Final Solutions

1) in fermion case:
~~occupancy is given by~~

$$N_{bound} = \sum_{states} \frac{1}{e^{\beta(\varepsilon-\mu)}+1} = N \frac{1}{e^{\beta(\varepsilon-\mu)}+1} = \overset{\text{given}}{\frac{N}{4}}$$

$$\text{so} \quad \frac{1}{e^{\beta(\varepsilon-\mu)}+1} = \frac{1}{4} \rightarrow e^{\beta(\varepsilon-\mu)} = 3$$

boson case

$$N_{bound} = \frac{N}{e^{\beta(\varepsilon-\mu)}-1} = \frac{N}{3-1} = \boxed{\frac{N}{2}}$$

2) Need to calculate ~~the~~ U_{elec}

$$U_{elec} = \int E(k)\rho(k)\,f(k)\,dk \qquad f(k) = \left(e^{-\beta(E-\mu)}+1\right)^{-1}$$

a) in low temperature limit $kT \ll \frac{2\pi E_0}{N a k_0}$ the temperature is less than the level spacing, so $f(k)$ is a step function

$$U_{elec} = \int_{-k_0}^{0}\left(-\Delta + \frac{E_0-\Delta}{k_0}k\right)\frac{Na}{2\pi}\,dk$$

$$= \frac{-Na\Delta}{2\pi}\left(0-(-k_0)\right) + \frac{Na}{2\pi}\left(\frac{E_0-\Delta}{k_0}\right)\left(\tfrac{1}{2}0^2 - \tfrac{1}{2}(-k_0)^2\right)$$

$$= -\frac{N a k_0\Delta}{2\pi} + \frac{-Na}{4\pi}(E_0-\Delta)\,k_0$$

$$= -\frac{Na}{4\pi}E_0 - \frac{Nak_0\Delta}{4\pi}$$

so the total energy is

$$U = N\frac{k_s}{2}u_0^2 - \frac{Na}{4\pi}E_0 - \frac{Nak_0\Delta}{4\pi}$$

$$\frac{U}{N} = \frac{k_s}{2}u_0^2 - \frac{E_0 a}{4\pi} - \frac{ak_0\alpha u_0}{\pi}$$

$$\frac{dU/N}{du_0} = k_s u_0 - \frac{ak_0\alpha}{\pi} \qquad \boxed{u_0 = \frac{ak_0\alpha}{k_s\pi}}$$

b) in the high temperature limit electrons are excited up to the positive energy states $k>0$. These electrons are pushed to higher energy by the distortion, which negates the favorable energy reduction from the $k<0$ states. Therefore, the lattice will relax back to the equilibrium state.

3) a) $Q_N = \sum_{n=0}^{N} e^{+\beta\varepsilon_z n}$

$$= \sum_{n=0}^{\infty} e^{+\beta\varepsilon_z n} - \sum_{n=N+1}^{\infty} e^{+\beta\varepsilon_z n}$$

sums diverge

$$= \frac{1}{1-e^{+\beta\varepsilon_z}} - e^{+\beta\varepsilon_z(N+1)} \sum_{n=0}^{\infty} e^{+\beta\varepsilon_z n}$$

$$= \frac{1-e^{+\beta\varepsilon_z(N+1)}}{1-e^{+\beta\varepsilon_z}}$$

To fix divergence in sum

$$\sum_{n=0}^{N} e^{\beta\varepsilon_z n} = e^{\beta\varepsilon_z N}\sum_{n=0}^{N} e^{-\beta\varepsilon_z n}$$

$$= e^{\beta\varepsilon_z N}\frac{1-e^{-\beta\varepsilon_z(N+1)}}{1-e^{-\beta\varepsilon_z}}$$

$$= \frac{e^{\beta\varepsilon_z(N+1)}-1}{e^{\beta\varepsilon_z}-1}$$

same answer

b)

Q = unblocked states + blocked states

$$= Q_N + e^{\beta(\mu_b+\varepsilon_b)} Q_{M-1}$$

$$= \frac{1-e^{+\beta\varepsilon_z(N+1)}}{1-e^{-\beta\varepsilon_z}} + e^{\beta(\mu_b+\varepsilon_b)}\frac{1-e^{+\beta\varepsilon_z M}}{1-e^{+\beta\varepsilon_z}}$$

$$= \frac{1-e^{+\beta\varepsilon_z(N+1)} + e^{\beta(\mu_b+\varepsilon_b)}\left(1-e^{-\beta\varepsilon_z M}\right)}{1-e^{+\beta\varepsilon_z}}$$

blocked probability

$$= \frac{e^{\beta(\mu_b+\varepsilon_b)}Q_{M-1}}{Q} = \frac{e^{\beta(\mu_b+\varepsilon_b)}\left(1-e^{+\beta\varepsilon_z M}\right)}{1-e^{+\beta\varepsilon_z(N+1)} + e^{\beta(\mu_b+\varepsilon_b)}\left(1-e^{+\beta\varepsilon_z M}\right)}$$

solve for μ_b:
free energy of ideal gas

$$F(N_b) = kTN_b\left(\ln\frac{N_b}{V}\lambda^3 - 1\right)$$

$$\mu_b = \frac{\partial F}{\partial N_b} = kT\ln\frac{N_b}{V}\lambda^3 = kT\ln\left(c_b\lambda^3\right)$$

so blocked probability

$$= \frac{c\lambda^3 e^{+\beta\varepsilon_b}\left(1-e^{+\beta\varepsilon_z M}\right)}{1-e^{+\beta\varepsilon_z(N+1)} + c\lambda^3 e^{\beta\varepsilon_b}\left(1-e^{+\beta\varepsilon_z M}\right)}$$

