Homework 11

Due in class Nov. 20

From Griffiths: 4.24, 4.31, 4.33, 4.35, 4.49

Extra Credit: 4.56, 4.61

1. Given $\ell_1 = 1$ and $\ell_2 = 4$ (note that you may calculate any Clebsch-Gordan coefficients by whatever method you like):

(a) Calculate the possible values of the total angular momentum $\mathbf{L} = \mathbf{L_1} + \mathbf{L_2}$.

(b) Write down the explicit expressions in terms of $|\ell_1 m_1\rangle|\ell_2 m_2\rangle$ for the states

  (*i*) $|(14)3-3\rangle$

  (*ii*) $|(14)4-3\rangle$

  (*iii*) $|(14)5-5\rangle$

  (*iv*) $|(14)54\rangle$.

(c) Turn the expansion around and write the state $|11\rangle|4-3\rangle$ as an expansion on eigenstates of total angular momentum.

2. Consider a hydrogen atom in a constant external electric field $\vec{\mathcal{E}} = \mathcal{E}\hat{z}$. The perturbation is

$$V = -\vec{d} \cdot \vec{\mathcal{E}}$$

where $\vec{d} = -e\vec{r}$ is the dipole operator and $-e$ is the charge of the electron.

(a) Find the matrix representation of the full, non-relativistic Hamiltonian in the basis composed of the $n = 1$ and $n = 2$ states.

(b) By comparing to Prob. 3 of HW #9, write down the explicit values of the matrix elements $d_{i,j}$ defined there.

(c) Calculate the correction to the ground state energy in first order perturbation theory.

(d) Calculate the corrections to the $n = 2$ states in first order perturbation theory.

(e) For the $m = 0$ states, make a physical argument supporting the direction of the energy shifts relative to the unperturbed energies. (A good starting point is the electron density described by the wave functions.)

The behavior here is called the Stark effect; and the states, the Stark states. (HINTS: Write z in terms of Y_{lm}'s and notice that

$$\int d\Omega\ Y_{00}^* Y_{lm} Y_{l'm'} = \frac{1}{\sqrt{4\pi}} \int d\Omega\ Y_{lm} Y_{l'm'}$$

since $Y_{00} = 1/\sqrt{4\pi}$.)

3. Since L^2 and L_z commute with Π (parity), they should share a basis with it. Verify that

$$\Pi Y_{\ell m} = (-)^\ell Y_{\ell m}.$$

(Hint: first show that $\theta \to \pi - \theta$ and $\phi \to \phi + \pi$ under parity. Then use the properties of spherical harmonics.)

4. Consider two *different* spin-$\frac{1}{2}$ particles whose Hamiltonian is completely specified as $H = c\mathbf{S}_1 \cdot \mathbf{S}_2$ where c is a real constant.

(a) Write down the direct product basis for this problem.

(b) By directly calculating the matrix elements of H in your basis from (a), find the matrix representation of H.

(c) Using the spin-$\frac{1}{2}$ matrix representations of $\mathbf{S}_1$ and $\mathbf{S}_2$, construct H with $\sum_i S_{1i} \otimes S_{2i}$. Make sure that you've done this product in the order consistent with your basis in (a). Does your matrix for H match what you found in (b)? Should it?

(d) Diagonalize H to find the energy eigenstates. What units should c have?

(e) You can also diagonalize H by transforming to the total $\mathbf{S}$ basis (where $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ is the total spin angular momentum). Carry out this procedure and compare your answers to what you found in (d)? Are they the same?

5. Using the same distinguishable spin-$\frac{1}{2}$ particles and Hamiltonian as above:

(a) What are the constants of motion?

(b) If at time $t = 0$ the spin of particle (1) along the z-axis is up and the spin of particle (2) along the z-axis is down, what is the wave function of the system at a later time t?

(c) At a given time $t > 0$, what values of the energy can be measured and with what probabilities?

(d) At $t > 0$, what is the probability that the system is in the same state it was at $t = 0$?