Return to main webpage of Mick O'Shea

# Some details of derivations for 'The most dangerous point in a climb may be just after you start',

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## Section A. Forces on the climber (no slack, no tension in rope)

### 1) Finding the average force on landing, *F*.

Define: height of climb *H*, spring constant (unit length of rope) $\kappa$, climber height *h* and mass *m*.

Define: $q \equiv mg/\kappa$

Length of rope being used = $2H - h$.

Spring constant $k = \dfrac{\kappa}{(2H-h)}$

Consider the case $h < h_1$ where:

$$h_1 = 4Hq/(1+2q). \qquad \text{equation (1)}$$

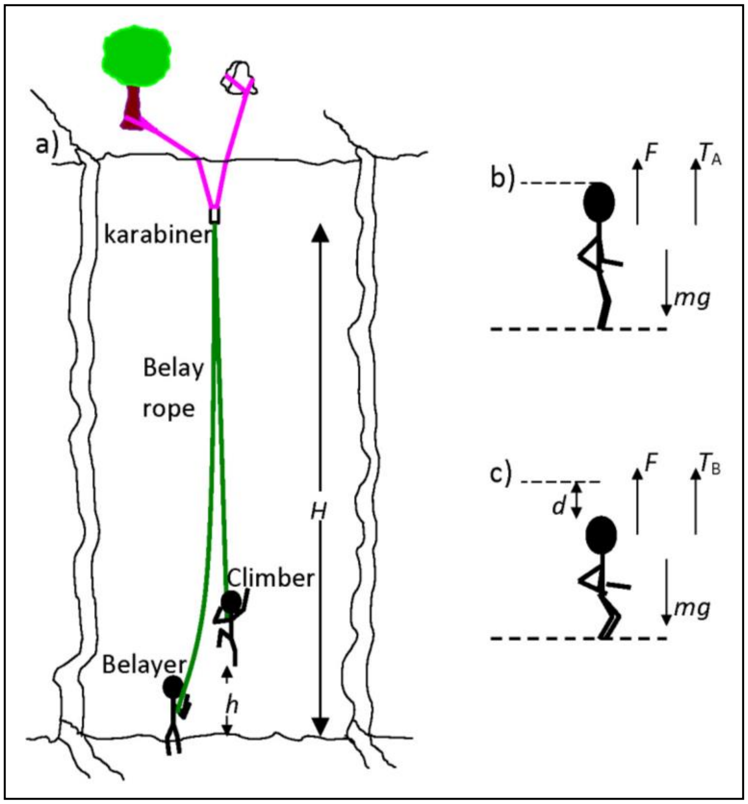


Figure 1. a) A top roped climber ascends a climb of height *H*. A belay rope passes through a double karabiner at the top of the climb and connects the climber to a belayer who controls the rope via a friction device. The climber slips and b) initially touches the ground as the belay rope stretches and c) comes to rest. During the landing the climber lowers their center of mass an amount *d* by bending their knees. The forces on the climber are shown to the right. [online colour]

Climber will impact the ground with some velocity *v* if they slip.

Finding the landing speed: energy conservation

*PE* + *KE* (just as climber touches ground)
= *PE* (before climber slips)

Change in length of rope when climber first touches the ground = *h*. So:

$$\frac{1}{2}kh^2 + \frac{1}{2}mv^2 = mgh$$

$$\Rightarrow v = \sqrt{g}\left(2h - \frac{h^2}{q(2H-h)}\right)^{1/2} \qquad \text{equation (2)}$$

Note: setting *v* to zero yields equation (1).

Model tension in belay rope during landing as:

$$T_{avg} \approx (T_A + T_B)/2 = k(h + d/2).$$

Finding the average value of the force exerted by ground on climber during landing: assume the climber center of mass drops a distance *d* during landing and the deceleration *a* is uniform so that $a = v^2/2d$. Then applying Newton's 2nd law:

$$F + T_{avg} - mg = \frac{mv^2}{2d} \qquad \text{equation (3)}$$

Substituting for $v$ and $T_{avg}$ and simplifying:

$$F = mg\left[1 + \frac{h}{d} - \frac{(h+d)^2}{2dq(2H-h)}\right]$$

**2) Value of $h$ ($\equiv h_{max}$) when $F$ is a max.**

$$\frac{dF}{dh}=0\Rightarrow \qquad (1+2q)h^2-4(1+2q)Hh+\left(8H^2q-4dH-d^2\right)=0$$

Solving this quadratic equation in $h$: $\quad h_{max}=2H\pm\dfrac{(2H+d)}{\sqrt{1+2q}}$ equation (4)

The minus sign must be selected since $h$ must be less than $2H$.

