Mid-term Exam

Due at the beginning of class Tuesday, Oct. 23

Instructions:

- 120 points are possible on this exam, so there are 20 points extra credit
- Unlike the homework, this exam is not to be discussed with your classmates (or other professors, students, post-docs, etc.). If you have questions, feel free to bring them to me, and I will do my best to answer them without telling you how to do the problem. You may use any other resource you like, but cite anything you didn't do yourself. If you use Mathematica, say so; if you pull an answer off the web, say so; if you use a result from a book or class, say so.
- Remember to write down everything you can about a problem! Even if you can't find the solution because of time or difficulty, you can write down what you think needs to be done and what physics you expect to result.
- Any plots, discussion, or physical insight beyond what is asked for has a good chance of becoming extra credit ...

(25 pts) **1.** A particle of mass m sees the potential

$$V(x) = V_0 \ell \left[\delta(x) + \delta(x-a) + \delta(x-2a)\right]$$

where all of the constants are positive and real.

(a) Discuss this system qualitatively. Does it possess any symmetries? Do you expect it to have bound states? What behavior do you expect for R and T? Are there any resonances? If there are resonances, sketch what you think T will look like and be semi-quantitative when placing the resonances.

(b) Calculate R and T. There will be a lot of algebra, so careful organization will help considerably. Do not try to simplify the results too far — there's not much to be gained by doing so.

(c) Assuming $\hbar = m = 1$, $\ell = a = 1$, and $V_0 = 5$, plot T and discuss it physically. Which resonances describe a metastable state with the longest lifetime? Do it support your predictions from (a)?

(d) We now add to this potential the following

$$V(x) = V_0 \ell \left[\delta(x-3a) + \delta(x-4a) + \delta(x-5a)\right].$$

Calculate R and T for the total potential with six delta functions. Plot T and discuss it physically, especially mention the connection to your results from (b) and (c). Use the same parameters as in (c).

(25 pts) **2.** The Hamiltonian for some system is given by

$$\mathbf{H} = \hbar\Omega \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with Ω a positive, real constant.

NOTE: Computing with Mathematica or any other similar software/device is neither acceptable nor necessary for this problem, so make sure that your work clearly shows the steps you follow. If you cannot see how to do it by hand, then go ahead and use Mathematica (or your favorite substitute), but expect a grade penalty. (If I don't see *at least* the main steps in your derivation, then I will assume you used Mathematica!)

(a) Find the eigenstates for this system and their energies. Identify the ground state and label each excited state in increasing energy order as usual.

(b) We now introduce three operators

$$\mathbf{A} = a \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{B} = b \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{C} = c \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Under what conditions could these operators represent physical observables? Be specific. For the remainder of this problem, assume these operators do represent physical observales. Can you construct simultaneous eigenstates for any combination of **H**, **A**, **B**, and **C**? Why or why not? If you can, do so. (You need not use this common basis for the questions that follow, however!)

(c) The system is initially in its ground state and **A** is measured. What values can be obtained and with what probabilities?

(d) Assume that the measurement in (c) yielded $-a$. Some time t later, the energy is measured. What values can be obtained and with what probabilities?

(e) Instead of the energy as in (d), **B** is measured at some time t for the system that yielded $-a$ in (c). What values can be obtained and with what probabilities?

(25 pts) **3.** Consider a one-dimensional simple harmonic oscillator.

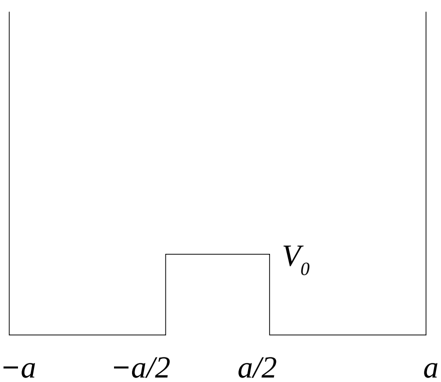
(a) Construct a linear combination of the states $|0\rangle$ and $|2\rangle$ such that ΔX is as small as possible.

(b) Is this minimum ΔX smaller than for the state $|0\rangle$ by itself? Explain how this is possible using the coordinate space wave functions. Sketches or plots would be highly desirable.

(c) Assume your state from (a) is the initial state at $t = 0$. What is the state vector for $t > 0$? Describe its behavior.

(d) Find $\Delta X(t)$ for your state from (c) and plot it. Is the result consistent with your discussion of the behavior in (c)? Explain why or why not.

(25 pts) **4.** Consider an infinite square well with a barrier (figure not to scale):



(a) Calculate the lowest two energy eigenstates and energies. You need only graphically indicate the solution of the transcendental equations you obtain. (You might want to take advantage of any symmetry present.) Sketch or plot both wave functions.

(b) At what value of $\beta = \sqrt{2mV_0/\hbar^2}\,\frac{a}{2}$ does the energy of the lowest state move above the barrier? How about for the first excited state? Give numerical values.

(c) How do the energies of the two states in (a) behave as a function of V_0? You can be qualitative here, but indicate the basic steps of any calculation or solution [using, for instance, your graphical solution from (a)] and be clear on the physical explanation. You can be quantitative in the limits $V_0 \to \infty$ and $V_0 \to 0$, right?

(d) Suppose we know that at $t = 0$, the particle is localized on the left side of the barrier assuming β is larger than both limits from (b). Using just the lowest two states, write down a total wave function $\psi(x, 0)$ that most closely describes this case. Plot your resulting wave function.

(e) Find $\psi(x, t)$ for the initial condition from (d) and describe the behavior of the system as a function of time. How long does the particle take to become maximally localized on the right side of the barrier?

(20 pts) **5.** A free particle of mass m has the following wave function at $t = 0$:

$$\psi(x, 0) = \alpha_1 \mathcal{N}_1 e^{ik_1 x} e^{-\frac{1}{2}\left(\frac{x-x_1}{\Delta_1}\right)^2} + \alpha_2 e^{i\varphi} \mathcal{N}_2 e^{ik_2 x} e^{-\frac{1}{2}\left(\frac{x-x_2}{\Delta_2}\right)^2}$$

where

$$\mathcal{N}_j = \frac{1}{\sqrt{\Delta_j \sqrt{\pi}}}.$$

You should assume that all constants are positive and real. You should also assume that:

$$|x_2 - x_1| \gg \Delta_1 \qquad |x_2 - x_1| \gg \Delta_2 \qquad x_2 > x_1.$$

(a) Sketch a $|\psi(x, 0)|^2$ consistent with the above assumptions and point out in your sketch what role each assumption plays.

(b) Determine the condition satisfied by α_1 and α_2. Be sure to clearly indicate any approximation you might make and justify it.

(c) Find $\psi(x, t)$. Is it normalized for all times? (Your answer to this question should require no computation!)

(d) Assuming $\alpha_1 = \alpha_2$, discuss the evolution of these wave packets with time from $t = 0$ to very large times. Be sure to consider $k_1 \ll k_2$, $k_1 \approx k_2$, and $k_1 \gg k_2$. What role does the width Δ_j of each wave packet play in this evolution? Your answers should involve $\langle X \rangle(t)$ and $\Delta X(t)$ of each wave packet.

(e) How does your answer to (d) change if α_1 and α_2 do not have the same magnitude? How will $|\psi(x, t)|^2$ look different?

(f) What role does φ play? In particular, what is its impact on the evolution of this state? Illustrations for a few φ would be helpful.