Homework 4

Due in class Sept. 25

1. Neutral K-mesons are created in one of two states: $|K^0\rangle$ or $|\bar{K}^0\rangle$. These states are not eigenstates of the Hamiltonian that describes the system. Instead,

$$H = m\left(|K^0\rangle\langle K^0| + |\bar{K}^0\rangle\langle\bar{K}^0|\right) + \varepsilon\left(|\bar{K}^0\rangle\langle K^0| + |K^0\rangle\langle\bar{K}^0|\right).$$

(a) Find the representation of the operator H in the $\{|K^0\rangle, |\bar{K}^0\rangle\}$ basis.

(b) Calculate the energy eigenvalues and corresponding eigenvectors.

(c) Show that if the state $|K^0\rangle$ is created at time $t = 0$, then the probability that the state is $|K^0\rangle$ at a later time t oscillates periodically between 0 and 1.

(d) Find the period of oscillation if m=495 MeV and ε=5 eV.

2. Consider the potential

$$V(x) = V_0 a\delta(x-a) + \begin{cases} V_1 & \text{if } x \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Sketch this potential and discuss qualitatively what features you expect for R and T.

(b) Calculate R and T for all possible energies E. Plot your results for both left and right incidence. Were your expectations in (a) correct?

(c) Do your expressions for R and T make sense? Verify by checking their limits in E and their limits for $V_0 \to 0$ and $V_1 \to 0$.

3. Consider a free electron whose initial wave function is given by a Gaussian wave packet.

(a) Use units such that $\hbar = m = 1$, and initial width of $\Delta = 1$, and an initial velocity of 0.1, find a way to plot or sketch the probability density $|\psi(x,t)|^2$ at several ($\approx$10) times t from -10 to 10.

(b) Describe the main features of the wave packet's evolution, especially the comparison with your expectations from the classical motion of a free particle and from the uncertainty principle.

(c) Plot $\langle X\rangle(t)$ and $\Delta X(t)$ under the conditions of (a). Explain whether (and why) these are consistent with the plots in (a).

(d) For the conditions in (a), plot the probability density $|\tilde{\psi}(p,t)|^2$ and discuss its features (*i.e.* $\langle P\rangle(t)$ and $\Delta P(t)$ and the connection with classical mechanics and the uncertainty principle).

(e) Calculate the mean energy of the wave packet $E(t)$ and the width of the wave packet in energy space $\Delta E(t)$.

(f) Now, imagine adding a potential step and scattering Gaussian wave packet from it. Discuss what you think will happen and try to support your expectations as quantitatively as possible. You should consider mean energies both above and below the step. HINT: Think about R and T for this problem.

4. An effective Hamiltonian for a hydrogen atom in a weak static electric field is given by

$$\begin{aligned} H =& \frac{\hbar\omega}{2}\left(-|1s_0\rangle\langle 1s_0| + |2p_0\rangle\langle 2p_0|\right) \\ &+ \frac{\hbar\Omega}{2}|1s_0\rangle\langle 2p_0| + \frac{\hbar\Omega^*}{2}|2p_0\rangle\langle 1s_0| \end{aligned}$$

neglecting all other levels. The zero of energy is chosen to be halfway between the two levels. The number Ω is a product of the dipole matrix element between $|1s_0\rangle$ and $|2p_0\rangle$, the magnitude of the electric field, and some constants.

(a) Find the representation of the operator H in the $\{|1s_0\rangle, |2p_0\rangle\}$ basis and calculate the energy eigenvalues and corresponding eigenvectors.

(b) The system is initially in the state $|2p_0\rangle$. Find $|\psi(t)\rangle$.

(c) If the energy is measured at some time $t > 0$, what values can be obtained and with what probabilities?

(d) What is the probability for the system to again be in the $|2p_0\rangle$ state at $t > 0$? Show that it oscillates between 0 and 1; calculate the period of oscillation.

(e) Using what you've learned in (a)–(d), describe an experiment that would let you measure the magnitude of the dipole matrix element between the $1s_0$ and $2p_0$ states. Could you also measure the phase of the matrix element and thus the relative phase between the $1s_0$ and $2p_0$ states?

5. A particle of mass m experiences the potential

$$V(x) = V_0 a\delta(x) + \begin{cases} \infty & |x| > a \\ 0 & |x| \leq a \end{cases}.$$

(a) List any symmetries this potential possesses.

(b) Find the bound state energies. You need only obtain the transcendental equations and indicate their solution graphically.

(c) What happens to the bound state energy in the limits $V_0 \to 0$ and $V_0 \to \infty$? Do these results make sense?

Extra Credit

A particle of mass m is confined to move around a ring of radius a. There is thus only one degree of freedom — the angle ϕ. The particle sees a potential given by

$$V(\phi) = \begin{cases} +V_0 & \text{if } 0 \leq \phi \leq \frac{\pi}{2} \\ 0 & \text{otherwise.} \end{cases}$$

What are the allowed energies for this system?