Homework 3

Due in class Sept. 18

1. Suppose $|i\rangle$ and $|j\rangle$ are eigenkets of some Hermitian operator $\hat{A}$. Under what condition(s) can we conclude that $|i\rangle + |j\rangle$ is also an eigenket of $\hat{A}$? Justify your answer.

2. The Hamiltonian for a two-state system is given by

$$\hat{H} = a\left(|1\rangle\langle 1| - |2\rangle\langle 2|\right) + b\left(|2\rangle\langle 1| + |1\rangle\langle 2|\right)$$

where a and b are constants with units of energy.

(a) Calculate the matrix representing $\hat{H}$ in the basis $\{|1\rangle, |2\rangle\}$.

(b) Find the energy eigenvalues and the corresponding eigenkets.

(c) Suppose there was a typo, and the Hamiltonian instead read

$$\hat{H} = a\left(|1\rangle\langle 1| - |2\rangle\langle 2|\right) + b|1\rangle\langle 2|.$$

What principle is now violated? Illustrate your point explicitly by attempting to solve the most general time-dependent problem using this Hamiltonian.

3. Consider a three-dimensional ket space. If a certain set of orthonormal kets — say $|1\rangle$, $|2\rangle$, and $|3\rangle$ — are used as the basis kets, the operators A and B are represented by

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \quad B = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}$$

with both a and b real.

(a) Obviously A exhibits a degenerate spectrum (why obviously?). Does B also have a degenerate spectrum?

(b) Show that A and B commute.

(c) Find a new set of orthonormal kets that are simultaneous eigenkets of both A and B. Specify the eigenvalues of A and B associated with each eigenket. Does your specification completely characterize each ket, *i.e.* is there a unique label for each ket?

4. Given the following Hamiltonians in 3D space, state (i) what physical system the Hamiltonian might describe and (ii) the conserved quantities (if any):

$$H = -\frac{\hbar^2}{2m}\nabla^2 \tag{1}$$

$$H = -\frac{\hbar^2}{2m}\nabla^2 + e\mathcal{E}z \tag{2}$$

$$H = -\frac{\hbar^2}{2m}\nabla^2 - \frac{1}{4\pi\epsilon_0}\frac{Ze^2}{r} \tag{3}$$

$$H = -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega^2(x^2+y^2+z^2) \tag{4}$$

$$H = -\frac{\hbar^2}{2m}\nabla^2 + \alpha\sin(\omega t)r^2 \tag{5}$$

where m, e, $\mathcal{E}$, Z, ω, and α are constants.

5. Show that for a real function $\psi(x)$, the expectation value of momentum is zero, $\langle\hat{P}\rangle = 0$.

6. Show that if $\psi(x)$ has mean momentum $\langle\hat{P}\rangle = P$, then the wave function $e^{ip_0x}\psi(x)$ has mean momentum $\langle\hat{P}\rangle = P + p_0$.

7. Consider the following operators:

$$L_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad L_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(a) What are the possible values one can obtain if L_z is measured?

(b) Take the state in which $L_z = 1$. In this state, what are $\langle L_y\rangle$, $\langle L_y^2\rangle$, and ΔL_y?

(c) Find the normalized eigenstates and the eigenvalues of L_y in the L_z basis.

(d) If the particle is in the state with $L_z = -1$ and L_y is measured, what are the possible outcomes and their probabilities?

(e) Consider the state

$$|\chi\rangle \equiv \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix}$$

in the L_z basis. If L_z^2 is measured in this state and a result $+1$ is obtained, what is the state after the measurement? How probable was this result? If L_z is measured instead, what are the outcomes and respective probabilities?

(f) A particle is in a state for which the probabilities are

$$P(L_z\!=\!1) = \frac{1}{4},\; P(L_z\!=\!0) = \frac{1}{2},\; P(L_z\!=\!-1) = \frac{1}{4}.$$

Verify by direct calculation that the most general normalized state consistent with these probabilities is

$$|\varphi\rangle = \frac{e^{i\delta_{+1}}}{2}|L_z\!=\!+1\rangle + \frac{e^{i\delta_0}}{\sqrt{2}}|L_z\!=\!0\rangle + \frac{e^{i\delta_{-1}}}{2}|L_z\!=\!-1\rangle.$$

We have discussed that if $|\psi\rangle$ is a normalized state then $e^{i\theta}|\psi\rangle$ is a physically equivalent state. Does this mean that the factors $e^{i\delta}$ multiplying the kets in $|\varphi\rangle$ are irrelevant for *all* measurements? Give an example to support your argument. HINT: The best example would be the computation of a physical observable like the probability.

