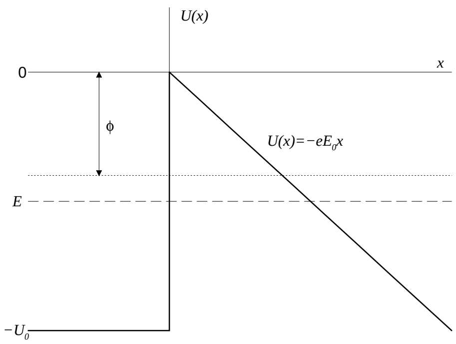
Homework 7

Due in class Oct. 16

From Griffiths: Prob. 2.42

1. Field emission occurs when an electron is pulled from a metal surface by a strong electric field E_0. This process can be modeled by the potential $U(x)$ in the figure. The electron is bound inside the metal, $x < 0$, with energy E. The work function of the metal ϕ is the energy required to extract an electron at the Fermi energy E_F, *i.e.* the least amount of energy that will free an electron in the absence of an electric field. Electrons with lower energies, however, can also tunnel out.



(a) Based on qualitative arguments, what total energy do you expect most electrons will have upon leaving the metal?

(b) Quantitatively estimate the tunneling probability as a function of energy.

(c) Plot your result from (b) and briefly discuss it physically.

2. So far in this class, we have considered wavepackets of free particles, wavepackets in an infinite square well, and wavepackets scattering from various 1D potentials. In this problem, we will consider the behavior of a wavepacket in an SHO. Pay special attention to how this wavepacket behaves compared to the others we have studied so far...

Can we construct quantum states of an SHO whose expectation values mimic the classical results? The answer is yes, and they are called "quasi-classical" or "coherent" states. To do so, we first note that the classical results can be written as ($\beta = \sqrt{\hbar/m\omega}$):

$$
\begin{aligned}
x(t) &= \frac{\beta}{\sqrt{2}} \left(\alpha_0 e^{-i\omega t} + \alpha_0^* e^{i\omega t}\right) \\
p(t) &= -\frac{i\hbar}{\beta\sqrt{2}} \left(\alpha_0 e^{-i\omega t} - \alpha_0^* e^{i\omega t}\right).
\end{aligned}
$$

(By the way, is there any problem with having an $\hbar$ in a classical expression like this?)

(a) Using the Heisenberg equation of motion for the matrix element $\langle \mathbf{a}\rangle(t) = \langle \psi(t)|\mathbf{a}|\psi(t)\rangle$ (with $\mathbf{a}$ the lowering operator),

$$i\hbar \frac{d}{dt}\langle \mathbf{a}\rangle(t) = \langle [\mathbf{a}, H]\rangle(t),$$

find $\langle \mathbf{a}\rangle(t)$. Similarly, find $\langle \mathbf{a}^\dagger\rangle(t)$, and thus $\langle X\rangle(t)$ and $\langle P\rangle(t)$. Note that it is necessary to have $\langle \mathbf{a}\rangle(0) = \alpha_0$ to obtain agreement with the classical results.

(b) By requiring the mean value of H to be the same as the classical total energy using $x(t)$ and $p(t)$ above, show that

$$\langle \mathbf{a}^\dagger \mathbf{a}\rangle(0) = |\alpha_0|^2.$$

Based on this relation, what does $|\alpha_0|^2$ represent physically?

(c) Introducing the new operator

$$\mathbf{b} = \mathbf{a} - \alpha_0$$

show that we must have

$$\mathbf{a}|\psi(0)\rangle = \alpha_0|\psi(0)\rangle$$

in order to satisfy the conditions $\langle \mathbf{a}\rangle(0) = \alpha_0$ and $\langle \mathbf{a}^\dagger \mathbf{a}\rangle(0) = |\alpha_0|^2$. (HINT: Evaluate $\langle \mathbf{b}^\dagger \mathbf{b}\rangle(0)$.)

(d) Find $|\alpha\rangle = |\psi(0)\rangle$, the eigenket of $\mathbf{a}$ assuming

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n(\alpha)|n\rangle,$$

i.e. find the expansion coefficients $c_n(\alpha)$. The state $|\alpha\rangle$ is exactly the coherent state we sought.

