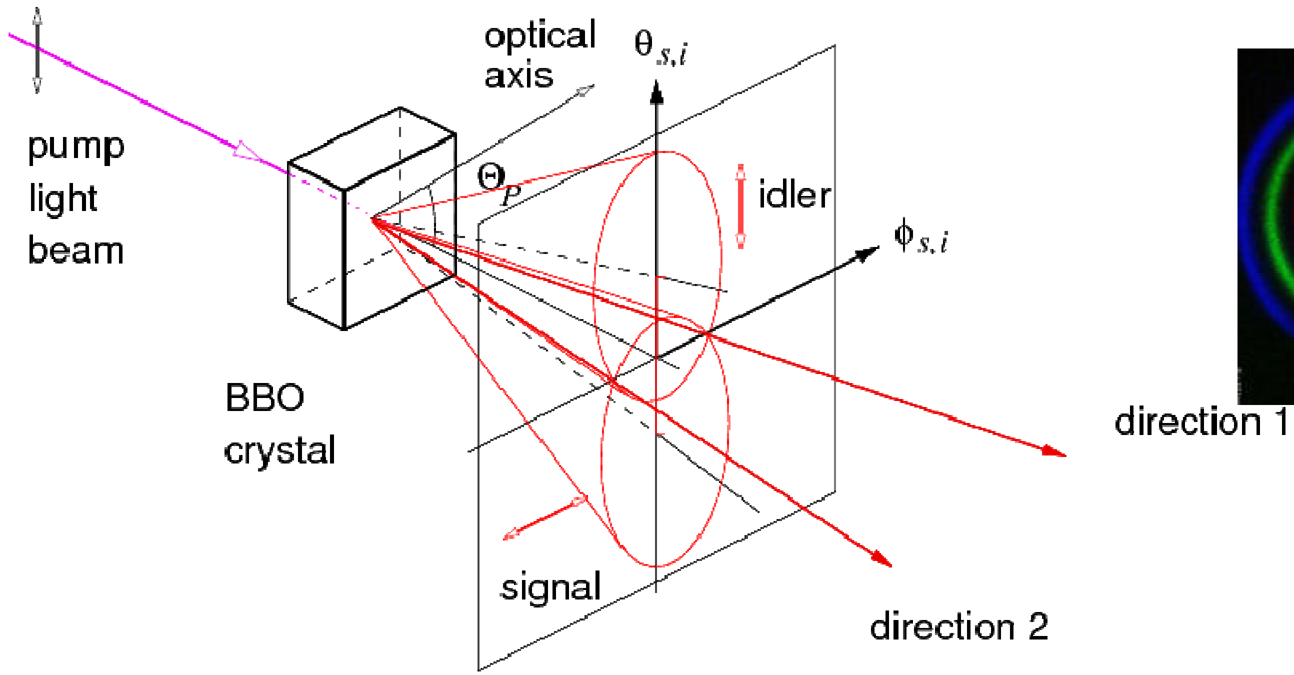


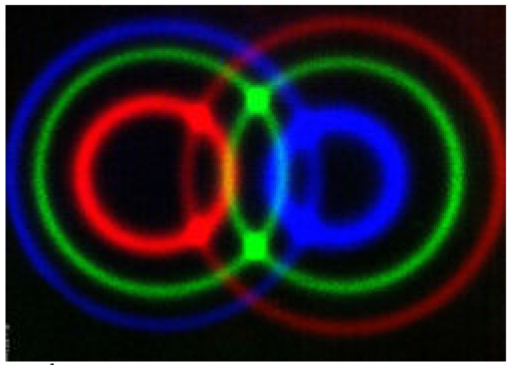
Type I PDC in BBO

$$\omega_1 + \omega_2 = \omega_p \qquad \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_p$$

Signal and idler have same polarization

Type II Spontaneous Down Conversion





Signal and idler have different polarization
Crystal birefringence gives two light cones

Measurement of Subpicosecond Time Intervals between Two Photons by Interference

C. K. Hong, Z. Y. Ou, and L. Mandel
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627
(Received 10 July 1987)

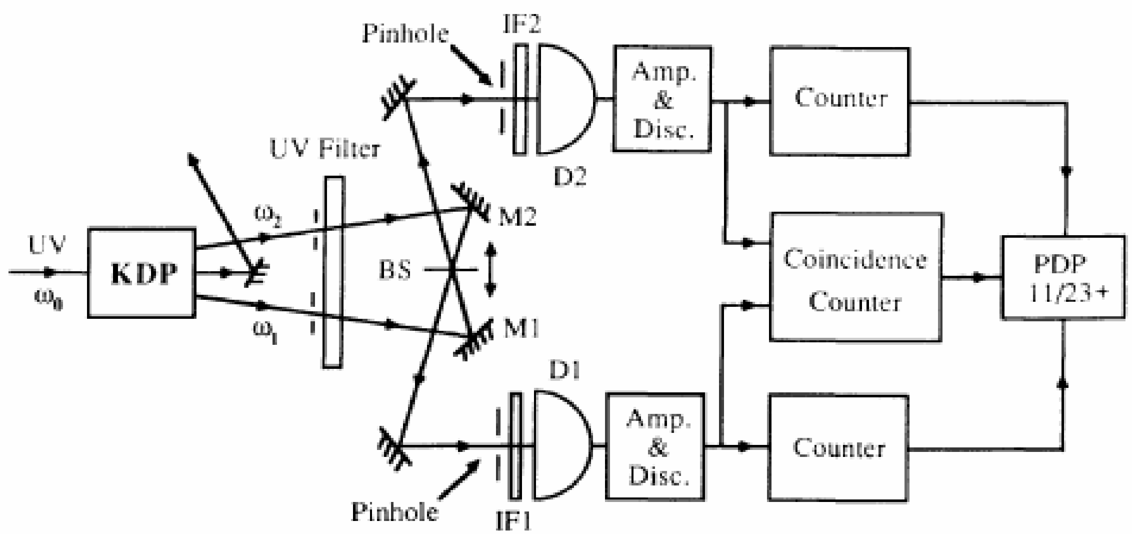


FIG. 1. Outline of the experimental setup.

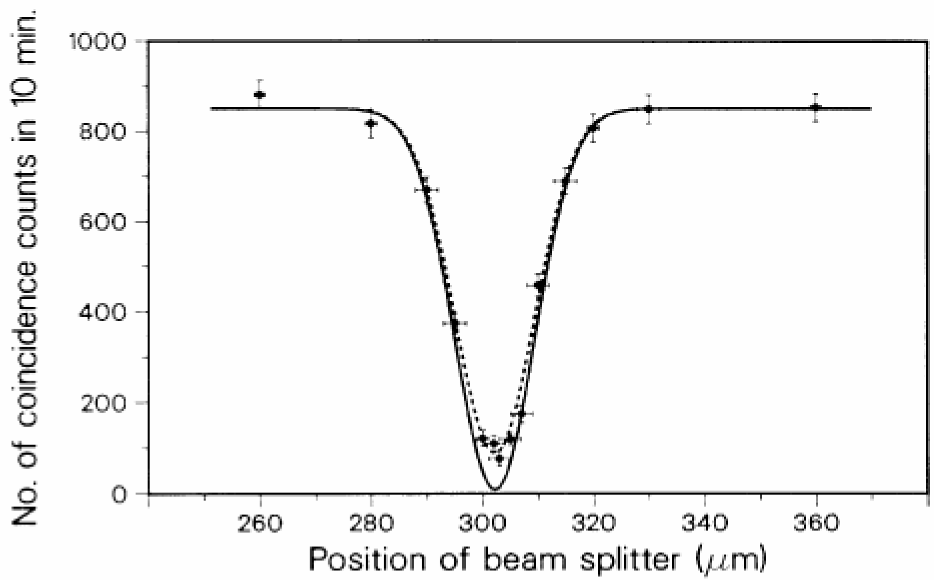


FIG. 2. The measured number of coincidences as a function of beam-splitter displacement $c\,\delta\tau$, superimposed on the solid theoretical curve derived from Eq. (11) with $R/T=0.95$, $\Delta\omega=3\times10^{13}$ rad s^{-1}. For the dashed curve the factor $2RT/(R^2+T^2)$ in Eq. (11) was multiplied by 0.9. The vertical error bars correspond to one standard deviation, whereas horizontal error bars are based on estimates of the measurement accuracy.

