HW 5 solutions

6.1 $$S = -\left(\frac{\partial \Omega}{\partial T}\right)_{V\mu} = k\frac{\partial (Tq)}{\partial T} = k\left(q + T\frac{\partial q}{\partial T}\right) = k\left[q - \beta\left(\frac{\partial q}{\partial \beta}\right)\right]$$

$$q = \mp \sum_\varepsilon \ln\left(1 \mp z e^{-\beta\varepsilon}\right) = \mp \sum_\varepsilon \left(1 \mp e^{-\beta(\varepsilon-\mu)}\right)$$

top sign for bosons, bottom sign for fermions

$$S = k\sum_\varepsilon\left[\mp \ln\left(1 \mp e^{-\beta(\varepsilon-\mu)}\right) + \frac{\beta(\varepsilon-\mu)e^{-\beta(\varepsilon-\mu)}}{1 \mp e^{-\beta(\varepsilon-\mu)}}\right]$$

define $e^{-\beta(\varepsilon-\mu)} \equiv W$

$$S = k\sum_\varepsilon\left[\mp \ln(1 \mp W) + \frac{-(\ln W)W}{1 \mp W}\right]$$

$$= k\sum_\varepsilon \frac{\mp(1 \mp W)\ln(1 \mp W) - W\ln W}{1 \mp W}$$

$$= k\sum_\varepsilon \frac{\mp 1}{1 \mp W}\ln(1 \mp W) + \frac{W}{1 \mp W}\ln\left(\frac{1 \mp W}{W}\right)$$

$$= k\sum_\varepsilon \frac{\pm 1}{1 \mp W}\ln\left(\frac{1}{1 \mp W}\right) - \frac{1}{W^{-1} \mp 1}\ln\left(\frac{1}{W^{-1} \mp 1}\right)$$

$$\langle n_\varepsilon\rangle = \frac{1}{e^{\beta(\varepsilon-\mu)} \mp 1} = \frac{1}{W^{-1} \mp 1}$$

$$= k\sum_\varepsilon \frac{W \pm (1 \mp W)}{1 \mp W}\ln\left(\frac{1 \mp W \pm W}{1 \mp W}\right) - \langle n_\varepsilon\rangle\ln\langle n_\varepsilon\rangle$$

$$= k\sum_\varepsilon\left(\langle n_\varepsilon\rangle \pm 1\right)\ln\left(1 \pm \langle n_\varepsilon\rangle\right) - \langle n_\varepsilon\rangle\ln\langle n_\varepsilon\rangle$$

6.1 (cont) Fermions:

$$\rho_\epsilon(n) = \begin{cases} 1-\langle n_\epsilon\rangle & \text{for } n=0 \\ \langle n_\epsilon\rangle & \text{for } n=1 \end{cases}$$

$$S = -k \sum_{\epsilon,n} \rho_\epsilon(n) \ln \rho_\epsilon(n)$$

$$= -k\left(\sum_\epsilon (1-\langle n_\epsilon\rangle)\ln(1-\langle n_\epsilon\rangle) + \langle n_\epsilon\rangle \ln\langle n_\epsilon\rangle\right) \checkmark$$

Bosons:

$$\langle n\rangle = \frac{1}{w^{-1}-1} \longrightarrow W = \frac{\langle n\rangle}{1+\langle n\rangle}$$

$$\rho_\epsilon(n) = \frac{W^n}{\sum_n W^n} \longleftarrow \text{partition function of occupation states}$$

$$= w^n(1-w) \qquad 1-W = \frac{1+\langle n\rangle-\langle n\rangle}{1+\langle n\rangle}$$

$$= \frac{\cancel{W^n}\langle n_\epsilon\rangle^n}{(1+\langle n_\epsilon\rangle)^{n+1}}$$

$$= \frac{\cancel{W^n}\ \langle n_\epsilon\rangle^n}{\langle n_\epsilon+1\rangle^{n+1}}$$

$$-\sum_n \rho_\epsilon(n)\ln\rho_\epsilon(n) = -\sum_n \rho_\epsilon(n)\ln\frac{\langle n_\epsilon\rangle^n}{\langle n_\epsilon+1\rangle^{n+1}}$$

$$= -\sum_n \rho_\epsilon(n)\, n \ln\langle n_\epsilon\rangle + \sum_n \rho_\epsilon(n)(n+1)\ln\langle n_\epsilon+1\rangle$$

$$= -\ln\langle n_\epsilon\rangle \underbrace{\sum_n n\rho_\epsilon(n)}_{\langle n_\epsilon\rangle} + \ln\langle n_\epsilon+1\rangle \underbrace{\sum_n \rho_\epsilon(n)(n+1)}_{\langle n_\epsilon+1\rangle} \checkmark$$

