Homework 10

Due in class Nov. 13

NOTE: For this assignment, you may neglect any spin degrees of freedom.

1. Given the bases $\{|+\rangle, |-\rangle\}_{(1)}$ and $\{|+\rangle, |-\rangle\}_{(2)}$ for spaces $\mathcal{V}_1$ and $\mathcal{V}_2$ and the operators

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \Omega = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

that operate in spaces (1) and (2), respectively:

(a) Under what conditions do Λ and Ω represent physical observables?

(b) Find the matrix representations of Λ and Ω in the full direct product space $(1)\otimes(2)$.

(c) Find the matrix representation of $(\Lambda\Omega)_{(1)\otimes(2)}$.

(d) Evaluate $\Lambda\Omega|\psi\rangle$ where

$$|\psi\rangle = \frac{1}{2}|+\rangle|-\rangle + \frac{1}{\sqrt{2}}|+\rangle|+\rangle + \frac{1}{2}|-\rangle|+\rangle.$$

Now, evaluate $\Omega\Lambda|\psi\rangle$ using the same $|\psi\rangle$. Should these answers be the same or different? Why? How do these two answers actually compare?

2. Two particles of mass m_1 and m_2 have positions $\mathbf{r}_1$ and $\mathbf{r}_2$ with respect to some laboratory-fixed coordinate system, and obey the following time-dependent Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r}_1, \mathbf{r}_2, t) = \left[-\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + V(|\mathbf{r}_1 - \mathbf{r}_2|)\right]\Psi(\mathbf{r}_1, \mathbf{r}_2, t).$$

(a) Find the relations between these lab-frame coordinates $(\mathbf{r}_1, \mathbf{r}_2)$ and the center of mass and relative coordinates $(\mathbf{R}, \mathbf{r})$. Also find the relations between the lab-frame momenta $(\mathbf{p}_1, \mathbf{p}_2)$ and the center of mass and relative momenta $(\mathbf{P}, \mathbf{p})$. Assuming that the lab-frame variables satisfy the canonical commutation relations $[\mathbf{r}_i, \mathbf{p}_j] = i\hbar\delta_{ij}$, verifty that your center of mass and relative variable are also properly conjugate.

(b) Using your results from (a), transform the above Schrödinger equation to center of mass and relative coordinates.

(c) What quantities are conserved in your equation from (b)?

(d) Solve your equation from (b) assuming

$$V(|\mathbf{r}_1 - \mathbf{r}_2|) = \frac{1}{2}\frac{\hbar\Omega}{a^2}\left(\mathbf{r}_1 - \mathbf{r}_2\right)^2.$$

Take all constants to be real and positive. Make sure to clearly indicate the relation between your answer to (c) and your solution.

(e) Under what conditions can this Schrödinger equation represent a system of two indistinguishable particles? Verify these conditions directly by evaluating $[P_{12}, H]$ where P_{12} is the particle permutation operator. Be sure to explain how your result does, in fact, support your answer. Does your answer depend on whether the particles are bosons or fermions?

3. When an energy measurement is made on a system of three indistinguishable bosons in a box of width a, the result $44\pi^2\hbar^2/(2ma^2)$ is obtained. Write down the most general wave function consistent with this observation.

4. Imagine a situation in which there are three particles and only three states $|a\rangle, |b\rangle, |c\rangle$ available to them. Show that the total number of allowed, distinct configurations for this system is

(a) 27 if they are distinguishable

(b) 10 if they are indistinguishable bosons

(c) 1 if they are indistinguishable fermions

5. Consider two identical particles of mass m in an infinite square well of width a.

   (a) Assuming the particles are indistinguishable *bosons*, write down the wave functions for the lowest three energy states of the system. What are their degeneracies?

   (b) Assuming the particles are indistinguishable *fermions*, write down the wave functions for the lowest three energy states of the system. What are their degeneracies?

   (c) Using Mathematica or any similar program, plot the wave functions you obtained in (a) and (b). These are functions of two variables, so you will need to plot contours or a surface. Comment on any general features you find.

6. Consider again two identical particles of mass m in an infinite square well of width a. Now, add the interparticle interaction

$$H_1 = \frac{\lambda}{a^2}(x_1 - x_2)^2$$

as a perturbation.

   (a) Assuming the particles are indistinguishable *bosons*, calculate the first order energy shift of the two lowest states.

   (b) Assuming the particles are indistinguishable *fermions*, calculate the first order energy shift of the two lowest states.

   (c) Where possible, compare the energy shifts from (a) and (b) and discuss the physical basis of their difference. That is, why do bosons shift more/less than fermions? You might want to refer back to your pictures in 7(c).