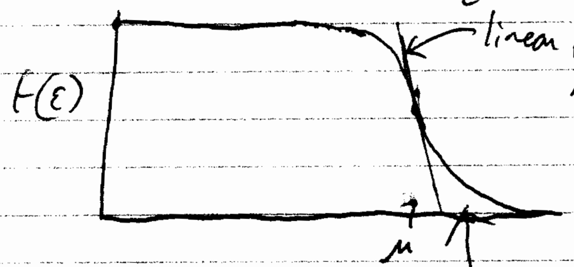
Pathria 8.1 Calculate low temp specific heat of Fermi gas
we need to calculate the ~~#~~ energy

$U = \int \varepsilon\, a(\varepsilon)\, F(\varepsilon)$ where $a(\varepsilon)$ = density of states $= \overset{\text{constant}}{A}\sqrt{\varepsilon}$

$F(\varepsilon)$ = Fermi function $= \dfrac{1}{e^{\beta(\varepsilon-\mu)}+1}$ $\qquad \dfrac{\partial F}{\partial \varepsilon} = \dfrac{-\beta e^{\beta(\varepsilon-\mu)}}{(e^{\beta(\varepsilon-\mu)}+1)^2}$




$F'(\mu) = \dfrac{-\beta}{4}$

approx Fermi function $F(\varepsilon) = \begin{cases} 1 & \varepsilon < \mu - 2kT \\ \frac{1}{2} - \frac{\beta}{4}(\varepsilon-\mu) & \text{in between} \\ 0 & \mu > \varepsilon + 2kT \end{cases}$

$\boxed{\text{explanation for } \frac{\pi^2}{4} \text{ factor: approximation misses electrons in this region}}$

using approximation

$U = \int \varepsilon\, a(\varepsilon)\, F(\varepsilon)\, d\varepsilon$

$$\simeq \int_0^{\mu-2kT} \varepsilon\, a(\varepsilon)\, d\varepsilon + \int_{\mu-2kT}^{\mu+2kT} \varepsilon\, a(\varepsilon)\left[\frac{1}{2} - \frac{\beta}{4}(\varepsilon-\mu)\right] d\varepsilon$$

$$= A\frac{2}{5}(\mu-2kT)^{5/2} + \int A\left(\frac{1}{2}+\frac{\beta}{4}\mu\right)\varepsilon^{3/2} d\varepsilon - \frac{A\beta}{4}\int \varepsilon^{5/2} d\varepsilon$$

$$= A\frac{2}{5}(\mu-2kT)^{5/2} + \frac{2A}{5}\left(\frac{1}{2}+\frac{\beta}{4}\mu\right)\left[(\mu+2kT)^{5/2} - (\mu-2kT)^{5/2}\right] - \frac{2A\beta}{7\cdot 4}\left[(\mu+2kT)^{7/2} - (\mu-2kT)^{7/2}\right]$$

$$= \frac{2A}{5}\mu^{5/2}(1-x)^{5/2} + \frac{2A\mu^{5/2}}{5}\left(\frac{1}{2}+\frac{\beta}{4}\mu\right)\left[(1+x)^{5/2} - (1-x)^{5/2}\right] - \frac{2}{7}\frac{A\beta}{4}\mu^{7/2}\left[(1+x)^{7/2} - (1-x)^{7/2}\right]$$

where $x = \dfrac{2kT}{\mu}$ expand: $(1+x)^{5/2} \simeq 1 + \frac{5}{2}x + \frac{15x^2}{8} + \frac{5x^3}{16} + \dots$

$(1+x)^{7/2} \simeq 1 + \frac{7x}{2} + \frac{35x^2}{8} + \frac{35x^3}{16} + \dots$

$$U \simeq \frac{2A}{5}\mu^{5/2}\left(1 - \frac{5}{2}x + \frac{15x^2}{8}\right) + \frac{2A\mu^{5/2}}{5}\left(\frac{1}{2} + \frac{1}{2x}\right)\left[5x + \frac{5x^3}{8}\right]$$

$$- \frac{2}{7}\frac{A}{2x}\mu^{5/2}\left[7x + \frac{35}{8}x^3\right]$$

$$= A\mu^{5/2}\left[\frac{2}{5} - x + \frac{3x^2}{4} + \frac{2}{5}\left(\frac{5x}{2} + \frac{5}{2} + \frac{5x^2}{16} + O(x^3)\right) - \frac{2}{7}\left(\frac{7}{2} + \frac{35}{16}x^2 + O(x^3)\right)\right]$$

$$= A\mu^{5/2}\left[\frac{2}{5} - \cancel{x} + \frac{3x^2}{4} + \cancel{x} + \cancel{1} + \frac{x^2}{8} - \cancel{1} - \frac{5}{8}x^2\right]$$

$$= A\mu^{5/2}\left[\frac{2}{5} + \frac{1}{4}x^2\right] = \frac{2}{5}A\mu^{5/2}\left(1 + \frac{5}{8}x^2\right)$$