Violation of Bell's Inequality and Classical Probability in a Two-Photon Correlation Experiment

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(Received 22 February 1988)

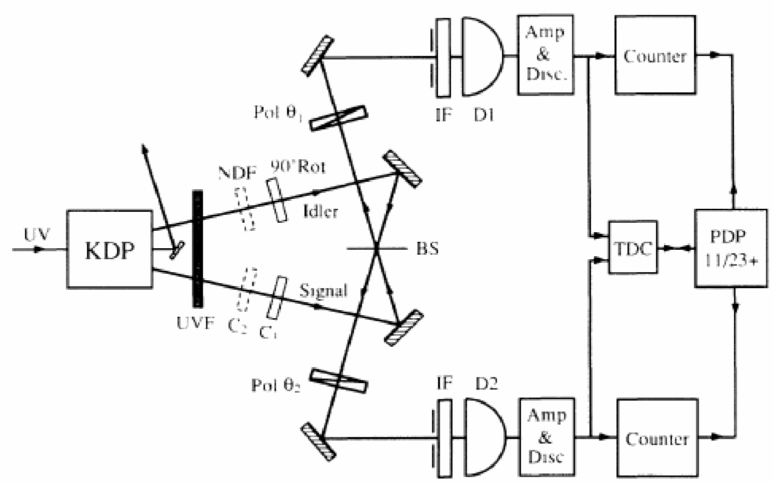


FIG. 1. Outline of the apparatus.

Local Hidden Variable Theory Gives

$$S = P(\theta_1, \theta_2) - P(\theta_1, \theta_2') + P(\theta_1', \theta_2')$$

$$+ P(\theta_1', \theta_2) - P(\theta_1', -) - P(-, \theta_2) \le 0. \qquad (1)$$

$P(\theta_1', -)$ and $P(-, \theta_2)$ are the corresponding probabilities with one or the other linear polarizer removed. We now calculate the joint probability $P(\theta_1, \theta_2)$ first by quantum mechanics and then by classical wave optics.

$$P(\theta_1, \theta_2) = C\{\langle I_s I_i \rangle [(T_x T_y)^{1/2} \cos\theta_1 \sin\theta_2 + (R_x R_y)^{1/2} \sin\theta_1 \cos\theta_2]^2$$
$$+ \langle I_s^2 \rangle R_x T_x \cos^2\theta_1 \cos^2\theta_2 + \langle I_i^2 \rangle R_y T_y \sin^2\theta_1 \sin^2\theta_2\},$$

$$P(\theta_1, \pi/4) = \tfrac{1}{4} C \langle I_s \rangle^2 [1 + \tfrac{1}{2} \sin 2\theta_1]. \qquad (18)$$

Quantum Theory Gives

If the polarizer angles are chosen so that

$$\theta_1 = \pi/8, \quad \theta_2 = \pi/4, \quad \theta_1' = 3\pi/8, \quad \theta_2' = 0, \qquad (6)$$

then one finds with the help of relations (1) and (5) for a 50%:50% beam splitter with $R_x = \frac{1}{2} = T_x$, $R_y = \frac{1}{2} = T_y$, that

$$S = \tfrac{1}{4} K(\sqrt{2} - 1) > 0, \qquad (7)$$

$$P(\theta_1, \theta_2) = K[(T_x T_y)^{1/2} \cos\theta_1 \sin\theta_2 + (R_x R_y)^{1/2} \sin\theta_1 \cos\theta_2]^2,$$

In particular, when $R_x = \frac{1}{2} = T_x$, $R_y = \frac{1}{2} = T_y$,

$$P(\theta_1, \pi/4) = \tfrac{1}{8} K[1 + \sin 2\theta_1]. \qquad (11)$$

Measurement Gives

$$\tilde{S} = \mathcal{R}(22.5°,45°) - \mathcal{R}(22.5°,0°) + \mathcal{R}(67.5°,45°) + \mathcal{R}(67.5°,0°) - \mathcal{R}(67.5°,-) - \mathcal{R}(-,45°)$$
$$= (11.5 \pm 2.0)/\text{min}.$$

TABLE I. Results of coincidence counting measurements for certain combinations of polarizer angles θ_1 and θ_2.

θ_1	θ_2	Coincidence rate per minute $\mathcal{R}$
67.5°	45°	28.3 ± 0.8
22.5°	45°	29.8 ± 0.8
67.5°	0°	29.9 ± 0.8
22.5°	0°	5.6 ± 0.7
67.5°	No polarizer	34.7 ± 0.9
No polarizer	45°	36.2 ± 0.9

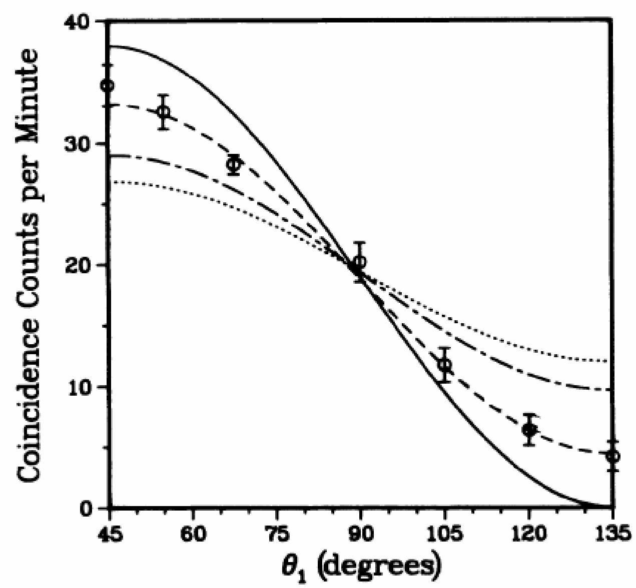


FIG. 2. Measured coincidence counting rate as a function of the polarizer angle θ_1, with θ_2 fixed at 45°. The full curve is the quantum prediction based on Eq. (11) and the dash-dotted curve is the classical prediction based on Eq. (18). The dashed and dotted curves are obtained by multiplication of the sinusoidal functions in Eqs. (11) and (18), respectively, by 0.76 to allow for reduced modulation caused by imperfect alignment.

