Homework 4, Kansas State University PHYS522, Mechanics Due: Tuesday, February 13, 2007 at beginning of class

1. Problem 3-40
2. Runge-Kutta solution for a damped harmonic oscillator

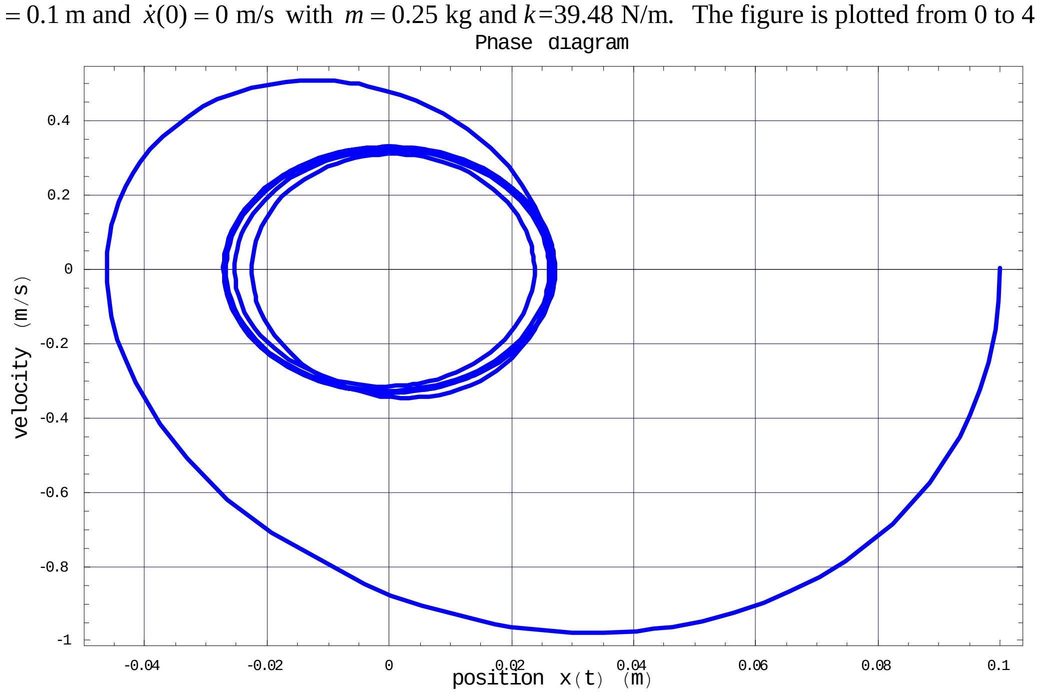
   Let us numerically solve the differential equation of a damped oscillator using the 4th order Runge-Kutta method. We wish to solve the equation for $x(t)$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

   where $\beta = 0.1\ \text{s}^{-1}$, $m = 0.1$ kg, k=1 N/m, $x(0) = 0.1$ m, and $\dot{x}(0) = -0.5$ m/s .

   a) Given the above parameters, in words describe the oscillatory motion.
   b) Write the above 2nd order differential equation as two coupled first order differential equations in terms of x_1 and x_2 where $x_1 \equiv x$, and $x_2 \equiv \dot{x}$.
   c) Using your favorite software (*Excel*, *Matlab*, *Mathematica*, *etc*) write a Runge-Kutta routine to solve these two coupled differential equations for $x(t)$. Find the solution over the time from 0 to 20 seconds. (To review Runge-Kutta go back to Tutorial 4 on Kstate online).
   d) Plot your numerical solution $x(t)$ from 0 to 20 seconds.
   e) Plot the analytic result of $x(t)$ we derived in class from 0 to 20 seconds.
   f) To compare your numerical result to the analytic result, compute the percent different between the two solutions as a function of time. What is the error?
   g) What number of steps did you use in b)? If your error is larger than 1%, increase the number of steps in b) and repeat. How many steps are needed to get better than a 1% error?