4) We need to compute the number of electrons in the excited state N_e

$$N_e = \int_0^\infty f_{FD}(\varepsilon)\, g(\varepsilon)\, d\varepsilon$$

where $f_{FD}(\varepsilon) = \dfrac{1}{z^{-1}e^{\beta\varepsilon} - 1}$

& $g(\varepsilon)$ is the density of states

To find the density of states:

free particles: $\varepsilon = \dfrac{p^2}{2m} = \dfrac{\hbar^2 k^2}{2m} = \dfrac{\hbar^2}{2m}(k_x^2 + k_y^2)$

$k_x = \dfrac{2\pi n_x}{L_x}$ where n_x is integer $\begin{pmatrix}1 \text{ state} \\ \text{per } n\end{pmatrix}$ for

so # of states with $|n| \le n'$ $\quad \left(|n| = \sqrt{n_x^2 + n_y^2}\right)$

$$2\pi \int_0^{n'} n\, dn = \pi n'^2$$

$\dfrac{d\varepsilon}{dp} = \dfrac{p}{m}$

$$dn = \frac{dn}{dk}\frac{dk}{dp}\frac{dp}{d\varepsilon}\, d\varepsilon = \left(\frac{L}{(2\pi)}\right)\left(\frac{1}{\hbar}\right)\left(\frac{m}{p}\right) d\varepsilon$$

$p = \dfrac{\hbar 2\pi n}{L}$ so $n\, dn = \dfrac{Lp}{\hbar 2\pi}\left(\dfrac{L}{2\pi}\right)\left(\dfrac{1}{\hbar}\right)\left(\dfrac{m}{p}\right) d\varepsilon$

$$= \frac{L^2 m}{(\underbrace{2\pi\hbar}_{h})^2}\, d\varepsilon$$

area $= A$

# of states with $\varepsilon < \varepsilon'$

$$\int \underbrace{2\pi \frac{A m}{h^2}}_{g(\varepsilon)} d\varepsilon \qquad g(\varepsilon) = \frac{2\pi A m}{h^2}$$

$$N_e = \int_0^\infty \frac{2\pi A m}{h^2} \frac{1}{z^{-1}e^{\beta\epsilon}-1} d\epsilon = \frac{2\pi A m}{h^2} \int \frac{1}{z^{-1}e^{\beta\epsilon}-1} d\epsilon$$

$$= \frac{2\pi A m kT}{h^2} \int \frac{1}{z^{-1}e^{x}-1} dx \qquad x = \beta\epsilon = \epsilon/kT$$

$$= \frac{2\pi A m kT}{h^2} \int \frac{e^{-x}}{z^{-1}-e^{-x}} dx = \frac{2\pi A m kT}{h^2} \int \frac{du}{-z^{-1}+u} \qquad u = e^{-x},\ du = -e^{-x}dx$$

$$= \frac{2\pi A m kT}{h^2} \left[+\ln(u - z^{-1})\right]_1^0 = \frac{2\pi A m kT}{h^2} \left[\underbrace{\ln(-z^{-1}) - \ln(1-z^{-1})}_{\ln \frac{-z^{-1}}{1-z^{-1}} = -\ln(1-z)}\right]$$

$$= \frac{-2\pi A m kT}{h^2} \ln(1-z)$$

As $T \to 0$ $z \to 1$ so the logarithm diverges. Thus, the number of excited particles is unbounded so there is no condensation in the ground state

$N_0 = \frac{1}{z^{-1}-1} = \frac{z}{1-z}$ if this is of order N we have

$$N \sim \frac{z}{1-z} \to N - zN \sim z \qquad z \sim \frac{N}{1+N}$$

For the excited state we have

$$N_e \sim N = \frac{-2\pi A m kT_c}{h^2} \ln(1-z) \quad \text{plug in}$$

$$= \frac{-2\pi A m kT_c}{h^2} \ln\left(1 - \frac{N}{1+N}\right) \qquad \ln\left(1-\frac{N}{1+N}\right) = \ln\left(\frac{1}{1+N}\right) \simeq -\ln N$$

$$\text{so} \quad T_c \simeq \frac{N h^2 N}{2\pi A m k \ln N}$$

$$\simeq \frac{N h^2}{2\pi \ell^2 m k \ln(N)} \qquad \ell^2 = \frac{A}{N}$$

3 d) For bosons $\langle n_\varepsilon \rangle = \dfrac{1}{z^{-1}e^{\beta\varepsilon}-1}$ note $\mu = 0$