Measurement with attenuator gives

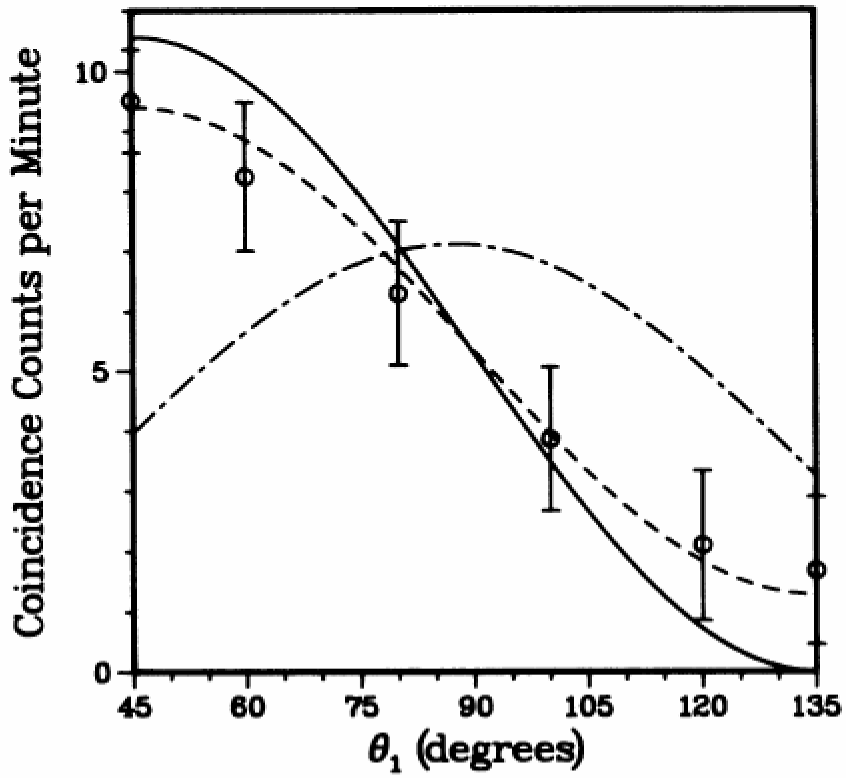


FIG. 3. Measured coincidence counting rate as a function of the polarizer angle θ_1, with θ_2 fixed at 45°, when an 8:1 attenuator is inserted into the idler beam. The full curve is the quantum prediction based on Eq. (11) and the dash-dotted curve is the classical prediction based on Eq. (15). The dashed curve is obtained by multiplication of the sinusoidal function in Eq. (11) by 0.76 to allow for reduced modulation caused by imperfect alignment.

If I was king…..

What would we cover if PHYS953 was two separate, semester long classes?

Nonlinear Optics

- The nonlinear optics of mode-locking in detail
- Resonant Systems
    - Alkali Gas Systems
    - Nonlinear optics and the two level approximation
    - Optical Block equations and Rabi Oscillations
- Semiconductor nonlinear optics
    - Third order effects
    - Two photon absorption
    - Absolute CEP detection using LT-GaAs
- Two photon absorption methods in biological materials
    - TPA imaging
    - Optical coherence tomography (OCT)
- Relativistic nonlinear optics
- Electro optics and acousto optic effects
- More on nonlinear pulses in fibers: solitons, dispersion managed solitons, etc.
- Applications of nonlinear optics for optical communications

What if this class were two, semester long classes …

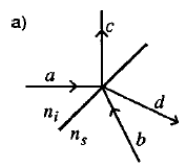
Quantum Optics

- More on quantum coherence functions
- Quadrature squeezing
  - Squeezing in optical fibers interferometers: reducing Shot noise
  - Using squeezing for gravity wave detection (LIGO)
- Quantum description of noise
- Quantum theory of solitons and squeezing
- Interactions of atoms and the quantized electric field
  - Jaynes-Cumming model
  - Dressed states
- Experiments in cavity QED
- Electromagnetic induced transparency (EIT) in resonant systems
- Quantum description of lasers and amplifiers
- More on photon detection methods: coincidence counting
- Quantum information
- Quantum crypography

The purpose of the mini-projects is to offer problems in nonlinear and quantum optics in a format that mimics problem-solving scenarios found in a research environment. Buried in the mini-projects are questions that I do not expect you to know or are the solution easily found in the book. This mini-project consists of problems that should be a review of topics that will be important for our initial introduction to nonlinear optics.

1. Classical treatment of a beam splitter

A beam splitter is used to combine or split electric fields. Consider a lossless beamsplitter reflectivities r_1 and r_2 and transmittivities t_1 and t_2 where these are complex numbers represented by $r_i = |r_i| \exp\left(i\theta_{r_i}\right)$ and $t_i = |t_i| \exp\left(i\theta_{t_i}\right)$ where $i = 1, 2$ (read Hamilton Am J. Phys. 68 (2) 2000). The actual values the reflectivity and transmittivity is given by the Fresnel equations for a specific dielectric material (amplitude and phase shifts). The incident electric fields E_a and E_b onto the beamsplitter are split into waves E_c and E_d using the following scattering matrix



$$\begin{pmatrix} E_c \\ E_d \end{pmatrix} = \begin{pmatrix} r_1 & t_2 \\ t_1 & r_2 \end{pmatrix} \begin{pmatrix} E_a \\ E_b \end{pmatrix}$$

The scattering matrix must by unitary if it satisfies energy conservation. This implies (read Ou and Mandel, Am. J. Phys 57 (1) 1988):

$$|t_1| = |t_2|, |r_1| = |r_2|$$
$$|t_1|^2 + |r_1|^2 = |t_2|^2 + |r_2|^2 = 1$$
$$r_1^* t_2 + t_1^* r_2 = 0$$

Furthermore, energy conservation dictates that the phase satisfy: $\theta_{t_1} - \theta_{r_1} + \theta_{t_2} - \theta_{r_2} = \pm\pi$.

1. Prove that the scattering matrix must be unitary if energy is conserved.
2. From now, consider a symmetric lossless beam splitter such that $r_1 = r_2$ and $t_1 = t_2$. Show that for the symmetric beam splitter, the phase is $\theta_t - \theta_r = \pm\pi/2$. How does this phase shift differ from the phase shift on reflection from glass window of index 1.5 (starting from air)?
3. Let the symmetric beam splitter be a 50/50 beamsplitter. This means that if intensity $I_a \equiv E_a E_a^* = 1$ and $I_b = 0$ (ignoring proper SI units) are incident onto the beamsplitter the output intensities will be $I_c = \frac{1}{2}$ and $I_d = \frac{1}{2}$. Come up with the correct scattering matrix for the symmetric 50/50 beam splitter. Show that your matrix computes the correct intensities I_c and I_d and the correct output phase shift.
4. Now have two input electric fields onto the beamsplitter with a phase shift $\Delta\varphi$, i.e.

$$E_a = \frac{1}{\sqrt{2}} \exp\left(i\omega t\right) \text{ and } E_b = \frac{1}{\sqrt{2}} \exp\left(i\omega t\right) \exp\left(i\Delta\varphi\right)$$

   Is it possible to determine a input phase shift $\Delta\varphi$ so that $I_c = 1$ and $I_d = 0$. If so, what is the correct phase?

2. Spectral response of an exponential decay

Consider the function

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ \exp(-t/\tau) & \text{for } t \geq 0 \end{cases}$$

1. Sketch this function. Find $f(\omega)$, the Fourier transform of the function $f(t)$.
2. Determine the real and imaginary portion of $f(\omega)$. Make comment on they symmetries of these function.
3. Plot the real and imaginary portions $f(\omega)$ as a function of ω. Does the mathematical form of either the real or imaginary portion look familiar? Do either of them have a name?
4. In class, we derived a model of the electric susceptibility starting from a damped driven oscillator. We derived the absorption and the index of refraction as a function of frequency. How do the real and imaginary portion of

the electric susceptibility compare to the real and imaginary portions of the Fourier transform? Can you provide a physical explanation why they look similar?