Now we need to determine $\mu$. This comes from the particle number condition

$N = \int a(\varepsilon)\, f(\varepsilon)\, d\varepsilon$ use same linear approx...

$$\simeq \int_0^{\mu-2kT} A\varepsilon^{1/2}\,d\varepsilon + \int_{\mu-2kT}^{\mu+2kT} A\varepsilon^{1/2}\left[\frac{1}{2} - \frac{\beta}{4}(\varepsilon-\mu)\right]d\varepsilon$$

$$= \frac{2A}{3}(\mu-2kT)^{3/2} + \frac{2A}{3}\left(\frac{1}{2}+\frac{\beta\mu}{4}\right)\left[(\mu+2kT)^{3/2}-(\mu-2kT)^{3/2}\right] - \frac{2}{5}\frac{A\beta}{4}\left[(\mu+2kT)^{5/2}-(\mu-2kT)^{5/2}\right]$$

$$= \frac{2A\mu^{3/2}}{3}(1-x)^{3/2} + \frac{2A}{3}\left(\frac{1}{2}+\frac{1}{2x}\right)\left[(1+x)^{3/2}-(1-x)^{3/2}\right]\mu^{3/2} - \frac{2A}{5}\frac{1}{2x}\mu^{3/2}\left[(1+x)^{5/2}-(1-x)^{5/2}\right]$$

$$= \frac{2A}{3}\mu^{3/2}\left[1-\frac{3}{2}x+\frac{3x^2}{8}\right] + \frac{2A}{3}\mu^{3/2}\left[\frac{1}{2}+\frac{1}{2x}\right]\left[3x-\frac{x^3}{8}\right] - \frac{2A}{\cancel{5}}\mu^{3/2}\frac{1}{2x}\left[\cancel{5}x+\frac{\cancel{5}x^3}{8}\right]$$

$$= 2A\mu^{3/2}\left[\frac{1}{3} - \cancel{\frac{1}{2}x} + \frac{x^2}{8} + \left(\cancel{\frac{x}{2}} + \cancel{\frac{1}{2}} - \frac{x^2}{3\cdot 16}\right) - \left(\cancel{\frac{1}{2}} + \frac{x^2}{16}\right)\right]$$

$$= 2A\mu^{3/2}\left[\frac{1}{3}+\frac{x^2}{8}-\frac{x^2}{12}\right] = 2A\mu^{3/2}\left[\frac{1}{3}+\frac{x^2}{24}\right] = \frac{2}{3}A\mu^{3/2}\left(1+\frac{x^2}{8}\right)$$

$$\frac{U}{N} = \frac{3}{5}\mu\,\frac{1+\frac{5}{8}x^2}{1+\frac{x^2}{8}} \simeq \frac{3}{5}\mu\left(1+\frac{5}{8}x^2-\frac{1}{8}x^2\right) = \frac{3}{5}\mu\left(1+\frac{1}{2}x^2\right)$$

But at $T=0$ we also have $N = \frac{2}{3}A\mu_0^{3/2}$ $T=0$

$$\cancel{\frac{2}{3}A}\mu^{3/2}\left(1+\frac{x^2}{8}\right) = \cancel{\frac{2}{3}A}\mu_0^{3/2}$$

$$\mu = \mu_0\left(1+\frac{x^2}{8}\right)^{-2/3} \simeq \mu_0\left(1-\frac{1}{12}x^2\right)$$

$$\frac{U}{N} = \frac{3}{5}\mu\left(1+\frac{1}{2}x^2\right) = \frac{3}{5}\mu_0\left(1+\frac{1}{2}x^2\right)\left(1-\frac{1}{12}x^2\right) = \frac{3}{5}\mu_0\left(1+\frac{5}{12}x^2\right)$$