**3) Finding the maximum force on landing, $F_{max}$**

Substituting $h = h_{max}$ into the equation for $F$ in Section A.1

$$F_{max}=mg\left(1+\frac{1}{d}\left(2H-\frac{(2H+d)}{\sqrt{1+2q}}\right)-\frac{\left(2H-\frac{(2H+d)}{\sqrt{1+2q}}+d\right)^2}{2dq\left(\frac{(2H+d)}{\sqrt{1+2q}}\right)}\right)=mg\left(1+\frac{1}{d}\left(2H-\frac{(2H+d)}{\sqrt{1+2q}}\right)-\frac{(2H+d)\left(1-\frac{1}{\sqrt{1+2q}}\right)^2}{\frac{2dq}{\sqrt{1+2q}}}\right)$$

$$=mg\left(1+\frac{2H}{d}-\frac{(2H+d)}{d\sqrt{1+2q}}+\frac{(2H+d)}{dq}-\frac{(2H+d)}{2dq}\left(\sqrt{1+2q}+\frac{\sqrt{1+2q}}{1+2q}\right)\right)$$

$$=mg\left(1+\frac{1}{q}\left(1-\sqrt{1+2q}\right)\right)\left(1+\frac{2H}{d}\right)$$

equation (5)

## <u>Section B. Slack ($x$) in the belay rope prior to the climber slipping</u>

**1) Finding the average force on landing, $F_x$.** Rope has slack amount $x$.

Length of rope being used = $2H-h+x$

Spring constant $k=\dfrac{\kappa}{(2H-h+x)}$

Consider the case: $x < h < h_{1,x}$ where $h_{1,x}=\dfrac{(q(2H+x)+x)+\sqrt{4xqH+(2H+x)^2q^2}}{(1+2q)}$ equation (6)

Climber will impact the ground with some velocity $v$ if they slip.

Finding the landing speed: energy conservation

PE + KE (just as climber touches the ground) = PE (before climber slips)

Change in length of rope when climber first touches the ground is $h - x$.

$$\Rightarrow\frac{1}{2}k(h-x)^2+\frac{1}{2}mv^2=mgh\Rightarrow \qquad v=\sqrt{g}\sqrt{2h-\frac{(h-x)^2}{q(2H-h+x)}}$$ equation (8)

Note: setting $v$ to zero yields equation (6).

Model tension in belay rope during landing as $T_{avg}\approx(T_A+T_B)/2=k(h-x+d/2)$

The average value of the force on the climber during landing is:

$$F_x=mg+\frac{mv^2}{2d}-T_{avg}=mg\left[1+\frac{h}{d}-\frac{(h-x+d)^2}{2dq(2H-h+x)}\right]$$ equation (9)

**2) Value of $h$ ($\equiv h_{\max,x}$) when $F_x$ is a max.**

Convenient to change variable: let $h' = h-x$

$$F_x = mg\left[1+\frac{h'+x}{d}-\frac{(h'+d)^2}{2dq(2H-h')}\right]$$

$$\frac{dF_x}{dh}=\frac{dF_x}{dh'}\frac{dh'}{dh}=mg\left[\frac{1}{d}-\frac{2(h'+d)}{2dq(2H-h')}-\frac{(h'+d)^2}{2dq(2H-h')^2}\right](1)=0$$

$$\Rightarrow (1+2q)h'^2-4(1+2q)Hh'+\left(8H^2q-4dH-d^2\right)=0$$

The minus sign must be selected since $h_{\max,x}$ (below) must be less than $H$.

Solving the quadratic equation in $h'$**:** $$h'_{\max,x}=2H\pm\frac{(2H+d)}{\sqrt{1+2q}}$$

From original definition of $h'$ :

$$h_{\max,x}=h'_{\max,x}+x=2H-\frac{(2H+d)}{\sqrt{1+2q}}+x=h_{\max}+x \qquad \text{equation (10)}$$

**3) Finding the maximum force on landing, $F_{\max,x}$**

Substituting $h = h_{\max,x}$ into the equation for $F_x$ in Section B.1

$$F_{\max,x}=mg\left[1+\frac{1}{d}\left(2H-\frac{(2H+d)}{\sqrt{1+2q}}+x\right)-\frac{\left(2H-\frac{(2H+d)}{\sqrt{1+2q}}+d\right)^2}{2dq\left(\frac{(2H+d)}{\sqrt{1+2q}}\right)}\right]$$

$$=mg\left[1+\frac{1}{d}\left(2H-\frac{(2H+d)}{\sqrt{1+2q}}\right)-\frac{\left(2H-\frac{(2H+d)}{\sqrt{1+2q}}+d\right)^2}{2dq\left(\frac{(2H+d)}{\sqrt{1+2q}}\right)}\right]+\frac{mgx}{d}$$

$$=F_{\max}+mg\frac{x}{d} \qquad \text{equation (11)}$$

(Used result for $F_{\max}$ from A.3.)

## <u>Section C. Tension ($T_1$) in belay rope prior to the climber slipping</u>

**1) Finding the average force on landing, $F_{T1}$**

The belayer applies tension $T_1$ so rope is <u>stretched</u> by an amount $x_1 = T_1/k$ prior to the climber slipping.

Length of rope being used if it is *untensioned* = $2H-h-T_1/k$ .