3. Anisotropic linear media

In this class, we will deal with crystals that are not isotropic and have nonlinear optical properties.

1. For a linear isotropic material, the direction of the flow of energy (given by the direction of $\mathbf{E}\times\mathbf{H}$) is in the same direction as the wavevector $\mathbf{k}$. This is **not** true for an anisotropic material. Furthermore, while $\mathbf{E}$ and $\mathbf{k}$ are mutually perpendicular in a linear isotropic material, they are **not** perpendicular in an anisotropic material. However, $\mathbf{H}$ is **always** perpendicular to $\mathbf{E}$ and $\mathbf{k}$. For an anisotropic material show that a) $\mathbf{E}$ is not in the same direction as $\mathbf{D}$, b) $\mathbf{E}$ and $\mathbf{k}$ are not perpendicular, and c) $\mathbf{H}$ is perpendicular to $\mathbf{E}$ and $\mathbf{k}$.
2. Draw a picture of the vectors, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{H}$, $\mathbf{E}\times\mathbf{H}$, and $\mathbf{k}$ for a plane wave in a anisotropic material. Sketch also the planes of constant phase (wavefronts), which will be perpendicular to $\mathbf{k}$. For a nonlinear process like second harmonic generation, the fact that $\mathbf{E}\times\mathbf{H}$ and $\mathbf{k}$ are not parallel in a crystal lead to *walkoff* between the fundamental and second harmonic electric fields.

4. Ultrashort pulse dispersion in fused silica

A train of ultrashort optical pulses is produced by a mode-locked Ti:sapphire laser. Each pulse has an electric field profile of hyperbolic secant, is transform limited, and each have a duration of 10 fs full-width half maximum (FWHM). The center wavelength is 800 nm. The laser's repetition rate is 100 MHz and the average power from the laser is 100 mW.

1. What is the pulse energy? The peak power?
2. Plot the temporal intensity and phase of the pulse.
3. Plot the spectral intensity and phase of the pulse.

The pulse propagates through a fused-silica window of thickness 1 cm. The dispersion of the fused-silica causes the pulse duration to increase. Consider only quadratic phase distortion (β_2) due to the fused-silica window.

4. Compute and plot final temporal intensity $I_{out}(t)$ and phase $\varphi_{out}(t)$ after propagation through the window.
5. Compute and plot final spectral intensity $I_{out}(\omega)$ and phase $\varphi_{out}(\omega)$ after propagation through the window.
6. Is the "chirp" of the pulse positive or negative?
7. Does the pulse have the same spectral bandwidth before and after the window?
8. What is the final pulse duration (FWHM) after the fused silica window?

Now, ignore the quadratic phase distortion but let the fused silica window have only cubic phase distortion β_3.

9. Compute and plot final temporal intensity $I_{out}(t)$ and phase $\varphi_{out}(t)$ after propagation through the window. Consider only quadratic phase distortion due to the fused-silica window.
10. Compute and plot final spectral intensity $I_{out}(\omega)$ and phase $\varphi_{out}(\omega)$ after propagation through the window.
11. Set β_3=-$\beta_{3,\text{fused silica}}$ and find $I_{out}(t)$. How is the temporal intensity different than in Question 9?

5. One dimensional anharmonic oscillator

The Lorentz model of the atom, which treats a solid as a collection of harmonic oscillators, is a good classical model that describes the linear optical properties of a dielectric material. This model can be extended to nonlinear optical media by adding anharmonic terms to the atomic restoring force. In the lecture we will look closely at this model but let's first solve the differential equations for a one-dimensional anharmonic oscillator.

Consider a one-dimension anharmonic oscillator of mass m under the influence of the nonlinear restoring force:

$$F(x) = -kx - \alpha x^2 - \beta x^3$$

where $\omega_0^2 = k/m$ is the natural frequency sans any anharmonic terms. Let $m = 1$ kg and k=1 N/m.

1. Plot the potential energy for the above force using $\alpha = 0.01$ N/m^2 and $\beta = 0$ N/m^3. Compare it to the potential energy of a simple harmonic oscillator.
2. Plot the potential energy for the above force using $\alpha = 0$ N/m^2 and $\beta = 0.01$ N/m^3.
3. Now, let $\alpha = 0.01$ N/m^2 and $\beta = 0.01$ N/m^3. Numerically solve the 2nd order differential equation of motion, solving for $x(t)$ for t=0 to 20 seconds assuming that $x_0 \equiv x(0) = 0.1$ m and $\dot{x}(0) = 0$ m/s. By plotting $x(t)$ determine the frequency of oscillation ω. How does it compare to ω_0?
4. Find $x(t)$ for $x_0 = 10$ m and $\dot{x}(0) = 0$ m/s, letting $\alpha = 0.01$ N/m^2 and $\beta = 0.01$ N/m^3. What is the new frequency of oscillation and how does it compare to ω_0.

5. An analytic approximation for $\omega(x_0)$, derived using the method of successive approximations (see Landau's *Mechanics*), is given by

$$\omega(x_0) = \omega_0 + \left(\frac{3\beta}{8\omega_0} - \frac{5\alpha^2}{12\omega_0^3}\right)x_0^2$$

Compare your numerical $\omega(x_0)$ to the analytic approximation expression for $x_0 = 0.1 \text{ m to } 10 \text{ m}$.

6. The Fourier series for $x(t)$ is given by the expression

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi}{T}t\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi}{T}t\right)$$

where the period $T = 2\pi/\omega$ and the Fourier series coefficients are given by

$$a_n \equiv \frac{2}{T}\int_0^T x(t)\cos\left(\frac{2n\pi}{T}t\right)dt \text{ and } b_n \equiv \frac{2}{T}\int_0^T x(t)\sin\left(\frac{2n\pi}{T}t\right)dt$$

Numerically solve for $x(t)$ with $x_0 = 10$ m using $\alpha = 0 \text{ N/m}^2$ and $\beta = 0.01 \text{ N/m}^3$. Find the first five Fourier series coefficients a_n (where n=0, … , 4) of the solution $x(t)$. Explain why $b_n = 0$ for all n.

7. Numerically solve for $x(t)$ with $x_0 = 10$ m using $\alpha = 0.01 \text{N/m}^2$ and $\beta = 0 \text{ N/m}^3$. Find the first five Fourier series coefficients a_n of the solution $x(t)$.

8. Compare the odd terms of a_n for the case where $\alpha = 0, \beta \neq 0$. Compare the even terms of a_n for the case where $\alpha \neq 0, \beta = 0$. How does the symmetry of the restoring force predetermine which order harmonics are produced by the nonlinear oscillator?

Informal Survey for PHYS953: Name:_______________________________

Please answer the following questions completely.

What classes in optics and quantum mechanics have you taken? Where have you taken these classes?

Briefly describe your research interests.

Why do want to take this class?

How many hours per week can you spend on homework for this class?

Which of the topics listed in the syllabus seem most interesting to you?

Are there other topics that we should cover in this class?

