Homework 5

Due in class Oct. 7

1. For HW #4, you calculated R and T for

$$V(x) = V_0 a\delta(x-a) + \begin{cases} V_1 & \text{if } x \le 0 \\ 0 & \text{otherwise} \end{cases}.$$

Re-calculate R and T for this potential, this time using the transfer matrix technique.

2. For the potential

$$V(x) = -V_0 a\left[\delta(x-a) + \delta(x+a)\right]$$

with a and V_0 positive, real constants:

(a) What do you expect the transmission coefficient to look like? Make a sketch. If you expect resonances, try to estimate what their energies will be.

(b) Calculate R and T for scattering from this potential. Plot R and T and discuss them physically. Do they match your predictions from (a)?

(c) What happens to R and T in the limit $V_0 \to 0$? $V_0 \to \infty$? Explain these results physically.

3. In Prob. 2, you examined the scattering states of the double δ-function potential. In this problem, you will calculate the bound states.

(a) What symmetries does $V(x)$ from Prob. 2 possess?

(b) Can you construct simultaneous eigenstates of energy and the symmetries you identified in (a)? Why or why not?

(c) For a single δ-function potential, you know there is only a single bound state, so it should be plausible that for two δ's there are at most two bound states. Find the transcendental equations that determine the energies of these states. Sketch or plot the corresponding wave functions and point out the symmetries from (a) and (b).

(d) What are the energies of each state in the limits $a \to 0$ and $a \to \infty$? Do your results make sense physically? Explain.

4. A particle of mass m is confined in an infinite square well of width a centered at the origin. At $t = 0$, the particle's wave function is given by

$$\psi(x,0) = \mathcal{N} \begin{cases} \cos^2\left[4\pi(x - \frac{a}{4})\right] & \frac{a}{8} \le x \le \frac{3a}{8} \\ 0 & \text{otherwise} \end{cases}.$$

(a) Find $\mathcal{N}$ and $\psi(x,t)$.

(b) At time t, what values of the energy can be measured and with what probabilities?

(c) Plot $|\psi(x,t)|^2$ for enough values of t to get a sense of the motion. Describe what is happening physically.

5. Consider the following potential:

$$V(x) = \begin{cases} \infty & x > 0 \\ 0 & x \le 0 \end{cases}.$$

The initial state of the system is

$$\psi(x,0) = \mathcal{N} e^{ik_0 x} e^{-\frac{1}{2}\left(\frac{x-x_0}{\Delta}\right)^2}$$

with $k_0 > 0$. Also, $\Delta \ll x_0$ and $x_0 < 0$ such that $\psi(x=0,0)$ is zero for all practical purposes.

(a) Sketch the potential. What boundary conditions should a wave function in this potential satisfy?

(b) Find the time-dependent wave function $\psi(x,t)$. Make sure it satisfies the boundary conditions!

(c) The potential is completely reflecting, so the wavepacket should simply bounce off, right? Prove that this is the case by calculating $\langle P\rangle$ long after the wavepacket reflects from the wall.

(d) Plot the probability density $|\psi(x,t)|^2$ for times when the wavepacket is being reflected. Explain what you see physically.

(e) Calculate $\langle X\rangle(t)$ and discuss the connection with the classical result. Doing this analytically may prove challenging, so you might consider evaluating the necessary integrals numerically for some choice of wavepacket parameters at enough times to plot a curve.

This question is just to get you thinking — you don't need to turn it in. When you have considered simple collisions of two particles classically, you have imagined two particles with well-defined momenta p_i and p_i' colliding and leaving with momenta p_f and p_f'. For elastic collisions, momentum and energy are conserved. In quantum mechanics, we get well-defined momenta only with momentum eigenstates. But, you know that momentum eigenstates are given by

$$\psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{i\frac{p}{\hbar}x}.$$

(a) What is the most likely position x of a quantum mechanical particle with well-defined momentum p? Given this, what are the problems with describing a collision of two such particles quantum mechanically?

(b) In Homework 3, you found the time-dependent wave function for a particle whose initial state was described by a Gaussian wavepacket. Would the problems you identified in (a) be solved by considering the collision of wavepackets instead of plane waves?

(c) Are there any problems with describing a collision in terms of wavepackets? Think about what would typically be measured in such a collision, what quantities they depend on, and any uncertainty in each.

(d) Is there a limit in which the wavepacket results should match the plane wave results? Given that plane waves are much easier to work with, what would be your strategy for solving quantum mechanical collision problems? What assumptions are you making in the strategy you chose?