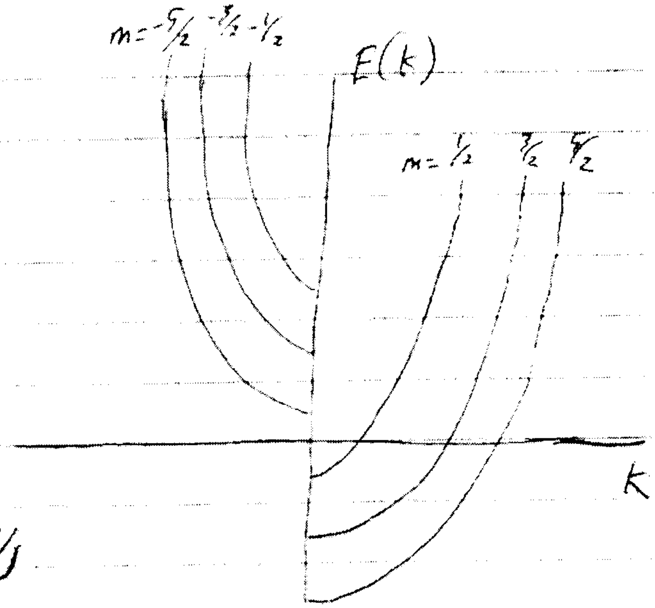
$$= \frac{3}{5}\mu_0\left(1+\frac{5}{12}\frac{4(kT)^2}{\mu_0^2}\right)$$

$$C_V = \frac{\partial U}{\partial T} = N\frac{\cancel{3}}{\cancel{5}}\mu_0\frac{\cancel{5}}{\cancel{12}}4\,\frac{2k^2T}{\mu_0^2} = 2Nk_B\frac{kT}{\mu_0}$$

compare to answer from lecture $C_V = Nk_B\frac{\pi^2}{2}\left(\frac{kT}{\mu_0}\right)$

# Pathria 8.14

spin $J = \frac{1}{2}$ or $\frac{3}{2}$ or ...
there are $2J+1$ spin states, so
there are $2J+1$ bands in the
dispersion plot ⟶




total # of particles

$$N = N_{m=-J} + N_{-J+1} + \dots N_{J-1} + N_J$$

↖ # of particles with $m=-J$

The total magnetic moment is the sum of the contributions from each band

$$M = \sum_m m g \mu_B N_m$$

$\mu_B$ = Bohr magneton
$g = 2$ for electrons

at $T=0$ + $B=0$ each band holds $\frac{N}{2J+1}$ particles

$$N_0 = \frac{N}{2J+1} = \int_0^{\mu_0} g(\varepsilon)\,d\varepsilon$$

density of states: $g(\varepsilon) = \frac{2\pi V}{h^3}(2m)^{3/2}\sqrt{\varepsilon}$

$$= \int_0^{\mu_0} \frac{2\pi V}{h^3}(2m)^{3/2}\sqrt{\varepsilon}\,d\varepsilon$$

$$= \frac{4\pi V}{3h^3}(2m)^{3/2}\mu_0^{3/2}$$

as the field is increased the filling level for each band changes

$$N_m = \frac{4\pi V}{3h^3}(2m)^{3/2}\left(\mu_0 + m g \mu_B B\right)^{3/2}$$

$$= \underbrace{\frac{4\pi V}{3h^3}(2m)^{3/2}\mu_0^{3/2}}_{N_0}\left(1 + \frac{3}{2}\frac{m g \mu_B B}{\mu_0} + \dots\right)$$

Note that this does not include the change in the chemical potential, but this correction does not contribute at 1st order

8.14 (cont) so: $M = \sum_m N_0 g \mu_B m \left(1 + \frac{3}{2} m g \mu_B B / \mu_0\right)$

$$= \frac{3}{2} N_0 g^2 \mu_B^2 \frac{B}{\mu_0} \sum_m m^2$$

Now evaluate sum

$$\sum_{m=-J}^{J} m^2 = 2 \sum_{n=0}^{J-1/2} (n+1/2)^2 = 2\left(\sum_n n^2 + n + 1/4\right)$$

$$= 2 \frac{(J-1/2)(J+1/2)(2J\cancel{-1}+1)}{6} + \cancel{2}\frac{1}{\cancel{2}}(J-1/2)(J+1/2) + 1/4\,(2J+1)$$

$$= (J+1/2)\left[\frac{2J}{3}(J-1/2) + (J-\cancel{1/2}) + \cancel{1/2}\right] = (J+1/2)\left[\frac{2J^2}{3} - \frac{J}{3} + J\right] = \frac{2}{3}(J+1/2)(J^2+J)$$

$$M = \frac{\cancel{3}}{\cancel{2}} N_0 g^2 \mu_B^2 \frac{B}{\mu_0} \frac{\cancel{2}}{\cancel{3}} (J+1/2)(J^2+J) \qquad N_0 = \frac{N}{2(J+1/2)}$$