(e) For an oscillator in the coherent state $|\alpha\rangle$, what values of energies can be obtained and with what probability? Which energy is most likely?

(f) Calculate $|\alpha(t)\rangle$. Is this still an eigenstate of $\mathbf{a}$? If so, what is its eigenvalue? If $\mathbf{a}$ were measured, what results could be obtained and with what probabilities?

(g) Calculate $\Delta X(t)$, $\Delta P(t)$, and $\Delta X(t)\Delta P(t)$ for $|\alpha(t)\rangle$. Contrast the behavior of this wavepacket in a SHO with the free Gaussian wavepackets you've worked with so far.

(h) Using the relation (where $|0\rangle = |n = 0\rangle$)

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha \mathbf{a}^\dagger} e^{-\alpha^* \mathbf{a}}|0\rangle,$$

find $\psi_\alpha(x) = \langle x|\alpha\rangle$. HINT: Rewrite $\mathbf{a}$ and $\mathbf{a}^\dagger$ in terms of the operators X and P, then put everything in the $\{|x\rangle\}$ representation. You might need the following relations:

$$
\begin{aligned}
\langle x|e^{-i\lambda \frac{P}{\hbar}} &= \langle x - \lambda| \\
e^A e^B &= e^{A+B} e^{\frac{1}{2}[A,B]}
\end{aligned}
$$

where λ is a real constant and A, B are operators. ($e^{-i\lambda \frac{P}{\hbar}}$ is called a translation operator.) Plot the probability density $|\psi_\alpha(x)|^2$ for two choices of α.

(i) Using your result from (f), show that $\psi_\alpha(x, t)$ is obtained simply by replacing α in $\psi_\alpha(x)$ by $\alpha(t)$. Calculate $|\psi_\alpha(x, t)|^2$ and discuss [consider especially (g)]. What is special about these coherent states? Plots might help.

3. A charged particle experiences the oscillator potential

$$V = \frac{1}{2}m\omega^2 X^2.$$

The whole system is then placed in a uniform electric field so that there is an additional potential

$$W = -q\mathcal{E}X$$

where q is the carge and $\mathcal{E}$ the magnitude of the electric field.

(a) Plot the total potential for this system with $\mathcal{E} \neq 0$.

(b) Calculate the eigenstates and energies for this system.

(c) This is a reasonable model of an electron bound to an atom placed in an electric field — the classical version is often used to explain the index of refraction in materials, for instance. A useful parameter for an atomic state is its polarizability. Calculate the polarizability and the induced dipole moment for the state $|n\rangle$. HINT: The polarizability α is defined from the energy as $E = -\frac{1}{2}\alpha\mathcal{E}^2$.

4. Let's place our charged oscillator from Prob. 3 in a time-varying electric field $\mathcal{E}(t)$. The total potential is thus

$$V(X,t) = \frac{1}{2}m\omega^2 X^2 - q\mathcal{E}(t)X.$$

Let's see if our coherent state still behaves like a classical particle.

(a) Sketch a picture of what you expect to happen as a function of time for $\mathcal{E}(t) = \mathcal{E}_0 \sin\omega' t$ for some choice of α.

(b) The number $\alpha(t) = \langle\psi(t)|\mathbf{a}|\psi(t)\rangle$ evolves according to [see Prob. 2(a)]

$$\frac{d}{dt}\alpha(t) = -i\omega\alpha(t) + i\lambda(t)$$

with

$$\lambda(t) = \frac{q}{\sqrt{2m\hbar\omega}}\mathcal{E}(t).$$

Integrate this equation to find $\alpha(t)$ and $\alpha^*(t)$.

(c) Calculate $\langle X\rangle(t)$ and $\langle P\rangle(t)$ and compare with the classical result. Does the coherent state still mimic a classical particle?

(d) Assume that at $t = 0$, $|\psi(t)\rangle = |n = 0\rangle$. Take

$$\mathcal{E}(t) = \mathcal{E}_0 \sin\omega' t$$

for $0 \leq t \leq T$ and zero otherwise. If the energy is measured for some time $t > T$, what results can be found and with what probabilities? Consider the case when $\omega' = \omega$ and when $\omega' \neq \omega$. Why are these results different?