Homework 4

Due in class Sept. 16

1. Suppose $|i\rangle$ and $|j\rangle$ are eigenkets of some Hermitian operator A. Under what condition(s) can we conclude that $|i\rangle + |j\rangle$ is also an eigenket of A? Justify your answer.

2. The Hamiltonian operator for a two-state system is given by

$$H = a\left(|1\rangle\langle 1| - |2\rangle\langle 2|\right) + b\left(|2\rangle\langle 1| + |1\rangle\langle 2|\right)$$

where a and b are constants with units of energy.

(a) Calculate the matrix elements of H in the basis $\{|1\rangle, |2\rangle\}$.

(b) Find the energy eigenvalues and the corresponding eigenkets.

(c) Suppose there was a typo, and the Hamiltonian instead read

$$H = a\left(|1\rangle\langle 1| - |2\rangle\langle 2|\right) + b|1\rangle\langle 2|.$$

What principle is now violated? Illustrate your point explicitly by attempting to solve the most general time-dependent problem using this Hamiltonian.

3. Consider a three-dimensional ket space. If a certain set of orthonormal kets — say $|1\rangle$, $|2\rangle$, and $|3\rangle$ — are used as the basis kets, the operators A and B are represented by

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \qquad B = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}$$

with both a and b real.

(a) Obviously A exhibits a degenerate spectrum (why?). Does B also have a degenerate spectrum?

(b) Show that A and B commute.

(c) Find a new set of orthonormal kets that are simultaneous eigenkets of both A and B. Specify the eigenvalues of A and B associated with each eigenket. Does your specification completely characterize each ket, *i.e.* is there a unique label for each ket?

4. Neutral K-mesons are created in one of two states: $|K^0\rangle$ or $|\bar{K}^0\rangle$. These states are not eigenstates of the Hamiltonian that describes the system. Instead,

$$H = m\left(|K^0\rangle\langle K^0| + |\bar{K}^0\rangle\langle \bar{K}^0|\right) + \varepsilon\left(|\bar{K}^0\rangle\langle K^0| + |K^0\rangle\langle \bar{K}^0|\right).$$

(a) Find the representation of the operator H in the $\{|K^0\rangle, |\bar{K}^0\rangle\}$ basis.

(b) Calculate the energy eigenvalues and corresponding eigenvectors.

(c) Show that if the state $|K^0\rangle$ is created at time $t = 0$, then the probability that the state is $|K^0\rangle$ at a later time t oscillates periodically between 0 and 1.

(d) Find the period of oscillation if m=495 MeV and ε=5 eV.

5. In the past, you have described collisions of two classical particles as two particles with well-defined momenta p_i and p_i' colliding and leaving with momenta p_f and p_f'. For elastic collisions, momentum and energy are conserved. In quantum mechanics, we get well-defined momenta only with momentum eigenstates. But, you know that momentum eigenstates are given by

$$\psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{-i\frac{p}{\hbar}x}.$$

(a) What is the most likely position x of a quantum mechanical particle with well-defined momentum p? Given this, what are the problems with describing a collision of two such particles quantum mechanically?

(b) In Homework 3, you found the time-dependent wave function for a particle whose initial state was described by a Gaussian wavepacket. Would the problems you identified in (a) be solved by considering the collision of wavepackets instead of plane waves?

(c) Are there any problems with describing a collision in terms of wavepackets? Think about what would typically be measured in such a collision, what quantities they depend on, and any uncertainty in each.

(d) Is there a limit in which the wavepacket results should match the plane wave results? Given that plane waves are much easier to work with, what would be your strategy for solving quantum mechanical collision problems? What assumptions are you making in the strategy you chose?