Solution of Prob. 2, HW 5: The Delta function potential

2. Given
$$V(x) = V_0 a\delta(x),$$
we need to calculate R and T.

(a) Well, we know
$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x \le 0 \\ Ce^{ikx} + De^{-ikx} & x \ge 0 \end{cases}.$$

Matching requires some care. To find the condition on ψ', we write
$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + V_0 a\delta(x)\psi = E\psi$$
$$-\frac{\hbar^2}{2m}\int_{-\epsilon}^{\epsilon} dx \frac{d^2\psi}{dx^2} + V_0 a\psi(0) = \int_{-\epsilon}^{\epsilon} dx E\psi = 0$$
by continuity of ψ. So,
$$\left.\frac{d\psi}{dx}\right|_{+\epsilon} - \left.\frac{d\psi}{dx}\right|_{-\epsilon} = \frac{2mV_0}{\hbar^2}a\psi(0).$$

Now define the unitless quantity
$$\gamma = \frac{2mV_0a^2}{\hbar^2}.$$

Finally, matching gives

for ψ $\quad A + B = C + D$

for ψ' $\quad -ik(A-B) + ik(C-D) = \frac{\gamma}{a}\psi(0) = \frac{\gamma}{a}(C+D).$

Rewriting in matrix form gives
$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1+i\frac{\gamma}{ka} & -1+i\frac{\gamma}{ka} \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix}$$
or
$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 1 \\ 1+i\frac{\gamma}{ka} & -1+i\frac{\gamma}{ka} \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix}.$$

We can easily invert the matrix to find
$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

So,
$$\begin{aligned}\begin{pmatrix} A \\ B \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1+i\frac{\gamma}{ka} & -1+i\frac{\gamma}{ka} \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} \\ &= \begin{pmatrix} 1+i\frac{\gamma}{2ka} & i\frac{\gamma}{2ka} \\ -i\frac{\gamma}{2ka} & 1-i\frac{\gamma}{2ka} \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix}.\end{aligned}$$

Defining the new quantity $\Gamma = \frac{\gamma}{2ka}$, we can write this simply as
$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1+i\Gamma & i\Gamma \\ -i\Gamma & 1-i\Gamma \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix}.$$

Finally, for left incidence, $D = 0$, and
$$R = \frac{|B|^2}{|A|^2} \quad \text{and} \quad T = \frac{|C|^2}{|A|^2}.$$

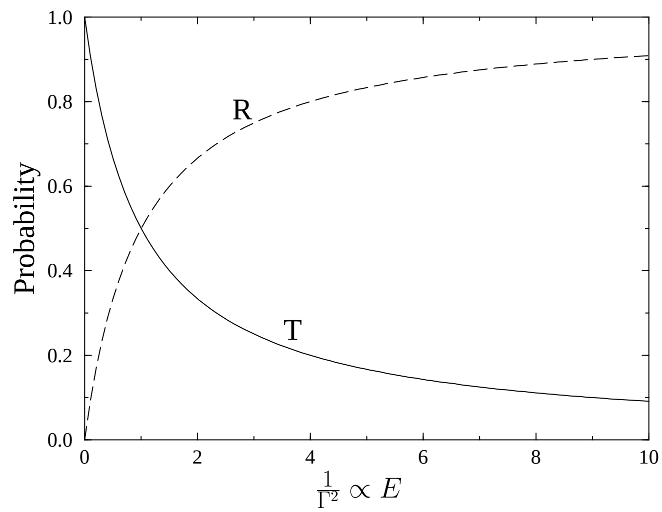
Solving for B and C,
$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1+i\Gamma & i\Gamma \\ -i\Gamma & 1-i\Gamma \end{pmatrix}\begin{pmatrix} C \\ 0 \end{pmatrix}$$
or
$$\begin{aligned} A &= (1+i\Gamma)C \\ B &= -i\Gamma C, \end{aligned}$$
giving
$$\begin{aligned} R &= \left|\frac{-i\Gamma}{1+i\Gamma}\right|^2 = \frac{\Gamma^2}{1+\Gamma^2} \\ T &= \left|\frac{1}{1+i\Gamma}\right|^2 = \frac{1}{1+\Gamma^2}. \end{aligned}$$

Clearly, $R + T = 1$.

(b) Both R and T depend only on
$$\Gamma^2 = (\frac{\gamma}{2ka})^2 = (\frac{2mV_0a^2}{\hbar^2}\frac{1}{2ka})^2 \propto V_0^2.$$

So, the sign of V_0 doesn't matter.

(c) The plots are



These behave as we expect both when $E \to 0$ and $E \to \infty$. In particular, $T \to 1$ for $\frac{1}{\Gamma^2} \gg 1$ which is equivalent to $\frac{E}{V_0} \gg 1$. Note that right incidence give the same answer by symmetry.