4) $H = \sum_i n_i(-U - \varepsilon\sigma_i) - \frac{J}{2}\sum_i n_i \sum_j^{\text{nearest neighbors}} n_j$

Mean field approx: $\sum_j n_j = q\langle n_i\rangle = q\theta$

$H \simeq \sum_i n_i(-U - \varepsilon\sigma_i - J\frac{q}{2}\theta)$

Grand partition function:

$$Q = \sum_{\{n_i, \sigma_i\}} e^{\beta\mu\sum n_i} e^{-\beta H}$$

$$= \sum_{\{n_i \sigma_i\}} e^{\beta\sum n_i(U + \varepsilon\sigma_i + J\frac{q}{2}\theta + \mu)}$$

in MF approx sites are independent so:

$Q = (Q_1)^N$

each site has 3 states

$$Q_1^N = \Big(\underset{n=0}{1} + \underset{n=1,\sigma=+1}{e^{\beta(U+\varepsilon+J\frac{q}{2}\theta+\mu)}} + \underset{n=1,\sigma=-1}{e^{\beta(U-\varepsilon+J\frac{q}{2}\theta+\mu)}}\Big)^N$$

$$\theta = \frac{\langle\sum n_i\rangle}{N} = \frac{1}{N}kT\frac{\partial \ln Q}{\partial\mu}$$

$$= \frac{\cancel{kT}}{\cancel{N}}\frac{\cancel{N}\left(\cancel{\beta}e^{\beta(U+\varepsilon+J\frac{q}{2}\theta+\mu)} + \cancel{\beta}e^{\beta(U-\varepsilon+J\frac{q}{2}\theta+\mu)}\right)}{1 + e^{\beta(U+\varepsilon+J\frac{q}{2}\theta+\mu)} + e^{\beta(U-\varepsilon+J\frac{q}{2}\theta+\mu)}}$$

$$= \frac{e^{\beta(U+J\frac{q}{2}\theta+\mu)}2\cosh(\beta\varepsilon)}{1 + e^{\beta(U+J\frac{q}{2}\theta+\mu)}2\cosh(\beta\varepsilon)}$$

$$\boxed{\theta = \frac{1}{e^{-\beta(U+J\frac{q}{2}\theta+\mu-\ln(2\cosh\beta\varepsilon))}+1}}$$

5) N_+ = right steps $\qquad N_+ + N_- = N$

N_- = left steps $\qquad N_+ - N_- = m$

$$N_+ + (N_+ - m) = N$$

$$N_+ = (N+m)/2 \qquad N_- = (N-m)/2$$

Probability of each trajectory w/ N_+ right steps

$$= p^{N_+}(1-p)^{N_-}$$

\# of such trajectories $\binom{N}{N_+} = \dfrac{N!}{N_+!\,(N-N_+)!}$

~~# of trajectories 2^N~~

so

$$P_N(m) = \cancel{\#}\ p^{\left(\frac{N+m}{2}\right)}(1-p)^{(N-m)/2}\ \cancel{\frac{N!}{N_+!\,(N-N_+)!}}\ \frac{N!}{\left(\frac{N+m}{2}\right)!\left(\frac{N-m}{2}\right)!}$$

check normalization

$$\sum_m P_N(m) = \cancel{\#} \sum_{N_+=0}^{N} p^{N_+}(1-p)^{N-N_+}\frac{N!}{(N_+)!\,(N-N_+)!}$$

$$= \frac{(1-p)^N}{\cancel{2^N}}\left(1+\frac{p}{1-p}\right)^N = \left(1-p+p\right)^N = 1 \ \checkmark$$

$$\langle m\rangle = \langle \textstyle\sum x_i\rangle \qquad x_i = \pm 1$$

$$= \textstyle\sum\langle x_i\rangle = N\left[p(+1) + (1-p)(-1)\right]$$

$$= N\left[2p-1\right] \quad \leftarrow \text{equals } 0 \text{ for } p=\tfrac{1}{2}$$

$$\langle m^2\rangle = \Big\langle \sum_{ij} x_i x_j\Big\rangle = \sum_i \langle x_i^2\rangle + \sum_{i\neq j}\langle x_i\rangle\langle x_j\rangle$$

$$= N\left[p(+1)^2 + (1-p)(-1)^2\right] + (N^2-N)\left[2p-1\right]^2$$

$$= N + (N^2-N)(4p^2-4p+1)$$