Homework 8

Due in class Oct. 30

1. Given the Hamiltonian

$$H = \begin{pmatrix} E_1 & \gamma \\ \gamma & E_2 \end{pmatrix}$$

with γ real, and $E_1 \neq E_2$,

(a) Find the exact eigenvalues of H.

(b) Find the approximate eigenvalues using perturbation theory to the lowest nonvanishing order.

(c) Expand your answers from (a) in a Taylor series in an appropriate small parameter and compare with your results from (b). Do your results make sense?

2. Consider the infinite square well:

$$V(x) = \begin{cases} 0 & \text{if } |x| \leq \frac{a}{2} \\ +\infty & \text{otherwise} \end{cases}.$$

(a) Calculate the energy shifts to first order for the three lowest states for a perturbation of the form

$$H_1 = \varepsilon \frac{2|x|}{a}.$$

State the conditions under which this result is valid.

(b) Calculate the energy shifts to first order for the three lowest states for a perturbation of the form

$$H_1 = \varepsilon \frac{2x}{a}.$$

State the conditions under which this result is valid.

(c) Calculate the energy shifts to first order for the three lowest states for a perturbation of the form

$$H_1 = \begin{cases} 0 & \text{if } x \leq 0 \\ \varepsilon \frac{2x}{a} & \text{otherwise} \end{cases}.$$

State the conditions under which this result is valid.

(d) Are your results from (a), (b), and (c) consistent? Do they make physical sense (in comparison with one another)? In particular, discuss the role of symmetry.

3. Consider the perturbation

$$\hat{H}_1 = \lambda \hat{X}^4$$

to a simple harmonic oscillator of mass m and frequency ω.

(a) Calculate the energy corrections to lowest non-vanishing order for each state $|n\rangle$.

(b) Argue that no matter how small λ is, the perturbation expansion will break down for some large enough n. Give both a mathematical and physical explanation.

4. Consider the Hamiltonian

$$\mathbf{H} = A\mathbf{S}_z^2 + B(\mathbf{S}_x^2 - \mathbf{S}_y^2)$$

where

$$\mathbf{S}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \mathbf{S}_y = i\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{S}_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

A and B are positive, real constants.

(a) Treating the B term as a perturbation, find the approximate eigenenergies of $\mathbf{H}$ to the lowest non-vanishing order of perturbation theory.

(b) Under what conditions are your results from (a) valid?

(c) Verify both (a) and (b) by calculating the exact eigenenergies and comparing.

5. Consider a particle of mass m and charge q in the infinite square well

$$V(x) = \begin{cases} 0 & \text{if } |x| \leq \frac{a}{2} \\ +\infty & \text{otherwise} \end{cases}$$

subject to the time-dependent perturbation

$$H_1(t) = -q\mathcal{E}(t)x$$

where

$$\mathcal{E}(t) = \mathcal{E}_0 e^{-\left(\frac{t}{\tau}\right)^2} \cos \omega t.$$

(a) Calculate the transition probabilities to the first, second, and third excited states assuming the system is in the ground state initially. Plot these probabilities as a function of ω assuming

$$\tau = 10\frac{2\pi}{\omega_{21}}$$

where $\hbar\omega_{21} = E_2 - E_1$. Explain the results in all cases. How would your results change for τ ten times longer?

(b) Next, set

$$\omega = \frac{8\pi^2\hbar^2}{2ma^2}$$

and plot your probabilities from (a) as a function of τ. What state is this resonant with? Are transitions to this state possible? Explain the τ dependence of your transition probabilities. How would your results change for

$$\omega = \frac{3\pi^2\hbar^2}{2ma^2}$$

(c) Assume now that the initial state of the system is an equal admixture of the ground and first excited state. Calculate the transition probabilities to the second and third excited states and contrast them to those you found in (a). Explain any differences you find.