Spring constant $k=\dfrac{\kappa}{(2H-h-T_1/k)}\Rightarrow \qquad k=\dfrac{(\kappa+T_1)}{(2H-h)}$.

Consider the case $h < h_{1,T1}$ where $h_{1,T1}=\dfrac{4H(q-t_1)}{(1+2q-t_1)}$, (defined $t_1\equiv T_1/\kappa$)

Climber will impact the ground with some velocity $v$ if they slip.

Finding the landing speed: energy conservation

PE + KE (just as climber touches the ground) = PE (before climber slips)

Rope extension before slipping: $x_1$ (= $T_1/k$);
Rope extension after climber reaches ground: $x_1 + h$.

$$\tfrac{1}{2}k(h+x_1)^2 + \tfrac{1}{2}mv^2 = mgh + \tfrac{1}{2}kx_1^2 \Rightarrow$$

$$v = \sqrt{g}\sqrt{2h - \frac{1}{q}\left(\frac{1+t_1}{2H-h}\right)h^2 - \frac{2ht_1}{q}} \qquad [\text{used} \quad kx_1 = T_1,\ \ q \equiv mg/\kappa,\ \ t_1 \equiv T_1/\kappa].$$

Note: setting $v$ to zero yields the previous equation for $h_{1,\mathrm{T1}}$.

Model tension in belay rope during landing as $T_{avg} \approx (T_A + T_B)/2 = k(h + T_1/k + d/2)$

Average value of the force on the climber during landing is:

$$F_{T1} = mg + \frac{mv^2}{2d} - T_{avg} = mg\left(1 + \frac{h}{d} - \frac{(h+d)^2(1+t_1)}{2dq(2H-h)} - \frac{t_1}{q}\left(1+\frac{h}{d}\right)\right)$$

**2) Value of $h$ ($\equiv h_{\max,\mathrm{T1}}$) when $F$ is a max.**

$$\frac{dF}{dh} = mg\left[\frac{1}{d} - \frac{2(h+d)(1+t_1)}{2dq(2H-h)} - \frac{(h+d)^2(1+t_1)}{2dq(2H-h)^2} - \frac{t_1}{qd}\right] = 0$$

$$\Rightarrow [1+2q-t_1]h^2 - [1+2q-t_1]4Hh + (q-t_1)8H^2 - (d^2+4dH)(1+t_1) = 0$$

Solving this quadratic equation in $h$: $\quad h_{\max,T1} = 2H \pm (2H+d)\sqrt{\dfrac{(1+t_1)}{(1+2q-t_1)}}$

The minus sign must be selected since $h_{\max,\mathrm{T1}}$ must be less than $H$.

**3) Finding the maximum force on landing, $F_{\max,\mathrm{T1}}$**

Substituting $h = h_{\max,\mathrm{T1}}$ into the equation for $F_{\mathrm{T1}}$ in Section C.1

$$F_{T1} = mg\left[1 + \frac{2H - (2H+d)\sqrt{\frac{(1+t_1)}{1+2q-t_1}}}{d} - \frac{\left(2H - (2H+d)\sqrt{\frac{(1+t_1)}{1+2q-t_1}} + d\right)^2(1+t_1)}{2dq\left((2H+d)\sqrt{\frac{(1+t_1)}{1+2q-t_1}}\right)} - \frac{t_1}{q}\left(1 + \frac{2H - (2H+d)\sqrt{\frac{(1+t_1)}{1+2q-t_1}}}{d}\right)\right]$$

$$= mg\left[1 + \frac{2H}{d} - \frac{(2H+d)}{d}\sqrt{\frac{(1+t_1)}{1+2q-t_1}} - \frac{(2H+d)\left(1 - \sqrt{\frac{(1+t_1)}{1+2q-t_1}}\right)^2(1+t_1)}{2dq\left(\sqrt{\frac{(1+t_1)}{1+2q-t_1}}\right)} - \frac{t_1}{dq}(2H+d)\left(1 - \sqrt{\frac{(1+t_1)}{1+2q-t_1}}\right)\right]$$

$$= mg\left[1 + \frac{1}{q} - \frac{1}{2q}\sqrt{(1+t_1)(1+2q-t_1)} - \frac{1}{2q}\left(\sqrt{\frac{(1+t_1)}{1+2q-t_1}} + 2q\sqrt{\frac{(1+t_1)}{1+2q-t_1}} - t_1\sqrt{\frac{(1+t_1)}{1+2q-t_1}}\right)\right]\left(1 + \frac{2H}{d}\right)$$

$$= mg\left[1 + \frac{1}{q} - \frac{1}{q}\sqrt{(1+t_1)(1+2q-t_1)}\right]\left(1 + \frac{2H}{d}\right) \qquad \text{equation (12)}$$

Note that the value of $h$ at which $F$ goes to zero is not given by $h_1$. The fall height for which $F$ is zero is found by setting $F$ = 0 in equation (3) yielding $h_{F=0} = (4Hq - d)/(1 + 2q)$.