1. Second Harmonic Generation in Potassium Dihydrogen Phosphate (KDP)

You wish to produce second harmonic generation (SHG) of a continuous wave Nd:YAG laser centered at 1064 nm. To do this you will use a KDP crystal that is cut to produce the second harmonic using Type $I^{(-)}$ (ooe) phase matching. A single laser provides the fundamental fields for the E_1 and E_2 fields at frequency $\omega=\omega_1=\omega_2$ (corresponding to 1064 nm), the second harmonic field will be the E_3 field at $\omega_3=2\omega$. Thus, $E_1=E_2$ and half of the total power is shared among these fields. The laser power is P=0.2 W and beam diameter (assuming a "top-hat" spatial profile) of the laser in the crystal is 10 μm. The length of the crystal is L=1.0 cm

Type $I^{(-)}$ phase matching implies that the fundamental fields ($E_1=E_2$) are both orientated along the ordinary (o) axis and the second harmonic (E_3) is orientated along the extraordinary (e) axis of the negative uniaxial KDP crystal. The ordinary and extraordinary indices of refraction as a function of wavelength for KDP are given by the following Laurent series expressions (where λ is expressed in μm):

$$n_o^2(\lambda) = 2.2576 + \frac{1.7623\lambda^2}{\lambda^2 - 57.898} + \frac{0.0101}{\lambda^2 - 0.0142} \qquad n_e^2(\lambda) = 2.1295 + \frac{0.7580\lambda^2}{\lambda^2 - 127.0535} + \frac{0.0097}{\lambda^2 - 0.0014}$$

$$n_e(\theta,\lambda) = \left[\frac{\sin^2\theta}{n_e^2(\lambda)} + \frac{\cos^2\theta}{n_o^2(\lambda)}\right]^{-1/2} \tag{1}$$

For this process d_{eff} will have the form $d_{eff} = d_{ooe} = d_{36}\sin\theta\sin 2\phi$ where d_{36} =0.39 pm/V for KDP.

1. What is the wavelength of the second harmonic generated field (E_3)?
2. Assuming the phase matching process is Type $I^{(-)}$ (ooe), find the phase matching angle θ_{pm} where $\Delta k=0$.
3. Assuming the phase matching process is Type $I^{(-)}$ (ooe), what are the values of n_1, n_2, n_3 where $n_j = n(\lambda_j)$. Make sure to use the proper index n_j (either $n_e(\theta,\lambda)$ or $n_o(\lambda)$) when computing n_1, n_2, n_3

You try to orientate the crystal for perfect phase matching, however you make an error and set the crystal at angles $\theta = 0.995\theta_{pm}$ and $\varphi = 45°$.

4. Find d_{eff} under these conditions in units of pm/V
5. Compute the phase mismatch Δk under these conditions. Use the proper n_1, n_2, n_3.
6. Determine the initial electric field amplitudes $A_1(z=0)$ and $A_2(0)$ in V/m from the given total input power of P=0.2 W. Remember that irradiance (intensity) has units of W/m^2 and is given by

$$I_j = 2\varepsilon_0 n_j c A_j A_j^* \text{ in units of W/m}^2 \tag{2}$$

7. What is the initial amplitude of $A_3(0)$?
8. Numerically solve the three coupled differential equations derived in class for the amplitudes $A_1(z)$, $A_2(z)$ and $A_3(z)$. Assume the possibility of pump depletion, $\theta = 0.995\theta_{pm}$ and $\varphi = 45°$. Plot $I_3(z)$ and $I_1(z)$ for z=0 to L.
9. Is the fundamental power depleted at $z=L$?
10. Using your numerical solution, determine the output SHG power in Watts at $z=L$=1.0 cm. Is the power at $z=L$ the maximum SHG power produced at any position z in the crystal?
11. Determine the SHG conversion efficiency $\eta_{SHG}(z) \equiv I_3(z)/[I_1(0)+I_2(0)]$ at $z=L$.

Now you set the angle θ for perfect phasematching $\theta = \theta_{pm}$ thus setting the phase mismatch Δk to zero.

12. Solve the coupled differential equations again with $\theta = \theta_{pm}$ and $\Delta k=0$, using the correct values of n_1, n_2, n_3.
13. What SHG power and SHG conversion efficient at $z=L$? Is it larger than before?

We can define a nonlinear length L_{NL} which is a length scale that determines the strength of the nonlinearity. Note that $\eta_{SHG}(z = L_{NL}) \simeq 0.58$ for perfect phase matching. A form for the nonlinear length is given by

$$L_{NL} = \frac{1}{4\pi d_{eff}}\sqrt{\frac{2\varepsilon_0 n_1 n_2 n_3 c \lambda_1^2}{I_1(0)}} \tag{3}$$

14. Compute L_{NL} using Eq. 3. How does it compare to L=1 cm?
15. Solve the coupled differential equations setting $L=4L_{NL}$ for ΔkL=10, ΔkL=1 and ΔkL=0. Plot the conversion efficiencies $\eta_{SHG}(z)$ and $\eta(z) \equiv [I_1(z)+I_2(z)]/[I_1(0)+I_2(0)]$ as a function of z for the three cases. Which case produced the most SHG power and the largest $\eta_{SHG}(L)$?

2. Second Harmonic Generation (SHG) of an Ultrashort Pulse

You wish to build a experiment to accurately measure the pulse duration of ultrashort pulses produced by a Chromium: Forsterite (Cr:F) laser. You do not need to know the details of the experiment, only that it needs second harmonic generated light to work. Thus a nonlinear crystal is needed to produce this SHG: the fundamental pulse (the pulse from the Cr:F laser) will be used to produce a SHG pulse using a nonlinear crystal. Phasematching in this nonlinear crystal will be obtained using angle tuning.

The Cr:F laser center wavelength is at 1275 nm, and it produces an average power of 0.5 W. The second harmonic light will be at 637.5 nm. The beam diameter is 50 μm in the crystal (assuming a "top-hat" spatial profile). A single pulse exits the laser every 10 ns thus the laser has a repetition rate of 100 MHz. An estimate of the pulse duration is roughly 20 fs full width at half maximum (FWHM).

Your job is choose a nonlinear crystal to generate second harmonic light at 637.5 nm from fundamental Cr:F laser pulses at 1275 nm.

1. What is the name of the crystal you would use? Find a common and easily purchased crystal that has the smallest absorption α (in units of 1/m) at the fundamental wavelength of 1275 nm.
2. Where could you buy this crystal? If you cannot find a vendor choose a different crystal. Use the internet.
3. Is the crystal uniaxial or biaxial? If your answer is biaxial, choose a different crystal.
4. Is the crystal negative or positive uniaxial?
5. What type of phase matching would you use? Type I or Type II? ooe or oeo or something else?
6. Given your choice of crystal and phase matching type, what would be the phase matching angle θ_{PM}?
7. What would be d_{eff} for your crystal in pm/V?

As discussed in class, each crystal has a finite phase matching bandwidth for pulsed SHG depending on the thickness of the crystal. This means that a given crystal cannot simultaneously phase match all spectral components of the pulse. For pulsed SHG you wish to have the longest crystal possible in order to get the most SHG power *but not at the cost of severely filtering the SHG spectrum*!

