Homework 13
Due in class Dec. 2

1. Consider two *different* spin-$\frac{1}{2}$ particles whose Hamiltonian is completely specified as $H = c\mathbf{S}_1 \cdot \mathbf{S}_2$ where c is a real constant.

(a) Write down the direct product basis for this problem.

(b) By directly calculating the matrix elements of H in your basis from (a), find the matrix representation of H.

(c) Using the spin-$\frac{1}{2}$ matrix representations of $\mathbf{S}_1$ and $\mathbf{S}_2$, construct H with $\sum_i S_{1i} \otimes S_{2i}$. Make sure that you've done this product in the order consistent with your basis in (a). Does your matrix for H match what you found in (b)? Should it?

(d) Diagonalize H to find the energy eigenstates. What units should c have?

(e) You can also diagonalize H by transforming to the total $\mathbf{S}$ basis (where $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ is the total spin angular momentum). Carry out this procedure and compare your answers to what you found in (d)? Are they the same?

2. Using the same distinguishable spin-$\frac{1}{2}$ particles and Hamiltonian as above:

(a) What are the constants of motion?

(b) If at time $t = 0$ the spin of particle (1) along the z-axis is up and the spin of particle (2) along the z-axis is down, what is the wave function of the system at a later time t?

(c) At a given time $t > 0$, what values of the energy can be measured and with what probabilities?

(d) At $t > 0$, what is the probability that the system is in the same state it was at $t = 0$?

3. Fort the distinguishable spin-$\frac{1}{2}$ particles and Hamiltonian in (a), the perturbation

$$H_1 = \mathcal{F}_0(S_{1y} + S_{2y})\cos\omega_0 t \qquad -\frac{T}{2} \leq t \leq \frac{T}{2}$$

is applied. The pulse is long, $T = 20\frac{2\pi}{\omega_0}$, and $\omega_0 = 2c\hbar$.

(a) Assuming the initial state is the ground state, calculate the probabilities to be in each of the excited states after the pulse in lowest order perturbation theory. Can you identify a selection rule for this perturbation?

(b) Calculate the probability to remain in the ground state to the same order in the perturbation. Is unitarity preserved?

(c) You should be able to obtain numerical answers to (a) and (b) (with $\mathcal{F}_0$). For what values of $\mathcal{F}_0$ are your answers valid?

(d) Repeat (a)–(c) if the perturbation is instead

$$H_1 = \mathcal{F}_0(S_{1z} + S_{2z})\cos\omega_0 t \qquad -\frac{T}{2} \leq t \leq \frac{T}{2}.$$

4. The motion of two *identical* spin-$\frac{1}{2}$ particles is governed by the Hamiltonian

$$H = c\mathbf{S}_1 \cdot \mathbf{S}_2.$$

(a) Find the eigenenergies and eigenstates. You may ignore the spatial degrees of freedom.

(b) The perturbation

$$H' = \gamma(S_{1x} + S_{2x})$$

is applied to the system. Calculate the energy shifts in lowest order perturbation theory.

(c) The perturbation

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is applied to the system. Calculate the energy shifts in lowest order perturbation theory. Compare your answers to (b) and comment on their relation.

5. Consider the spin-orbit interaction in the hydrogen atom,

$$W = \frac{e^2}{2m^2c^2}\frac{\mathbf{L} \cdot \mathbf{S}}{r^3}$$

where e and m are the charge and mass of the electron, $\mathbf{L}$ is its orbital angular momentum, $\mathbf{S}$ is its spin, and r is its distance from the proton.

(a) Calculate the first-order energy shift of the ground state.

(b) Calculate the first-order energy shifts of the $n = 2$ states.

This correction is one of the relativistic fine-structure corrections to the simple hydrogen energy spectrum that you're used to. (HINT: Notice that $J^2 = L^2 + S^2 + 2\mathbf{L} \cdot \mathbf{S}$.)

6. If we take the e-e interaction to be a perturbation in the helium atom, then the zero-th order eigenstates are just hydrogenic solutions with $Z = 2$ when spin-orbit and higher-order interactions are neglected.

(a) Write down the ground singlet and triplet states (properly symmetrized and angular momentum eigenstates!). Argue that orbital and spin angular momenta are separately conserved.

(b) Calculate to first order the shifts to the states in (a) due to the electron-electron interaction. Do the directions of the shifts make sense? (HINT: Consider the two-electron density.)

(c) Repeat (a) and (b) for the H^- ion. Is H^- stable in this approximation, or is it more favorable to break up into $H+e$? (The one with lower energy is more favorable.)