$E = \int \varepsilon\, a(\varepsilon) n(\varepsilon)\, d\varepsilon$

need density of states $a(\varepsilon)$ in d-dimension

$\varepsilon = \hbar c k = \dfrac{\hbar c n\pi}{L}$ $\qquad k = \dfrac{n\pi}{L}$ ← quantization condition

$d\varepsilon = \dfrac{\hbar c \pi}{L}\, dn$

$\#$ of states w/ $|n| < n'$ $= \displaystyle\int_0^{|n|} d\Omega\, n'^{d-1}\, dn'$ $\qquad \displaystyle\int d\Omega = \frac{2(\pi)^{d/2}}{\Gamma\left(\frac{d}{2}\right)}$ ← lecture 3

$$= \frac{2(\pi)^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \int n'^{d-1}\, dn'$$

$$= \frac{2(\pi)^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \left(\frac{L}{\hbar c \pi}\right)^d \int \varepsilon^{d-1}\, d\varepsilon$$

$$a(\varepsilon) = \frac{2}{\Gamma\left(\frac{d}{2}\right)} \left(\frac{L}{\hbar c \sqrt{\pi}}\right)^d \varepsilon^{d-1}$$

$$E = \frac{2}{\Gamma\left(\frac{d}{2}\right)} \left(\frac{L}{\hbar c \sqrt{\pi}}\right)^d \int \frac{\varepsilon^d\, d\varepsilon}{z^{-1}e^{\beta\varepsilon}-1} \qquad x = \beta\varepsilon$$

$$= \frac{2}{\Gamma\left(\frac{d}{2}\right)} \left(\frac{L}{\hbar c \sqrt{\pi}}\right)^d \frac{1}{\beta^{d+1}} \int \frac{x^d\, dx}{e^x - 1} \qquad \mu = 0 \text{ so } z = 1$$

$$\propto T^{d+1}$$

4) $p\,1/2$

First find density of states

$\#$ of states $=\left(\frac{L}{2\pi}\right)^d \int d^d k$

$=\left(\frac{L}{2\pi}\right)^d (\text{angular part}) \int k^{d-1}\,dk$

$= \text{constant} \int p^{d-1}\,dp$

$= \text{const} \int \frac{p^{d-1}}{A s p^{s-1}}\,d\varepsilon$

$= \text{const} \int p^{d-s}\,d\varepsilon$

$= \text{consts} \int \varepsilon^{d/s-1}\,d\varepsilon$ so $a(\varepsilon) = \underbrace{\text{const}}_{=C}\,\varepsilon^{(d/s-1)}$

$\varepsilon = A p^s$

$p = \hbar k$

$d\varepsilon = A s p^{s-1}\,dp$

$p = \text{const}\ \varepsilon^{1/s}$

$E = \int \frac{a(\varepsilon)\varepsilon}{z^{-1}e^{\beta\varepsilon}-1}\,d\varepsilon$

$= C\int \frac{\varepsilon^{d/s}}{z^{-1}e^{\beta\varepsilon}-1}\,d\varepsilon$ $\qquad x = \beta\varepsilon$

$= C\beta^{-(d/s+1)} \int \frac{x^{d/s}}{z^{-1}e^{x}-1}\,dx$

$= C\beta^{-(d/s+1)}\,\Gamma\left(\frac{d}{s}+1\right) g_{d/s+1}(z)$

$-PV = \phi = -kT \ln Q$ $\qquad Q = \prod_{\text{states}} \frac{1}{z e^{-\beta\varepsilon_s} - 1}$

$= kT \sum \ln\left(z e^{-\beta\varepsilon} - 1\right)$

$= kT \int a(\varepsilon) \ln\left(z e^{-\beta\varepsilon} - 1\right) d\varepsilon$

$= kTC \int \varepsilon^{(d/s-1)} \ln\left(z e^{-\beta\varepsilon} - 1\right) d\varepsilon$

$u = \ln(z e^{-\beta\varepsilon} - 1)$ $\qquad v = \frac{s}{d}\varepsilon^{d/s}$

$du = \frac{-z e^{-\beta\varepsilon}\,d\varepsilon\,\beta}{z e^{-\beta\varepsilon} - 1}$ $\qquad dv = \varepsilon^{d/s-1}\,d\varepsilon$