8. Given that the pulse duration approximately 20 fs FWHM, estimate the transform-limited spectral FWHM bandwidth of the fundamental pulse spectrum $I(\lambda)$ in nanometers?
9. Using the above pulse as the fundamental, what is the SHG spectral bandwidth (FWHM) in nanometers. The SHG spectrum $I_{SHG}(\omega)$ is proportional to the autoconvolution of the fundamental spectrum:

$$I_{SHG}(\omega) \propto \int I(\eta-\omega)I(\eta)d\eta \tag{4}$$

10. Make an educated guess for the optimal crystal thickness L needed for proper phase matching. Make your choice based on the longest crystal that does not severely filter the SHG spectrum. (Hint: the thickness should be between 0.001 and 1 mm). Remember, the spectral filter function $H(\omega)$ due to the phase mismatch is given by

$$H(\lambda) = \left(\frac{\sin\left(\Delta k(\theta,\lambda)L\right)}{\Delta k(\theta,\lambda)L}\right)^2 \text{ where } L \text{ is the crystal thickness.} \tag{5}$$

11. Determine the spectral width of the filtered SHG spectrum $H(\lambda)I_{SHG}(\lambda)$ in nanometers.

1. Soliton Propagation in a Single-Mode Optical Fiber

An optical soliton forms due to the interplay of anomalous group velocity dispersion (GVD) and self-phase modulation (SPM) in an optical fiber. For an ultrashort pulse injected into the fiber, GVD causes the pulse temporal envelope to broaden while SPM causes the spectral width to increase. A soliton forms when the two effects are balanced, which happens when the total amount of dispersion and nonlinearity is just right. We can define the nonlinear length (L_{NL}) and dispersion length (L_D) in the fiber in terms of the peak power P_0, the pulse duration FWHM Δt, group velocity dispersion β_2, and the effective nonlinearity γ by

$$L_{NL} = \frac{1}{\gamma P_0} \text{ and } L_D = \frac{T_0^2}{|\beta_2|} \quad \text{where } \mathrm{T}_0 = \frac{\Delta t}{2\ln(1+\sqrt{2})} \text{ and } \gamma = \frac{n_2 \omega}{c\pi r^2}.$$

A first order soliton occurs when $L_{NL}/L_D = 1$.

A hyperbolic secant pulse with center wavelength λ_0=1550 nm and pulse duration Δt=100 fs FWHM propagates through a length L_D of a single-mode optical fiber. The optical fiber has a core radius of r=4.1 μm and an index difference Δn=0.008 between the core and cladding index or refraction. The value for the nonlinear index of refraction is n_2=3 10^{-20} $\mathrm{m^2/W}$. The fiber core consists of germanium-doped fused silica whose index of refraction is given by the three term Sellmeier equation (valid for wavelength in μm):

$$n^2(\lambda) = 1 + \sum_{i=1}^{3} \frac{B_i \lambda^2}{\lambda^2 - C_i^2} \text{ where } \begin{array}{l} \mathrm{B}_1 = 0.711040,\ \mathrm{B}_2 = 0.451885,\ \mathrm{B}_3 = 0.704048 \\ \mathrm{C}_1 = 0.064270,\ \mathrm{C}_2 = 0.129408,\ \mathrm{C}_3 = 9.45478 \end{array} \tag{1}$$

(The fiber cladding consists of fused silica, which has a smaller index of refraction than germanium-doped fused silica. We will not need to use its Sellmeier equation for the problem.) The wave guiding due to the fiber geometry changes the total dispersion that the pulse experiences. The propagation constant $\beta(\omega)$ for the fiber, which represents the z component of the wavevector $\mathbf{k}(\omega)$, is given by

$$\beta(\omega) = \frac{n(\omega)\omega}{c}\sqrt{1 + 2\Delta n b(\omega)} \ .$$

The propagation constant is expressed where Δn is the index difference between core and cladding, r is the core radius, and $b(\omega)$ is the normalized mode propagation constant due to the fiber geometry given in terms of the normalized frequency $V(\omega)$. An approximate form for $b(\omega)$ is given by

$$b(\omega) = 1 - \left(\frac{1+\sqrt{2}}{1+\sqrt[4]{4+V(\omega)}} \right)^2 \text{ where } V(\omega) \equiv \frac{r\omega}{c} n(\omega)\sqrt{2\Delta n}.$$

1. Show that the value of the second order propagation constant β_2 (i.e. group velocity dispersion) at λ_0=1550 nm is -0.0000180 $\mathrm{fs^2/nm}$. β_2 can be determined from

$$\beta_2(\omega_0) = \left. \frac{d^2\beta(\omega)}{d\omega^2} \right|_{\omega=\omega_0}$$

2. What is L_D? Determine the peak power P_0 for where $L_{NL}/L_D = 1$.
3. Consider the pulse propagating through L_D of fiber experiencing only group velocity dispersion (no nonlinear effects). Plot the temporal chirp $\omega_{GVD}(t) = \omega_0 - \partial_t \varphi_{GVD}(t)$ of the pulse due only to GVD after L_D.
4. Consider the pulse propagating through L_D of fiber experiencing only self-phase modulation (no dispersion). Plot the temporal chirp $\omega_{SPM}(t) = \omega_0 - \partial_t \varphi_{SPM}(t)$ of the pulse due only SPM after L_D.
5. By comparing $\omega_{GVD}(t)$ and $\omega_{SPM}(t)$, explain how the interaction of SPM and GVD leads to soliton formation.

2. Partially Degenerate Four Wave Mixing in a Single-Mode Optical Fiber

We wish to determine the pump, signal, and idler frequencies for partially degenerate four-wave mixing (FWM) in an optical fiber. Partially degenerate FWM is described by

$$2\omega_p - \omega_i - \omega_s = 0$$

where we use the terms pump (p), signal (s), and idler (i) as for difference frequency generation. Here we define $\omega_i > \omega_s$.

A strong continuous wave laser serves as the pump at ω_p of power P_0=0.5 MW. The pump is injected into an optical fiber with a germanium-doped fused silica core. The fiber has a core radius 4.1 μm and index difference Δn=0.008 between the core and cladding indices (as in Problem 1).

1. Determine the signal and idler wavelengths produced through partial degenerate four wave mixing for pump wavelengths from λ_p=900 to 2000 nm. To determine this for a given pump frequency ω_p you will need to find the signal ω_s and idler ω_i frequencies that satisfies both energy conservation and phase matching:

$$2\omega_p - \omega_i - \omega_s = 0$$
$$\Delta k = \Delta k_m + \Delta k_w + \Delta k_{NL} = 0$$

where

$$\Delta k_m = c^{-1}\left(n(\omega_s)\omega_s + n(\omega_i)\omega_i - 2n(\omega_p)\omega_p\right)$$
$$\Delta k_w = \Delta n c^{-1}\left(b(\omega_s)\omega_s + b(\omega_i)\omega_i - 2b(\omega_p)\omega_p\right)$$
$$\Delta k_{NL} = 2\gamma P_0$$

   The phase mismatch Δk has contributions due to material dispersion (Δk_m), waveguide dispersion (Δk_w), and the fiber nonlinearity (Δk_{NL}). To determine the phase mismatch, you will need to use the Sellmeier equation and $b(\omega)$ from the previous problem.
2. Plot λ_s and λ_i versus λ_p.

