Homework 8

Due in class Oct. 15

1. Can we construct quantum states of an SHO whose expectation values mimic the classical results? The answer is yes, and they are called "quasi-classical" or "coherent" states. To do so, we first note that the classical results can be written as ($\beta = \sqrt{\hbar/m\omega}$):

$$x(t) = \frac{\beta}{\sqrt{2}}\left(\alpha_0 e^{-i\omega t} + \alpha_0^* e^{i\omega t}\right)$$
$$p(t) = -\frac{i\hbar}{\beta\sqrt{2}}\left(\alpha_0 e^{-i\omega t} - \alpha_0^* e^{i\omega t}\right).$$

(By the way, is there any problem with having an $\hbar$ in a classical expression like this?)

(a) Using the Heisenberg equation of motion for the matrix element $\langle \mathbf{a}\rangle(t) = \langle\psi(t)|\mathbf{a}|\psi(t)\rangle$ (with $\mathbf{a}$ the lowering operator),

$$i\hbar\frac{d}{dt}\langle \mathbf{a}\rangle(t) = \langle[\mathbf{a}, H]\rangle(t),$$

find $\langle \mathbf{a}\rangle(t)$. Similarly, find $\langle \mathbf{a}^\dagger\rangle(t)$, and thus $\langle X\rangle(t)$ and $\langle P\rangle(t)$. Note that it is necessary to have $\langle \mathbf{a}\rangle(0) = \alpha_0$ to obtain agreement with the classical results.

(b) By requiring the mean value of H to be the same as the classical total energy for $x(t)$ and $p(t)$ above, show that

$$\langle \mathbf{a}^\dagger \mathbf{a}\rangle(0) = |\alpha_0|^2.$$

(c) Introducing the new operator

$$\mathbf{b} = \mathbf{a} - \alpha_0$$

show that we must have

$$\mathbf{a}|\psi(0)\rangle = \alpha_0|\psi(0)\rangle$$

in order to satisfy the conditions $\langle \mathbf{a}\rangle(0) = \alpha_0$ and $\langle \mathbf{a}^\dagger\mathbf{a}\rangle(0) = |\alpha_0|^2$. (HINT: Evaluate $\langle \mathbf{b}^\dagger\mathbf{b}\rangle(0)$.)

(d) Find $|\alpha\rangle = |\psi(0)\rangle$, the eigenket of $\mathbf{a}$ assuming

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n(\alpha)|n\rangle,$$

i.e. find the expansion coefficients $c_n(\alpha)$. The state $|\alpha\rangle$ is exactly the coherent state we sought.

(e) For an oscillator in the coherent state $|\alpha\rangle$, what values of energies can be obtained and with what probability? Which energy is most likely?

(f) Calculate $|\alpha(t)\rangle$. Is this still an eigenstate of $\mathbf{a}$? If so, what is its eigenvalue?

(g) Calculate $\Delta X(t)$, $\Delta P(t)$, and $\Delta X(t)\Delta P(t)$ for $|\alpha(t)\rangle$. Contrast the behavior of this wavepacket in a SHO with the free Gaussian wavepackets you've worked with so far.

(h) Using the relation (where $|0\rangle = |n=0\rangle$)

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha\mathbf{a}^\dagger} e^{-\alpha^*\mathbf{a}}|0\rangle,$$

find $\psi_\alpha(x) = \langle x|\alpha\rangle$. HINT: Rewrite $\mathbf{a}$ and $\mathbf{a}^\dagger$ in terms of the operators X and P, then put everything in the $\{|x\rangle\}$ representation. You might need the following relations:

$$\langle x|e^{-i\lambda\frac{P}{\hbar}} = \langle x-\lambda|$$
$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}$$

where λ is a real constant and A, B are operators. $e^{-i\lambda\frac{P}{\hbar}}$ is called a translation operator.

(i) Using your result from (f), show that $\psi_\alpha(x,t)$ is obtained simply by replacing α in $\psi_\alpha(x)$ by $\alpha(t)$. Calculate $|\psi_\alpha(x,t)|^2$ and discuss [consider especially (g)].

2. Let's place our charged oscillator from HW 7 in a time-varying electric field $\mathcal{E}(t)$. The total potential is thus

$$V(X,t) = \frac{1}{2}m\omega^2 X^2 - q\mathcal{E}(t)X.$$

(a) Let's see if our coherent state still behaves like a classical particle. The number $\alpha(t) = \langle\psi(t)|\mathbf{a}|\psi(t)\rangle$ evolves according to [see Prob. 1(a)]

$$\frac{d}{dt}\alpha(t) = -i\omega\alpha(t) + i\lambda(t)$$

with

$$\lambda(t) = \frac{q}{\sqrt{2m\hbar\omega}}\mathcal{E}(t).$$

Integrate this equation to find $\alpha(t)$ and $\alpha^*(t)$.

(b) Calculate $\langle X\rangle(t)$ and $\langle P\rangle(t)$ and compare with the classical result. Does the coherent state still mimic a classical particle?

(c) Assume that at $t=0$, $|\psi(t)\rangle = |n=0\rangle$. Take

$$\mathcal{E}(t) = \mathcal{E}_0 \sin\omega' t$$

for $0 \le t \le T$ and zero otherwise. If the energy is measured for some time $t > T$, what results can be found and with what probabilities? Consider the case when $\omega' = \omega$ and when $\omega' \ne \omega$.

From Shankar: Exercises 17.2.1, 17.3.2.

Note that the spin matrices needed for 17.3.2 are

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad S_y = i\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$S_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Don't worry about the spin aspect of this problem, you need only construct the 3×3 Hamiltonian matrices H_0 and H_1. Also, the language "stable under the perturbation" just means to do degenerate perturbation theory if you need to.