$= kTC\left[\underbrace{\ln\left(z e^{-\beta\varepsilon} - 1\right)\frac{s}{d}\varepsilon^{d/s}\Big|_0^{\infty}}_{=0} - \frac{s}{d}\beta \int \frac{(-1)\varepsilon^{d/s}\,d\varepsilon}{1 - z^{-1}e^{\beta\varepsilon}}\right]$

P 2/2

$$-PV = -kTC\,\frac{s}{d}\int \beta\,\frac{\varepsilon^{d/s}\,d\varepsilon}{z^{-1}e^{\beta\varepsilon}-1} \qquad x=\beta\varepsilon$$

$$PV = \cancel{kT}C\,\frac{s}{d}\,\beta^{-(\frac{d}{s}+1)}\int \frac{x^{d/s}\,dx}{z^{-1}e^{x}-1}$$

$$= \cancel{kT}C\,\frac{s}{d}\,\beta^{-(\frac{d}{s}+1)}\,\Gamma\!\left(\tfrac{d}{s}+1\right) g_{d/s+1}(z)$$

Compare $E = C\beta^{-(\frac{d}{s}+1)}\,\Gamma\!\left(\tfrac{d}{s}+1\right) g_{d/s+1}(z)$

so $PV = \frac{s}{d}E$ ✓

Calculate N

$$N = \int \frac{a(\varepsilon)}{z^{-1}e^{\beta\varepsilon}-1}\,d\varepsilon = C\int \frac{\varepsilon^{d/s-1}\,d\varepsilon}{z^{-1}e^{\beta\varepsilon}-1}$$

$$= C\beta^{-d/s}\int \frac{x^{d/s-1}\,dx}{z^{-1}e^{x}-1}$$

$$= C\beta^{-d/s}\,\Gamma\!\left(\tfrac{d}{s}\right) g_{d/s}(z)$$

$$\frac{E}{N} = \frac{1}{\beta}\,\frac{\Gamma(\frac{d}{s}+1)}{\Gamma(\frac{d}{s})}\,\frac{g_{d/s+1}(z)}{g_{d/s}(z)} = kT\,\frac{d}{s}\,\frac{g_{d/s+1}(z)}{g_{d/s}(z)}$$

as $T\to\infty$ $\mu\to-\infty$ so $z\to 0$ $\quad g_n(z)\xrightarrow[z\to 0]{} z+O(z^2)$

so $\frac{E}{N}\simeq kT\frac{d}{s}\frac{z}{z} = kT\frac{d}{s}$ $\quad \boxed{C_v = \frac{\partial E}{\partial T} = Nk\frac{d}{s}}$

$$\frac{C_p}{N} = \frac{\partial}{\partial T}\left(\frac{E+PV}{N}\right)$$

$$\frac{E+PV}{N} = \frac{1}{\beta}\,\frac{d}{s}\,\frac{g_{d/s+1}(z)}{g_{d/s}(z)} + \frac{s}{d}\,\frac{1}{\beta}\,\frac{\Gamma(\frac{d}{s}+1)}{\Gamma(\frac{d}{s})}\,\frac{g_{d/s+1}(z)}{g_{d/s}(z)}$$

$$\xrightarrow[T\to\infty]{z\to 0} kT\frac{d}{s} + kT \qquad C_p = N\frac{\partial}{\partial T}\,kT\left(\frac{d}{s}+1\right) = Nk_B\left(\frac{d}{s}+1\right) ✓$$

8.7 $\varepsilon = \frac{p^2}{2m} \longrightarrow p = \sqrt{2m\varepsilon}$

$u = \frac{p}{m} = \sqrt{\frac{2\varepsilon}{m}}$

density of states
$a(\varepsilon) = \underbrace{C}_{\text{constant}}\sqrt{\varepsilon}$

$$\langle u \rangle = \frac{\int u(\varepsilon)\, a(\varepsilon)\, n(\varepsilon)\, d\varepsilon}{\int a(\varepsilon)\, n(\varepsilon)\, d\varepsilon}$$

$$= \frac{C\int \frac{\sqrt{\frac{2\varepsilon}{m}}\,\sqrt{\varepsilon}}{z^{-1}e^{\beta\varepsilon}+1}\, d\varepsilon}{C\int \frac{\sqrt{\varepsilon}}{z^{-1}e^{\beta\varepsilon}+1}\, d\varepsilon}$$

$$= \frac{\sqrt{\frac{2}{m}}\,\frac{1}{\beta^2}\int \frac{x\, dx}{z^{-1}e^{x}+1}}{\frac{1}{\beta^{3/2}}\int \frac{x^{1/2}\, dx}{z^{-1}e^{x}+1}}$$