The zero group velocity dispersion wavelength λ_{zGVD} is ~ 1345 nm for this fiber, which is determined using $\beta(\omega)$. Notice that the behavior of λ_s versus λ_p and λ_i versus λ_p is different on the long and short wavelength sides of λ_{zGVD}

The purpose of this Mini-project is to expose you to groundbreaking, highly cited paper in nonlinear optics, and to see how this significant paper lead to new research and discoveries. There will be two parts to this Mini-project: Writing the Summary and Reviewing the Summary

1. Writing the Summary

You will need to write a short summary of two journal papers. This first paper you will have chosen (by random ballot) from the list below. You will need to pick the second paper, however the second paper must be a relatively recent paper that cites the first paper in its reference section. Example:

Paper 1: Franken, P.A. *et al*, "Generation of Optical Harmonics", Phys Rev Lett, Vol. 7, 4, 1961
Time Cited: 564

Paper 2 which references Paper 1: Deng L, Hagley EW, Wen J, *et al*., "Four-wave mixing with matter waves", Nature, Vol. 398, 6724 Pages: 218-220 Published: MAR 18 1999
Times Cited: 260

When writing this summary, your target audience will be your fellow classmates and not your instructor. The Summary will consist of a one or two page summary of Paper 1 and a one or two page summary of Paper 2. In the first summary, you must discuss the major results of Paper 1 and the importance of the paper. In the second summary you must discuss the major results Paper 2 and how the results of Paper 1 contributed to these results. The format of the paper should be as follows:

Summary Format
Page 1: Title page with your name
Pages 2-3: Summary of Paper 1 (summary may be one page only)
Pages 4-5: Summary of Paper 2 (summary may be one page only)
The Summary needs to be typed and turned in electronically as a PDF file to me at washburn@phys.ksu.edu. Use 10 or 12 pt font, Times New Roman Font, 1 inch margins. Only put your name on page 1.

Please pay attention to the Review Criteria before writing your Summary. See below.

2. Reviewing the Summary

For Part Two you will evaluate your classmate's summary in a similar fashion as for the review of a journal. The manuscript will be given to you in an anonymous fashion and you must complete your review in an anonymous fashion. You will judge the Summary using the criteria below.

Review Criteria
How well does the Summary cover the important results of Paper 1?
How well does the Summary cover the important results of Paper 2?
How well does the Summary show a connection (or show a lack of a connection) between the results of Paper 1 to the result of Paper 2?
Are there any significant formatting, spelling or grammatical errors?

Then make a final decision on the Summary:
____ Summary is excellent, accept as is with no revisions
____ Summary needs minor revision
____ Summary needs major revision
____ Summary is poor, reject

Complete your review by writing a brief statement answering the following questions and then make a final decision on the Summary. Email the review to me. To be a responsible referee, you will need to read (or at least skim) the papers that the Summary is reviewing. Do not put your name on the review since it will go back to the author. Grades will be given based on the result of the Summary Review and on the quality of your review.

3. Due dates

Summary Due: 11/11/10
Review Due: 11/18/10

Paper List

1. Probing single molecules and single nanoparticles by surface-enhanced Raman scattering
Author(s): Nie SM, Emery SR
Source: SCIENCE Volume: 275 Issue: 5303 Pages: 1102-1106 Published: FEB 21 1997

2. PLASMA PERSPECTIVE ON STRONG-FIELD MULTIPHOTON IONIZATION
Author(s): CORKUM PB
Source: PHYSICAL REVIEW LETTERS Volume: 71 Issue: 13 Pages: 1994-1997 Published: SEP 27 1993

3. SUPERCONTINUUM GENERATION IN GASES
Author(s): CORKUM PB, ROLLAND C, SRINIVASANRAO T
Source: PHYSICAL REVIEW LETTERS Volume: 57 Issue: 18 Pages: 2268-2271 Published: NOV 3 1986

4. SURFACE-PROPERTIES PROBED BY 2ND-HARMONIC AND SUM-FREQUENCY GENERATION
Author(s): SHEN YR
Source: NATURE Volume: 337 Issue: 6207 Pages: 519-525 Published: FEB 9 1989

5. OBSERVATION OF SELF-PHASE MODULATION AND SMALL-SCALE FILAMENTS IN CRYSTALS AND GLASSES
Author(s): ALFANO RR, SHAPIRO SL
Source: PHYSICAL REVIEW LETTERS Volume: 24 Issue: 11 Pages: 592-& Published: 1970

6. OPTICAL INVESTIGATION OF BLOCH OSCILLATIONS IN A SEMICONDUCTOR SUPERLATTICE
Author(s): FELDMANN J, LEO K, SHAH J, et al.
Source: PHYSICAL REVIEW B Volume: 46 Issue: 11 Pages: 7252-7255 Published: SEP 15 1992

7. QUASI-PHASE-MATCHED OPTICAL PARAMETRIC OSCILLATORS IN BULK PERIODICALLY POLED LINBO3
Author(s): MYERS LE, ECKARDT RC, FEJER MM, et al.
Source: JOURNAL OF THE OPTICAL SOCIETY OF AMERICA B-OPTICAL PHYSICS Volume: 12 Issue: 11 Pages: 2102-2116 Published: NOV 1995

8. Phase-matched generation of coherent soft X-rays
Author(s): Rundquist A, Durfee CG, Chang ZH, et al.
Source: SCIENCE Volume: 280 Issue: 5368 Pages: 1412-1415 Published: MAY 29 1998

9. DISCRETE SELF-FOCUSING IN NONLINEAR ARRAYS OF COUPLED WAVE-GUIDES
Author(s): CHRISTODOULIDES DN, JOSEPH RI
Source: OPTICS LETTERS Volume: 13 Issue: 9 Pages: 794-796 Published: SEP 1988

10. MODE-LOCKING OF TI-AL2O3 LASERS AND SELF-FOCUSING - A GAUSSIAN APPROXIMATION
Author(s): SALIN F, SQUIER J, PICHE M
Source: OPTICS LETTERS Volume: 16 Issue: 21 Pages: 1674-1676 Published: NOV 1 1991

11. EXPERIMENTAL-OBSERVATION OF PICOSECOND PULSE NARROWING AND SOLITONS IN OPTICAL FIBERS
Author(s): MOLLENAUER LF, STOLEN RH, GORDON JP
Source: PHYSICAL REVIEW LETTERS Volume: 45 Issue: 13 Pages: 1095-1098 Published: 1980

12. 2-PHOTON EXCITATION IN CAF2 - EU2+
Author(s): KAISER W, GARRETT CGB
Source: PHYSICAL REVIEW LETTERS Volume: 7 Issue: 6 Pages: 229-& Published: 1961

13. HIGH-ORDER HARMONIC-GENERATION FROM ATOMS AND IONS IN THE HIGH-INTENSITY REGIME
Author(s): KRAUSE JL, SCHAFER KJ, KULANDER KC
Source: PHYSICAL REVIEW LETTERS Volume: 68 Issue: 24 Pages: 3535-3538 Published: JUN 15 1992

14. Compression of high-energy laser pulses below 5 fs
Author(s): Nisoli M, DeSilvestri S, Svelto O, et al.
Source: OPTICS LETTERS Volume: 22 Issue: 8 Pages: 522-524 Published: APR 15 1997

Mini-Project 5: Nonlinear Optical Processes in Quantum Optics
KSU PHYS953, NQO Due 12/7/07

Nonlinear Processes for the Generation of Quadrature Squeezed Light

This project investigates the use of a nonlinear optical process for the generation of nonclassical light. Do the first three questions for full credit. The other questions will be extra credit.

Consider the superposition state $\left|\psi\right\rangle = a\left|0\right\rangle + b\left|1\right\rangle$ where *a* and *b* are complex and satisfy the relationship $\left|a\right|^2 + \left|b\right|^2 = 1$.

1. Calculate the variances of the quadrature operators $\hat{X}_1$ and $\hat{X}_2$ (see Eq. 2.52 and Eq. 2.53). The variance of an operator is given by
$$\left\langle (\Delta\hat{X}_i)^2 \right\rangle = \left\langle \hat{X}_i^{\,2} \right\rangle - \left\langle \hat{X}_i \right\rangle^2$$
Remember that $\hat{X}_1$ is called the in-phase component and $\hat{X}_2$ is the in-quadrature component.
2. Show that there exits values of the parameters *a* and *b* for which either of the quadrature variances become *less* than for a vacuum state. Hint: let $b = \sqrt{1-\left|a\right|}e^{i\varphi}$ and $a^2 = \left|a\right|^2$ (this is done without the loss of generality). Plot the variance as a function of $\left|a\right|^2$ for different φ.
3. For the cases where the quadrature variances become less than for a vacuum state, check to see if the uncertainty principle is violated.
4. Verify that the quantum fluctuations of the field quadrature operators are the same for the vacuum when the field is in coherent state (*i.e.* verify Eq. 3.16).

