Midterm 1 Solutions

1) $Q_N = \frac{Q_1^N}{N!}$ $\qquad H = \frac{p^2}{2m} + mgh$

$$Q_1 = \frac{1}{h^3}\int d^3p\, d^3q\, e^{-\beta H(p,q)}$$

$$= \frac{1}{h^3}\int e^{-\beta\frac{p^2}{2m}} d^3p \int e^{-\beta mgh}\, dh\, dx\, dy$$

$$= \underbrace{\frac{1}{h^3}\left(\sqrt{\frac{2\pi m}{\beta}}\right)^3}_{\frac{1}{\lambda^3}} A \int_0^H e^{-\beta mgh}\, dh$$

from x-y integration

$$= \frac{A}{\lambda^3}\frac{1}{-\beta mg}\left[e^{-\beta mgH} - 1\right]$$

$$= \frac{A}{\lambda^3 \beta mg}\left(1 - e^{-\beta mgH}\right)$$

check $g \to 0$ $\quad \frac{A}{\lambda^3 \beta mg}\left(\cancel{1} - \cancel{1} + \beta mgH\right) = \frac{AH}{\lambda^3} = \frac{V}{\lambda^3}$ ✓

to get C_v we need $\langle E \rangle$ $\qquad C_v = \frac{\partial E}{\partial T}$

$$\langle E \rangle = \frac{\sum E e^{-\beta E}}{\sum e^{-\beta E}} = -\frac{\partial \ln Q}{\partial \beta}$$

$$\ln Q = \cancel{\ln\left[\frac{V^N}{\lambda^{3N} N!}\right] = N \ln}$$

$$= \ln\left[\frac{A^N}{\lambda^{3N}(\beta mg)^N N!}\left(1 - e^{-\beta mgH}\right)^N\right]$$

$$= N \ln\left[\frac{A}{\lambda^3 \beta mg}\left(1 - e^{-\beta mgH}\right)\right] - \ln N!$$

$$-\frac{\partial \ln Q}{\partial \beta} = -\frac{\partial}{\partial \beta}\left[-N\ln \frac{\lambda^3 \beta m g}{A} + N\ln\left(1-e^{-\beta m g H}\right) - \ln N!\right]$$

$$= +N\frac{\partial}{\partial \beta}\ln \underbrace{\frac{h^3\, mg}{(2m\pi kT)^{3/2}\, kT\, A}}_{\propto \beta^{5/2}} - N\frac{mgH\, e^{-\beta m g H}}{1-e^{-\beta m g H}}$$

$$= \boxed{+\frac{N\frac{5}{2}}{\beta} - \frac{N mgH}{1-e^{+\beta m g H}}}$$

check $g \to 0$ $\quad +\frac{5N}{2\beta} - \frac{NmgH}{\cancel{1}-\cancel{1}-\beta m g H} = +\frac{3}{2}\frac{N}{\beta} = \frac{3}{2}NkT$ ✓

$$C_V = \frac{\partial E}{\partial T}$$

$$= \frac{\partial}{\partial T}\left[\frac{5}{2}NkT + \frac{NmgH}{1-e^{\beta m g H}}\right]$$

$$= \frac{5}{2}Nk + \frac{N(mgH)^2 e^{\beta m g H}}{(1-e^{\beta m g H})^2}\left(\frac{-1}{kT^2}\right)$$

$$\boxed{= \frac{5}{2}Nk_B - \frac{N(mgH)^2}{kT^2}\frac{e^{\beta m g H}}{(1-e^{\beta m g H})^2}}$$

2) 1^{st} Consider states where big spin is "up"

$\uparrow$ need remaining spins to add to $-\varepsilon$

-2ε therefore 3 up + 2 down

there are $\binom{5}{3}$ ways to do this

$$= \frac{5!}{3!\,2!} = 10$$

Now consider states where big spin is "down"

$\downarrow$ need all five spins to point up

2ε $\quad 2\varepsilon + (-5\varepsilon) = -3\varepsilon$

$$\binom{5}{5} = 1$$

so total multiplicity $\Omega = 10 + 1 = 11$

$$S = k_B \ln(11)$$

Now compute $\langle N_+ \rangle$

we have 10 states with 4 up spins
1 state with 5 up spins

$$N_+ = \frac{10 \times 4 + 1 \times 5}{11} = \boxed{\frac{45}{11} \simeq 4.09}$$

---

will also accept $\langle N_- \rangle$

$$\frac{10 \times 2}{11} + \frac{1 \times 1}{11} = \frac{21}{11} \simeq 1.91$$

3) The ions are distinguishable, therefore there are $N \times N = N^2$ total states

there are N states where the ions occupy the same site

and $N^2 - N$ states where they are on different sites

$$Q = \sum_{\text{states}} e^{-\beta E}$$

$$= (N^2 - N)e^{-0} + N e^{+\beta\delta}$$

$$= N^2 - N + N e^{\beta\delta}$$

$$\langle E \rangle = -\frac{\partial \ln Q}{\partial \beta} = \frac{-\delta N e^{\beta\delta}}{N^2 - N + N e^{\beta\delta}}$$

Probability they are on same site

$$P_{same} = \frac{\langle E \rangle}{(-\delta)} = \frac{N e^{\beta\delta}}{N^2 - N + N e^{\beta\delta}}$$

$T = 0 \quad \beta \to \infty \qquad P_{same} \to 1$

$T \to \infty \quad \beta = 0 \qquad P_{same} = \frac{N}{N^2 - N + N} = \frac{1}{N}$