Homework 10

Due in class Nov. 4

1. Consider an infinite square well

$$V(x) = \begin{cases} 0 & \text{if } |x| \le \frac{a}{2} \\ +\infty & \text{otherwise} \end{cases}$$

subject to the time-dependent perturbation

$$H_1(t) = \lambda X \sin \omega t \qquad 0 \le t \le T.$$

(a) Calculate and plot (as a function of ω) the transition probabilities to the first, second, and third excited states assuming the system in the ground state initially. Set

$$\omega = \frac{8\pi^2\hbar^2}{2ma^2}$$

and plot the same probabilities as a function of T. Explain the results in all cases.

(b) For the cases in (a), what is the probability (to the same order as in (a)) that the ground state survives the perturbation?

(c) Assume now that the initial state of the system is an equal admixture of the ground and first excited state. Repeat part (a), making sure to explain the results and contrast them to those in (a).

2. A particle is in the ground state of the infinite square well

$$V(x) = \begin{cases} 0 & 0 < x < a \\ \infty & \text{otherwise.} \end{cases}$$

At $t = 0$ the wall at $x = a$ is suddenly moved to $x = 2a$. This process happens very fast — approximately instantaneously.

(a) Calculate the probability that long after $t = 0$ the system is in the ground state of the new potential. What is the earliest time t for which your result is valid?

(b) How fast must the change take place for this "instantaneous" assumption to be valid?

3. A delta function potential is actually not a bad model for the H^- atom: both have only one bound state and the potential between the "outer" electron and H is short ranged like the delta function. So, let's explore this model a little, but reduced to 1D for simplicity.

When working with electrons in atoms, it's handy to use atomic units. In these units, $\hbar$=m_e=e=1. So, H^- is represented by the Hamiltonian (in atomic units)

$$H_0 = -\frac{1}{2}\frac{d^2}{dx^2} - \kappa\delta(x)$$

where κ is a real positive number. This Hamiltonian describes the motion of the electron in the presence of an H atom. We will expose our H^- to a laser pulse whose electric field is given by

$$\mathcal{E}(t) = \mathcal{E}_0 e^{-(\frac{t}{\tau})^2} \cos \omega_0 t.$$

The perturbation felt by the H^- is thus

$$H_1 = x\mathcal{E}(t)$$

in the dipole approximation.

(a) Show that there is only one bound state with energy

$$E_0 = -\frac{\kappa^2}{2}$$

where κ depends on constants in H_0. Find the associated wave function. Does it possess any symmetry?

(b) We know that evaluating the matrix elements in perturbation theory is simpler if we can use symmetry. So, let's construct continuum states for H_0 with the highest symmetry possible. In particular, assume the following forms for the continuum states:

$$\psi_E^o(x) = \sqrt{\frac{2}{\pi k}} \begin{cases} \sin(kx + \phi^o) & x < 0 \\ \sin(kx - \phi^o) & x > 0 \end{cases},$$

$$\psi_E^e(x) = \sqrt{\frac{2}{\pi k}} \begin{cases} \cos(kx + \phi^e) & x < 0 \\ \cos(kx - \phi^e) & x > 0 \end{cases},$$

where the subscript E indicates these are energy eigenstates, and the superscript e and o indicates the even and odd solutions, respectively. The constant prefactor is required to energy normalize these states (just keep it along, but you don't need to worry about it). Verify that these states are indeed even and odd. Calculate ϕ^e and ϕ^o.

(c) Using first order perturbation theory, calculate the ionization probability for our model H^- as a function of energy. Plot your probability as a function of energy for $\omega < E_0$, $\omega \approx E_0$, and $\omega > E_0$. In each case discuss what you see physically. In an experiment, this spectrum is exactly what you would measure if you collected all ionized electrons and measured their kinetic energy, right? For what values of $\mathcal{E}_0$ are your results valid?

(d) In 3D, we can ask the question, "What is the angular distribution of electrons?" In 1D, though, the best we can ask is whether the ionized electrons go left or right. It is not so easy to answer this question with the continuum states above since they don't move ($\langle P \rangle = 0$ for real functions, right?). So, let's construct new states that represent electrons moving right and left:

$$\psi_E^L(x) \to \sqrt{\frac{2}{\pi k}} e^{-ikx} \qquad \text{for } x < 0,$$

$$\psi_E^R(x) \to \sqrt{\frac{2}{\pi k}} e^{ikx} \qquad \text{for } x > 0.$$

(We don't care about ψ_E^L for $x > 0$ or ψ_E^R for $x < 0$ since we won't be trying to detect them there!) These states must be related to the even and odd solutions above — find the relations. Using the orthogonality of ψ_E^e and ψ_E^o, are your ψ_E^L and ψ_E^R also orthogonal?

(e) Calculate, in first order perturbation theory, the left- and right- ionization probabilities $P_L(E)$ and $P_R(E)$. Plot these as a function of E and discuss them physically.

(f) **(10 pts) Extra Credit:** Consider what the ionization probabilities will be in second order perturbation theory. Be as quantitative as possible. If you can't do the calculation explicitly, say why not and discuss what you expect to happen qualitatively (supported by quantitative estimates).