The above result illustrate a case where the expectation value of the quadrature operator becomes less than a vacuum state, even though the quadrature operators must satisfy the minimum uncertainty relationship. Squeezing is the process when one canonical (conjugate) variable has a variance less than the vacuum state but the other canonical variable will have a larger variation in order to satisfy the uncertainty principle. The quadrature operators $\hat{X}_1$ and $\hat{X}_2$ are canonical variables and do not commute, thus they have an uncertainty relationship given by Eq. 2.56. Quadrature squeezing occurs when

$$\left\langle (\Delta\hat{X}_1)^2 \right\rangle < \frac{1}{4} \text{ or } \left\langle (\Delta\hat{X}_2)^2 \right\rangle < \frac{1}{4}.$$

We can plot a phase space diagram of a normal and squeezed state (see below and on page 154). The area in phase space remains must constant to maintain the minimum uncertainty relationship. However, we can "squeeze" the circle into an ellipse while keeping the area constant (like squeezing the Charmin done in class).

Not Squeezed Squeezed

Quadrature squeezed light can be produced by the second order nonlinear effect known as **degenerate parametric down-conversion**. This process involves two signal (*s*) waves produced by one pump wave (*p*), *i.e.* $\omega_s + \omega_s - \omega_p = 0$. This process is a degenerate form of difference frequency generation with the signal wave equal to the idler wave. The Hamiltonian for this degenerate parametric down-conversion is given by

$$\hat{H} = \frac{\hbar}{2}\left(\chi^{(2)}\hat{a}_s^\dagger \hat{a}_p \hat{a}_s^\dagger + \chi^{*(2)}\hat{a}_s \hat{a}_p^\dagger \hat{a}_s\right)$$

5. Use the Heisenberg equations of motion (Eq. 2.19) to derived two coupled first order differential equation for $\frac{d\hat{a}_s}{dt}$ and $\frac{d\hat{a}_s^\dagger}{dt}$.
6. What are the solutions to these differential equations if we assume a non-depleted pump? Integrate from time 0 to *T*.
7. Show that the quadrature operators have the solution

$$\begin{bmatrix} \hat{X}_1(T) \\ \hat{X}_2(T) \end{bmatrix} = \begin{bmatrix} e^{-\delta T} & 0 \\ 0 & e^{\delta T} \end{bmatrix} \begin{bmatrix} \hat{X}_1(0) \\ \hat{X}_2(0) \end{bmatrix} \text{ where } \delta \equiv i\chi^{(2)}\hat{a}_p$$

8. Consider a coherent state $|\alpha\rangle$ Show the mean square fluctuations (variance) result in

$$\begin{bmatrix} \langle\alpha|(\hat{X}_1(T))^2|\alpha\rangle - \langle\alpha|\hat{X}_1(T)|\alpha\rangle^2 \\ \langle\alpha|(\hat{X}_2(T))^2|\alpha\rangle - \langle\alpha|\hat{X}_2(T)|\alpha\rangle^2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} e^{-2\delta T} \\ e^{+2\delta T} \end{bmatrix}$$

This result states that the mean square fluctuations of the in-phase component $\hat{X}_1$ is exponentially smaller by $e^{-2\delta T}$ and mean square fluctuations of the in-quadrature component $\hat{X}_2$ are exponentially larger by $e^{2\delta T}$. So the above picture depicts the squeezing performed by the nonlinear process. The bizarre thing about this analysis is that it is also true for a **vacuum state**. One can have vacuum and squeezed vacuum.

9. A third order nonlinear can also be used to produce squeezed light instead. Name a third order nonlinear process that will give rise to squeezed light (Hint: we discussed a third order process that "looks" like difference frequency generation. What was that process?).

Squeezed light and squeezed vacuum has many important applications, specifically for light detection at levels below the quantum noise (*i.e.* shot noise) level. See Henry *et al*, Amer. J. Phys. Vol 56 (4) p. 318 (1988) for more information.

Final Project: Research Paper **KSU PHYS953, NQO**

The final project will be an investigation of a topic or problem in the areas of nonlinear and quantum optics, that will involve a literature search and some original work. The purpose of the paper is to pose a question about your chosen topic and try to answer that question. Please keep in mind you do not to answer the question you have posed. Your paper will be evaluated on a complete literature search, a good discussion on the question, and a well-executed attempt in answering the question.

The final project will consist of three parts:

Part 1: Abstract and bibliography	Due November 17, 2010
Part 2: Six to eight page paper	Due December 2, 2010
Part 3: 10 minute presentation	Starting December 9, 2010

1. Part 1

For Part 1, you will need to provide a draft title, abstract, and bibliography. In your abstract, you will need to state a draft question that the paper will try to answer. I will look over your topic and approve it so you can do the rest of the project. On the back is a short list of research topics. Feel free to pick any topic in quantum and nonlinear optics you wish.

2. Part 2

Part 2 is a six to eight page paper on your topic. The paper should include:

The title and abstract
An introduction to the topic
A discussion of prior work
A section stating the question your paper wishes to answer
A section of your own work investigating the question
A summary comparing your conclusions with respect to prior work
A list of references

Paper Format: 10 or 12 pt font, Times New Roman Font, 1 inch margins

3. Part 3

For Part 3 you will need to give a 10 minute talk about your topic, with 3 minutes for questions. The talk will be given in class at the times listed below. For your talk, **you will only have the white/black board at your disposal**; do not prepare a computer-based talk. You are encouraged to provide handouts to the class for your talk. Also, you are encouraged to practice your talk using a white/black board before you give your talk. Your talk will be evaluated on the clarity of presentation as well as the use of time (in other words, do not go over time!).

List of Sample Topics and Questions

- Self focusing in a rare gas with estimations of focusing versus pulse intensity
- Explaining how to describe the Compton effect semi-classically
- Quantum mechanically description of stimulated Raman scattering
- Explaining how to describe the photoelectric effect semi-classically
- Discuss how quantum entanglement can be used for secure communications
- Discuss the theory and operation of an optical parametric chirped-pulse amplification, OPCPA
- Investigate the thermodynamics of laser mode-locking and how nonlinear effects are involved
- Self similar behavior in optical fiber and the third order nonlinearity
- Quantum optics in cold atoms: how does one generate entangled states?
- How to generate entangled light using second order nonlinear processes in crystals.
- What is the quantum eraser and how can one demonstrate this?
- How is two-photon absorption used for biological imaging?
- Discuss the role of electrons and holes in the nonlinear optics of III-V semiconductors
- The quantum mechanics of electromagnetic noise: Shot and thermal noise
- The quantum description of heterodyne and homodyne optical detection
- The quantum theory of a laser: the master equation.
- The role of higher order nonlinear effects in laser mode-locking
- Quantum optical description of electromagnetically induced transparency in atomic systems
- Quadrature squeezing in optical fibers
- Applications of squeezed noise in gravity wave detection
- Nonlinear spectroscopy of gases: theory of saturated absorption
- Entanglement and quantum teleportation of states

Ask me if you want more topics.