Class 7: Solving PDEs numerically

Introduction

There are a lot of different types of partial differential equations (PDEs), and a lot of ways of solving them. The subject is too broad to be covered in a day or a week.

Let's consider just a few commonly encountered cases and some solution techniques:

- Linear problems where the solution is a superposition of known solutions.
- Homogeneous problems where the solution at one point can be determined from nearby points in a discrete grid.
- Problems where the initial condition is given and there is a simple time evolution as a function of the values.

Notation

The general PDE involves a function ϕ of multiple variables $\underline{x} = (x_1, x_2, ..., x_n)$.

$$\phi = \phi(x_1, x_2, ...x_n) = \phi(\underline{x})$$

The most general PDE can be any equation involving any partial derivatives of ϕ.

A linear PDE has the form $D\phi = f(\underline{x})$, where D is some linear differential operator, such as ∇^2.

More generally, there could be multiple coupled functions, $\underline{\phi} = \underline{\phi}(\underline{x})$. (E.g., Maxwell's equations.)

When solving on a grid, the location of k-th tic along the j-th dimension will be denoted x_{jk}. (See figure.) For simplicity of presentation, the equations in these notes assume equal number of points n and equal grid spacing h along all dimensions; the replacements $n \to n_j$ and $h \to h_j$ may be made.

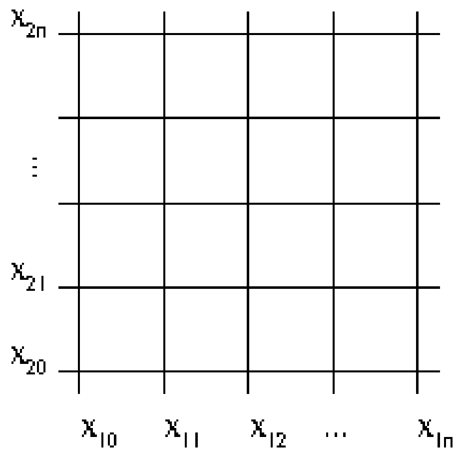


Figure 1: Two-dimensional grid with positions denoted as described in the text.

Linear equations, superposable solutions

Techniques:

- Orthogonal functions when the PDE is homogeneous between limits.
- Image charge method for simple inhomogeneous cases.
- Fourier transform methods for many cases.
- More generally, Green's function method, if you can derive the Green's functions analytically.

Superposition of orthogonal functions

Linear problems have solutions like this:

$$\phi = a_1\phi_1 + a_2\phi_2 + ...$$

If the ϕ_i are known, we only need to find a_i. This can be done using the orthogonality properties of the basis functions ϕ_i.

In particular, if we are working on a discrete grid, then we replace the integral of the analytic dot product with a sum. This works *if* the functions are orthogonal when summed in this discrete way too. (E.g., a discrete Fourier series.)

For example, suppose ϕ is known everywhere for some particular value of $x_j = x_{j0}$, for a particular dimension j. (This might be a particular point in time, or one wall of a box.) Then

$$\phi(x_{1...j-1}, x_{j0}, x_{j+1..n}) = F_0(x_{1...j-1}, x_{j+1..n}).$$

Further suppose the x_{j0} dependence separates out, so

$$\phi_i(\underline{x}) = f_i(x_{1...j-1}, x_{j+1..n})g_i(x_j).$$

Using $< \cdot\cdot >$ to denote the dot product,

$$< f_i F_0 >= a_i < f_i f_i >$$

follows from the orthogonality condition $< f_i f_j >= 0$ for $i \neq j$. Calculating the coefficients a_i becomes as simple as

$$a_i = \frac{< f_i F_0 >}{< f_i f_i >}.$$

... see hand-written notes for more ....

For slightly more general B.C.,

if φ_i functions arise from sep. of vars,

then $\varphi_i = \varphi_{i_1 i_2 i_3 \ldots i_n} = X_{1i_1}(x_1)\, X_{2i_2}(x_2) \ldots X_{ni_n}(x_n)$.

If the X_{ji} are orthogonal ($\langle X_{ji}\, X_{ji'} \rangle = 0$ for $i \neq i'$),

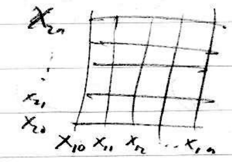
then can find the coefficients.

This is same as in analytic methods (linear algebra). [Assume students know this.]

Orthog. condition on uniform grid is

$$\sum_{k=0}^{n} X_{ji}(x_{jk})\, X_{ji'}(x_{jk}) = 0 \text{ for } i \neq i'$$

and x_{jk} are the k^{th} position of the j^{th} dimension.



The grid. (2d)

Fourier transform approach:

Poisson's equation: given ρ, find φ that solves

$$\nabla^2 \varphi = \rho$$
$$-k^2 \tilde{\varphi} = \tilde{\rho}$$

If b.c. on φ are equivalent to periodic bdy. cond., we can use discrete F.T. (E.g., conducting walls.)