8. Consider a particle of mass m in the potential:

$$V(x) = \begin{cases} 0 & \text{if } 0 \le x \le a \\ +\infty & \text{otherwise.} \end{cases}$$

$|\phi_n\rangle$ are the eigenstates of the Hamiltonian H of the system, and their eigenvalues are

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}, \quad n = 1, 2, 3, \ldots$$

The state of the particle at the instant $t = 0$ is:

$$|\psi(0)\rangle = a_1|\phi_1\rangle + a_2|\phi_2\rangle + a_3|\phi_3\rangle.$$

(a) What is the probability, when the energy of the particle in the state $|\psi(0)\rangle$ is measured, of finding a value smaller than $\frac{2\pi^2\hbar^2}{ma^2}$?

(b) What is the mean value and what is the root-mean-square deviation of the energy of the particle in the state $|\psi(0)\rangle$?

(c) Calculate the state vector $|\psi(t)\rangle$ at the instant t. Do the results found in (a) and (b) above at the instant $t = 0$ remain valid at an arbitrary time t?

(d) When the energy is measured, the result $\frac{\pi^2\hbar^2}{2ma^2}$ is found. After the measurement, what is the state of the system? What is the result if the energy is measured again?

9. Consider a free particle whose initial wave function is given by the Gaussian wave packet

$$\psi(x, 0) = \mathcal{N} e^{ip_0 x} e^{-\frac{1}{2}\left(\frac{x}{\Delta}\right)^2}$$

In solving this problem, you might want to refer to Prob. 2.22 in Griffiths.

(a) Find the normalization constant $\mathcal{N}$.

(b) What are $\langle x\rangle$ and $\langle p\rangle$ for this wave function? You should not need to do any lengthy integrals for your answer — use instead symmetry and Probs. 5 and 6.

(c) Calculate the corresponding momentum space wave function and write the coordinate space wave function as an expansion on momentum space basis functions.

(d) Using your expansion from (c), write down the wave function at an arbitrary time t. Then, carry out the Fourier transform (do the momentum space integral) and find the explicit form of $\psi(x, t)$.

(e) Use your wave function from (d) to calculate $\langle x\rangle(t)$ and $\Delta x(t)$ (you can compare with the answers in Prob. 2.22 in the text). Does $\langle x\rangle(t = 0)$ agree with your answer to (b)? Sketch both functions and interpret them physically.

(f) Now, calculate $\langle p\rangle(t)$. HINT: You can answer this with an easy integral or a hard integral — look through your answers to the parts above to find the way to the easy integral. Does $\langle p\rangle(t = 0)$ agree with your answer to (b)? Interpret your result physically.

(g) Find $|\psi(x, t)|^2$. Sketch it at $t = 0$ and at some large t. Based on these, describe the qualitative behavior of $|\psi(x, t)|^2$.

10. This problem is loosely based on Prob. 3.38 from Griffiths. The Hamiltonian for some three-level system is represented by the matrix

$$H = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Two other observables, A and B, are represented by the matrices

$$A = a \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad B = b \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The constants ω, a, and b are positive and real.

(a) Find the eigenvalues and normalized eigenvectors of all three operators. To the maximum extent possible, find common eigenvectors.

(b) Suppose the system starts out in the state

$$|\chi(0)\rangle \equiv \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

where the c_i are complex constants chosen such that $|\chi\rangle$ is normalized. Calculate the expectation values of H, A, and B.

(c) What is the state of the system $|\chi(t)\rangle$ at an arbitrary time t? If you measured the energy at some time t, what values could be obtained and with what probabilities? Answer the same questions for A and B.

(d) Suppose that B is measured at $t = 0$ and the result b is obtained. The energy is then immediately measured. What values are possible and with what probabilities? How do your answers change if the energy is not immediately measured, but measured instead at some later time t? Briefly explain your answers in words.

(e) Repeat (d), but now assume that A was measured at $t = 0$, giving the result $-a$.

Extra credit: What quantitities are conserved for the following Hamiltonian

$$H = -\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + \frac{1}{2}\mu\omega^2|\mathbf{r}_1 - \mathbf{r}_2|^2$$

with m_i and ω constants?