$x = \beta\varepsilon$
$dx = \beta\, d\varepsilon$

$$= \frac{\sqrt{\frac{2}{m}}\,\frac{1}{\sqrt{\beta}}\, f_2(z)}{\frac{\sqrt{\pi}}{2}\, f_{3/2}(z)}$$

$$\left\langle \frac{1}{u} \right\rangle = \frac{\int \frac{1}{u}\, a(\varepsilon)\, n(\varepsilon)\, d\varepsilon}{\int a(\varepsilon)\, n(\varepsilon)\, d\varepsilon}$$

$$= \frac{\cancel{C}\int \frac{\sqrt{\frac{m}{2\cancel{\varepsilon}}}\,\cancel{\sqrt{\varepsilon}}}{z^{-1}e^{\beta\varepsilon}+1}\, d\varepsilon}{\cancel{C}\int \frac{\sqrt{\varepsilon}}{z^{-1}e^{\beta\varepsilon}+1}}$$

$$= \frac{\sqrt{\frac{m}{2}}\,\frac{1}{\beta}\, f_1(z)}{\frac{\sqrt{\pi}}{2}\,\frac{1}{\beta^{3/2}}\, f_{3/2}(z)}$$

$$\langle u \rangle \left\langle \frac{1}{u} \right\rangle = \frac{\sqrt{\frac{2}{\cancel{m}}}\,\cancel{\frac{1}{\sqrt{\beta}}}\, f_2(z)}{\frac{\sqrt{\pi}}{2}\, f_{3/2}(z)}\;\frac{\sqrt{\frac{\cancel{m}}{2}}\,\cancel{\sqrt{\beta}}\, f_1(z)}{\sqrt{\pi}/2\;\, f_{3/2}(z)} = \frac{4\, f_2(z)\, f_1(z)}{\pi\, \left[f_{3/2}(z)\right]^2}$$

8.7 cont. In class we showed that

$$I = \int_0^\infty h(\varepsilon)\, f_{FD}(\varepsilon)\, d\varepsilon \cong \int_0^\mu h(\varepsilon)\, d\varepsilon + \frac{(kT)^2 \pi^2}{6} \left.\frac{dh}{d\varepsilon}\right|_{\varepsilon=\mu}$$

in dimensionless form this becomes

$$\int_0^\infty \frac{\phi(x)\, dx}{e^{x-\xi}+1} = \int_0^\xi \phi(x)\, dx + \frac{\pi^2}{6} \left.\frac{d\phi}{dx}\right|_{x=\xi}$$

Pathria Eq. 16 in appendix E (Eq 18 in ver. 3)

where $x = \beta\varepsilon \quad \xi = \beta\mu$

we have

f_1: $\phi = 1$ so $I = \int_0^\xi dx = \xi$

f_2: $\phi = x$ $\quad I = \frac{1}{2}\xi^2 + \frac{\pi^2}{6}$

$f_{3/2}$: $\phi = \frac{2}{\sqrt{\pi}}\sqrt{x}$ $\quad I = \frac{2}{\sqrt{\pi}}\left(\frac{2}{3}\xi^{3/2} + \frac{\pi^2}{12}\xi^{-1/2}\right)$

at low temps $\xi \gg 1$ so expand in powers of $(1/\xi)$

$$\langle u\rangle\left\langle\frac{1}{u}\right\rangle = \frac{4}{\pi}\frac{f_1 f_2}{(f_{3/2})^2} \simeq \frac{4}{\pi}\frac{\xi\left(\frac{1}{2}\xi^2 + \frac{\pi^2}{6}\right)}{\frac{4}{\pi}\left(\frac{2}{3}\xi^{3/2} + \frac{\pi^2}{12}\xi^{-1/2}\right)^2}$$

$$= \frac{\frac{1}{2}\xi^3\left(1 + \frac{\pi^2}{3\xi^2}\right)}{\left(\frac{2}{3}\xi^{3/2}\right)^2\left(1 + \frac{\pi^2}{8\xi^2}\right)^2} \cong \frac{9}{8}\frac{\left(1 + \frac{\pi^2}{3\xi^2}\right)}{\left(1 + \frac{\pi^2}{4\xi^2}\right)}$$

$$= \frac{9}{8}\left(1 + \frac{\pi^2}{3\xi^2}\right)\left(1 - \frac{\pi^2}{4\xi^2}\right) = \frac{9}{8}\left(1 + \frac{\pi^2}{12\xi^2}\right)$$

$$= \frac{9}{8}\left(1 + \frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2\right)$$

agrees w/ problem 6.6