Homework 12

Due in class Nov. 17

1. Consider a particle in a state described by

$$\psi(x) = \mathcal{N}\left(x + ye^{i\frac{\pi}{4}} - z\right)e^{-\alpha r}$$

where $\mathcal{N}$ is a normalization constant and α is a positive real constant. Following the general method outlined in Exercise 12.5.13, calculate:

(a) The relative probability that a measurement will yield $\ell = 0, 1, 2, \ldots$.

(b) The relative probability that a measurement will yield $m = \ldots, -1, 0, 1, \ldots$.

(c) The relative probability that a measurement will yield $\ell = 1$ and $m = -1$.

2. Since L^2 and L_z commute with Π (parity), they should share a basis with it. Verify that

$$\Pi Y_{\ell m} = (-)^{\ell} Y_{\ell m}.$$

(Hint: first show that $\theta \to \pi - \theta$ and $\phi \to \phi + \pi$ under parity. Then use the properties of spherical harmonics.)

3. Given $\ell_1 = 1$ and $\ell_2 = 4$ (note that you may calculate any Clebsch-Gordan coefficients by whatever method you like):

(a) Calculate the possible values of the total angular momentum $\mathbf{L} = \mathbf{L_1} + \mathbf{L_2}$.

(b) Write down the explicit expressions in terms of $|\ell_1 m_1\rangle|\ell_2 m_2\rangle$ for the states

  (*i*) $|(14)3-3\rangle$

  (*ii*) $|(14)4-3\rangle$

  (*iii*) $|(14)5-5\rangle$

  (*iv*) $|(14)54\rangle$.

(c) Turn the expansion around and write the state $|11\rangle|4{-}3\rangle$ as an expansion on eigenstates of total angular momentum.

4. Consider a hydrogen atom in a constant external electric field $\vec{\mathcal{E}} = \mathcal{E}\hat{z}$.

(a) Calculate the correction to the ground state energy in first order perturbation theory.

(b) Calculate the corrections to the $n = 2$ states in first order perturbation theory.

(c) For the $m = 0$ states, argue that the direction of the energy shift relative to the unperturbed energies makes sense. (A good starting point is the electron density described by the wave functions.)

(d) Sketch the energies you obtained in (a) and (b) as a function of the magnitude of the electric field $\mathcal{E}$. For what values of $\mathcal{E}$ are these results valid?

The behavior here is called the Stark effect; and the states, the Stark states. (HINTS: Write z in terms of Y_{lm}'s and notice that

$$\int d\Omega\; Y_{00}^{*} Y_{lm} Y_{l'm'} = \frac{1}{\sqrt{4\pi}} \int d\Omega\; Y_{lm} Y_{l'm'}$$

since $Y_{00} = 1/\sqrt{4\pi}$.)

5. Consider the spin-orbit interaction in the hydrogen atom,

$$W = \frac{e^2}{2m^2c^2}\frac{\mathbf{L}\cdot\mathbf{S}}{r^3}$$

where e and m are the charge and mass of the electron, $\mathbf{L}$ is its orbital angular momentum, $\mathbf{S}$ is its spin, and r is its distance from the proton.

(a) Calculate the first-order energy shift of the ground state.

(b) Calculate the first-order energy shifts of the $n = 2$ states.

This correction is one of the relativistic corrections to the simple hydrogen energy spectrum that you're used to. (HINT: Notice that $J^2 = L^2 + S^2 + 2\mathbf{L}\cdot\mathbf{S}$.)