Then $\vec{k} = \left(\frac{2\pi n_x}{L_x}, \frac{2\pi n_y}{L_y}, \ldots\right)$

$$\tilde{\varphi} = -\frac{\tilde{\rho}}{k^2}$$

Ignore $\tilde{\varphi}(0)$, it just adds a constant.

$$\varphi = \mathcal{F}^{-1}(\tilde{\varphi}).$$

Relaxation method

... see hand-written notes for more ....

Image charges: If you know analytic solution is
Sum of free space solution plus image
charges, just use computer to do the sum

Relaxation method:

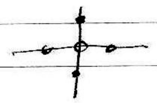
$$\nabla^2 \varphi = \rho$$

Approximate ∇^2 in finite differences: (2d case)

$$\nabla^2 \varphi = \frac{\varphi(\underline{x}+h\hat{x}_1)+\varphi(\underline{x}-h\hat{x}_1)-2\varphi(\underline{x})}{h^2} + \frac{\varphi(\underline{x}+h\hat{x}_2)+\varphi(\underline{x}-h\hat{x}_2)-2\varphi(\underline{x})}{h^2} + O\left(\frac{1}{h^3}\right) = \rho$$

$$\frac{4}{h^2}\varphi(\bar{x}) = \frac{\sum_{4\,\text{points}} \varphi(\underline{x}+\text{offset})}{h^2} - \rho + O\left(\frac{1}{h^3}\right)$$

$$\varphi(\bar{x}) = \frac{1}{4}\sum_{4\,\text{points}} \varphi(\underline{x}+\text{offset}) - \frac{h^2}{4}\rho + O\left(\frac{1}{h}\right)$$



The "4 points"

Start with some guess for φ and repeatedly
replace guess with new value from above
until it converges.

Notes: slowly converges, more slow for large scale
errors, slowness increases w/ # of points in grid.
Various tricks for improving convergence, notably
solving first on coarse grid, using interpolation
from that as guess for finer grid, repeat.

Finite differences, with initial condition

- This sort of problem can be integrated on a finite difference grid in which each point is a variable in a system of ODEs.

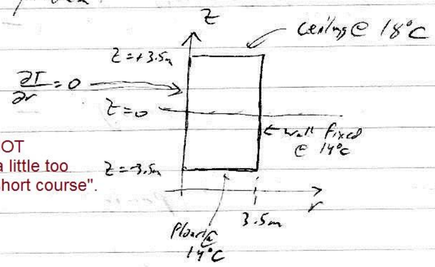
... see hand-written notes for more ....

"Time" evolution of grid starting w/ initial state, some bdy condition given.

Can be treated as ODE in N vars where n is # points in grid not on boundary, impose boundary conditions on each step. (Easiest case: Dirichlet bdy cond.)

Too tough.

X Assigned problem?



This problem NOT assigned. It's a little too much for the "short course".

$$\nabla^2 T = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2}{\partial z^2}T = 0$$

$$\frac{\partial^2}{\partial z^2}T = \frac{T(z+h)+T(z-h)-2T(z)}{h^2} + O(h^{-3})$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}$$
$$= \frac{T(r+h)+T(r-h)-2T(r)}{h^2} + \frac{1}{r}\cdot\frac{T(r+h)-T(r-h)}{2h} + O(h^{-3})$$

$T = \alpha r^2$

$\frac{\partial T}{\partial r} = 2\alpha r$

$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = 4\alpha r$

$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = 4\alpha$

Near r = 0, assume symmetry about r = 0. Avoid singularity @ r = 0 by not having r grid start @ r = 0.

Hmm, this is tricky due to cylindrical sym.

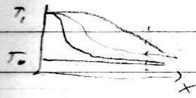
$\frac{180}{9} \times 2$

Assignment

The question to answer using a numerical solution to the correct PDE: How long can a space shuttle thermal tile (LI-900 material) withstand the heat of re-entry, keeping temperature at the back of the tile less than annealing point of aluminum (350 deg. F).

... see hand-written notes for more ....

Easier assigned problem: 1d slab
heated to T_1 @ 1 end starting @ $t=0$,
find temp as fn of t, x



$$k' \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}, \qquad T = T(x,t).$$

$T_0 = 300K$, $T_1 = 1000°K$ $(540°R \to 1800°R)$

$c_p \simeq 0.20 \frac{BTU}{lb \cdot °R}$, $k \simeq 10^{-5}\ BTU/(ft \cdot s \cdot °R)$, $\rho = 9\ lb/ft^3$

$$k' = \frac{k}{\rho c_p} = \frac{10^{-5}}{1.8} \frac{BTU}{ft \cdot s \cdot \cancel{°R}} \cdot \frac{lb \cdot \cancel{°R} \cdot ft^3}{BTU\ lb} \simeq 0.5 \times 10^{-5}\ ft^2/s.$$

Aluminum annealing @ 350°F. (N.B. Units)

NASA-2000-tm 210289, appendix C.
Downloaded from tpub.com, www.cs.odu.edu/~mln

Critical question: how long can 1"-thick tile withstand
heat of reentry ($T(x=1") < 350°F$)?
How long can 5"-thick tile withstand?

* Advanced case (see NASA Tech note): k' is fn of T!
* Apply heat, not fixed temp, and include effect
of radiation of heat from surface.
* Time-dep. heating (see Fig. 6 of NASA note)