$$= \frac{N}{2} g^2 \mu_B^2 \frac{B}{\mu_0} J(J+1)$$

~~So~~ low temp $\chi = \frac{N g^2 \mu_B^2}{2\mu_0} J(J+1)$

Now take $T \to \infty$

The occupation of each band is (at $B=0$)

$$N_0 = \frac{N}{2J+1} = \frac{V}{\lambda^3} f_{3/2}(z) \underset{T\to\infty}{\simeq} \frac{V}{\lambda^3} z$$

now we want to know how these numbers change as the field is turned on

$$\frac{\partial N_0}{\partial \mu} = \frac{V}{\lambda^3} \frac{\partial z}{\partial \mu} = \frac{V}{\lambda^3} \beta z = \beta N_0 \qquad \boxed{z = e^{\mu/kT}}$$

$$M = \sum_m m g \mu_B N_m = \sum m g \mu_B \left(N_0 + \frac{\partial N_0}{\partial \mu} m g \mu_B B\right)$$ change in filling level

$$= g^2 \mu_B^2 B \beta N_0 \sum_m m^2$$

$$= \frac{g^2 \mu_B^2 B N}{kT\, 3} J(J+1) \qquad \chi = \frac{g^2 \mu_B^2 N}{3kT} J(J+1)$$

3) The van der Waals eqn of state (in reduced variables) is

$$p = \frac{8T}{3v-1} - \frac{3}{v^2}$$

$$\frac{\partial p}{\partial v} = \frac{-3\times(8T)}{(3v-1)^2} + \frac{6}{v^3}$$

$$K_T = \left(\frac{-1}{v}\,\frac{\partial v}{\partial p}\right)_T$$

$$= -\left(-v\,\frac{24T}{(3v-1)^2} + \frac{6}{v^2}\right)^{-1}$$

$$= -\left(\frac{-24Tv^3 + 6(3v-1)^2}{v^2(3v-1)^2}\right)^{-1}$$

$$= -\left(\frac{v^2(3v-1)^2}{-24Tv^3 + 6(3v-1)^2}\right)$$

at critical point $v \to 1$

$$\to -\frac{(2^2)}{-24T + 6\cdot 2^2} = +\frac{1}{6(T-1)}$$

converting to non-reduced units $T \to T/T_c$

$$K_T \sim \frac{1}{6(T/T_c - 1)} \sim \frac{T_c}{6(T - T_c)} \qquad \text{so } \gamma = -1$$

4) From lecture the equation of state is

$$\tanh(\beta h) = \frac{m - \tanh(\tau m)}{1 - m\tanh(\tau m)} \qquad \tau = T_c/T$$

expand for small $h$ & $\tau$

$$\beta h \cong \frac{m - \tau m + (\tau m)^3/3}{1 - m\tau m}$$

$$\simeq m - \tau m + \frac{(\tau m)^3}{3} + \tau m^3 - m^3\tau^2 \qquad (1)$$

$$\beta\, dh = dm(1-\tau) + 3m^2\, dm\left(\frac{\tau^3}{3} + \tau - \tau^2\right)$$

$$\chi = \left(\frac{\partial m}{\partial h}\right)_T \qquad (\text{for } h \to 0)$$

so

$$\beta = \chi(1-\tau) + 3m^2\chi\left(\tau - \tau^2 + \tau^3/3\right) \qquad (2)$$

for $T > T_c$ $m = 0$

so

$$\beta = \chi(1-\tau) \to \chi = \frac{1}{kT(1 - T_c/T)} \sim \frac{1}{(T - T_c)} \checkmark$$

for $T < T_c$ we need to know $m$

setting $h = 0$ in Eq (1)

$$0 = m(1-\tau) + m^3\left(\tau - \tau^2 + \tau^3/3\right)$$

$$m^2 = \frac{\tau - 1}{\tau - \tau^2 + \tau^3/3} \xrightarrow{\tau \to 1} \frac{\tau - 1}{\tau}\,\frac{1}{(1 - 1 + 1/3)} = 3\frac{(T_c - T)}{T}$$

plug into (2)

$$\beta = \chi(1-\tau) + 3\chi\cancel{\frac{\tau - 1}{\tau - \tau^2 + \tau^3/3}\left(\tau - \tau^2 + \tau^3/3\right)} \to \beta = -2\chi(1-\tau)$$

$$\chi \sim \frac{1}{kT(1-\tau)} \sim \frac{1}{T - T_c} \checkmark$$