3. Consider the phase space diagram of a forced damped harmonic oscillator below. The particle starts at $x(0) = 0.1$ m and $\dot{x}(0) = 0$ m/s with $m = 0.25$ kg and k=39.48 N/m. The figure is plotted from 0 to 4 seconds.



   a) Sketch $x(t)$ and $v(t)$ by hand over four cycles.
   b) Determine the initial energy of the oscillator.
   c) What is the lowest energy of the oscillator
   d) What is the steady state amplitude? What is the steady state energy?
   e) Estimate the number cycles does it take for the oscillator to reach steady state? (more questions below)
   f) Estimate β from the figure.
   g) If you ignore the driving force, how much energy is lost per cycle?
   h) The energy of the oscillator is not zero after 4 seconds even with the damping force. Where does the energy come from to keep this oscillation going? How much energy per cycle is acquired from this source?

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4. Consider the LRC circuit of Figure 3-18 under a sinusoidal driving force of $V_0\sin(\omega_d\, t)$ where $L = 1$ H, $C = 1\mu$F, $R = 1$ kΩ, and $V_0 = 1$V .
   a) Find the steady state voltages $V_L(t)$, $V_R(t)$, and $V_C(t)$.
   b) What is the resonance frequency?
   c) Determine the Q factor for the circuit.
   d) Let L=0 and sketch by hand $h(\omega_d)$ for $V_R(t)$.
   e) Describe a potential use for this circuit as a filter for L=0.

5. Consider a forced damped harmonic oscillator under the influence of a sawtooth wave driving force

$$F(t) = F_0 t \qquad \text{for } -T/2 < t < T/2$$

and repeat over all t

   Let $\beta = 0.9$ s^{-1}, $m = 0.15$ kg, k=0.1 N/m, $x(0) = 0$ m, $\dot{x}(0) = -0.5$ m/s, $T = 2\pi/\omega_0$.
   a) Describe in words the motion of the oscillator without the driving force. Make a sketch of $x_c(t)$..
   b) Make a sketch of $F(t)$ and derive the Fourier series of $F(t)$. What are the units of F_0?
   c) Using a Fourier expression for $F(t)$, solve the differential equation $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = F(t)/m$ for $x(t)$. (Hint: solve this differential equation numerically using ~100 terms in your Fourier series.)
   d) Using a computer, plot $x_c(t)$ and $x(t)$.
   e) Generate a phase space plot for this oscillator for five cycles.

6. A harmonic oscillator is under the periodic square-wave forcing function $F(t)$

$$F(t) = \begin{cases} F_0 \text{ for } (T-w) < t < (T+w) \\ -F_0 \text{ for } \left(\frac{T}{2} - w\right) < t < \left(\frac{T}{2} + w\right) \end{cases}$$

Let $\beta = 0.1$ s^{-1}, $m = 0.25$ kg, $k = 4\pi^2$ N/m, $x(0) = 0.1$ m, $\dot{x}(0) = 0$ m/s, $T = 2\pi/\omega_0$, and $2w = 0.1T$.
   a) Sketch by hand $F(t)$ over two periods. Is this an even or odd function?
   b) Find the Fourier series expression for $F(t)$.
   c) Using your Fourier expression for $F(t)$, solve the differential equation $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = F(t)/m$ for $x(t)$.
   d) Using a computer, plot $x(t)$.
   e) Generate a phase space plot for this oscillator for five cycles.

7. You have an oscillator that you wish to stop oscillating at the first instance of $x(t)$=0 by giving it a Gaussian impulse at some time t_0 the form

$$F(t) = F_0 \exp\left(-\frac{(t-t_0)^2}{\sigma^2}\right)$$

where $\sigma = 0.05$ s, $\beta = 0.1$ s^{-1}, $m = 0.1$ kg, k=0.1 N/m, $x(0) = 1$ m, and $\dot{x}(0) = 0$ m/s.
   a) Describe in words the motion of the oscillator without the driving force. Make a sketch of the transient solution $x_c(t)$.
   b) Plot F(t).
   c) At which time t_0 would you apply the impulse to dampen the oscillations in the shortest time? Explain your choice of t_0.
   d) What force F_0 would you apply to dampen the oscillations. How did you come to choose that value (Hint: think in terms of impulse delivered by the applied force)?
   e) Plot the motion $x(t)$ for your choice of t_0 and F_0. Plot this on top of the transient solution x_c(t). Did the motion completely stop with your choice of t_0 and F_0? (It should if you did things correctly)