Applied Quantum Mechanics: Final Exam

Due by 6:00 PM Friday, Dec. 19

Instructions:

- 130 points are possible on this exam, so 30 of these will be extra credit.
- Unlike the homework, this exam is not to be discussed with your classmates (or other professors, students, postdocs, etc.). If you have questions, feel free to bring them to me, and I will do my best to answer them. You may use any other resource you like, but cite anything you didn't do yourself. If you use Mathematica, say so; if you pull an answer off the web, say so; if you use a result from a book or class, say so.
- Remember to write down everything you can about a problem! Even if you can't find the solution because of time or difficulty, you can write down what you think needs to be done and what physics you expect to result.
- Any plots, discussion, or physical insight beyond what is asked for has a good chance of becoming extra credit ...

(*10 pts*) 1. If $j_1 = 4$ and $j_2 = \frac{5}{2}$, then

(a) list the possible values of the total angular momentum j where $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$;

(b) write the explicit expansion of $|(4\frac{5}{2})\frac{7}{2}\frac{7}{2}\rangle$ in terms of uncoupled states;

(c) repeat (b) for $|(4\frac{5}{2})\frac{13}{2}\frac{13}{2}\rangle$ and $|(4\frac{5}{2})\frac{9}{2}\frac{3}{2}\rangle$;

(d) write the explicit expansion of $|4\,-3\rangle|\frac{5}{2}\,-\frac{3}{2}\rangle$ in terms of coupled states.

(*15 pts*) 2. Consider a hydrogen atom initially in the $2s$ state. A uniform electric field, $\mathcal{E}(t)$, is applied in the z-direction:

$$\mathcal{E}(t) = \begin{cases} 0 & t < 0 \\ \mathcal{E}_0 e^{-\frac{t}{\tau}} & t > 0. \end{cases}$$

(a) Apply first-order time-dependent perturbation theory to calculate the probability that the atom is in a state $n\ell m$ at $t \to \infty$.

(b) Give explicit expressions for the probability of being in (*i*) the $1s$ state, (*ii*) the $2p$ state, and (*iii*) the $3p$ state. Do your answers make sense?

(b) Under what conditions is your answer valid?

(*15 pts*) 3. Consider now a spin-1 particle in an external magnetic field. The Hamiltonian is simply the magnetic dipole interaction

$$H = -\mathbf{M} \cdot \mathbf{B} = -\gamma \mathbf{S} \cdot \mathbf{B}$$

where γ is a positive, real constant. We take the magnetic field to be $\mathbf{B} = B_0 \hat{\mathbf{y}}$.

(a) Calculate the matrix representing H.

(b) Calculate the eigenvalues and eigenstates.

(c) At $t = 0$, the system is in the state $|10\rangle$. Write down the state vector $|\psi(t)\rangle$ at an arbitrary time t later.

(d) If S_z is measured at t, what values can be obtained and with what probabilities?

(e) What is the expectation value of S_z and what does this quantity represent physically?

(*20 pts*) 4. Variational principle.

(a) We said in class that a trial function of the form

$$\psi(x) = \sum_{n=1}^{N} a_n \phi_n(x)$$

yields the minimum energy for an arbitrary Hamiltonian when $\mathbf{a}$ (the vector of expansion coefficients) satisfies an eigenvalue equation after applying the variational principle. The basis functions $\phi_n(x)$ are an arbitrary, but orthonormal, basis set. Prove this statement.

(b) Apply the above result to find an approximation to the ground state of the potential

$$V(x) = \begin{cases} \infty & |x| > a \\ 10\frac{\hbar^2}{ma^2}\left(1 - \left(\frac{x}{a}\right)^2\right) & |x| \leq a \end{cases}$$

First, sketch a picture of the potential with what you expect the ground state to look like. Then, carry out the variational calculation for N=1,2,3. Use the infinite square well solutions for your basis and make sure the basis states satisfy all of the requirements we discussed for a good trial function. (Add the infinite square well energy levels to your sketch.)

(c) Estimate the error in your best answer from (b). When you compare the three ground state energies you obtained, do they follow the pattern you expect for variational approximations?

(d) Plot the wave function you obtained in (b) for N=3. Does it make sense?

(e) Treat the above potential in first order perturbation theory (for the ground state only) and compare to your results from (b). Explain whether your results make sense.

(f) What is the smallest N that would give the ground state energy accurate to four digits?

(*15 pts*) 5. Consider two *different* spin-$\frac{1}{2}$ particles whose Hamiltonian is completely specified as $H = c\mathbf{S}_1 \cdot \mathbf{S}_2$ where c is a real constant.

(a) What are the constants of motion?

(b) Calculate the eigenvalues and eigenvectors of the Hamiltonian. What units should c have?

(c) If at time $t = 0$ the spin of particle (1) along the z-axis is down and the spin of particle (2) along the z-axis is up, what is the wave function of the system at a later time t?

(d) At a given time $t > 0$, what values of the energy can be measured and with what probabilities?

(e) At $t > 0$, what is the probability that the system is in the same state it was at $t = 0$?

(*15 pts*) 6. For an electron moving in a 3D simple harmonic oscillator potential, add the spin-orbit interaction:

$$H' = \frac{\omega^2}{m_e c^2}\mathbf{L} \cdot \mathbf{S}.$$

Here, $\mathbf{L}$ is the electron's orbital angular momentum, $\mathbf{S}$ is its spin angular momentum, and ω is the oscillator frequency.

(a) Calculate the energy shift of the ground state in first order perturbation theory.

(b) Calculate the energy shift of the first excited state in first order perturbation theory.

(c) If this oscillator is, in fact, a quantum dot in GaAs, then the electron moves as though it had a mass of $0.067m_e$. Typical quantum dots have $\hbar\omega$=5 meV. Taking these factors into account, is perturbation theory valid for the spin-orbit interaction? Is its effect large enough that it should be taken into account in experiments?

(*20 pts*) 7. An electron is moving in a 3D simple harmonic oscillator of frequency ω. A constant magnetic field of magnitude $\mathcal{B}_0$ is applied along the z-axis, and we will consider the effects of the diamagnetic term:

$$H' = \frac{e^2\mathcal{B}_0^2}{8m_e}(x^2 + y^2).$$

(a) Calculate the energy shift of the ground state in first order perturbation theory.

(b) Calculate the energy shifts for the first excited state in first order perturbation theory.

(c) Calculate the energy shifts for the second excited state in first order perturbation theory.

(d) For what values of $\mathcal{B}_0$ are your results in (a)–(c) valid? If this oscillator is the quantum dot from Prob. 6(c), for what range of $\mathcal{B}_0$ are your results applicable? If the experimentalist doesn't want to worry about this effect, will they have to shield their quantum dots from, say, the earth's magnetic field?

(e) Interpret your results in (a)–(c) physically. Do the directions of the energy shifts make sense? Are they consistent with the corresponding wave functions?

(*20 pts*) 8. Let's now place two electrons into our 3D simple harmonic oscillator (frequency ω). The electron-electron interaction will be treated in first order perturbation theory.

(a) Write down all possible wave functions for the lowest two energy levels neglecting the electron-electron interaction. Make sure the wave functions are properly symmetrized and include spin.

(b) Calculate the energy shift due to the electron-electron interaction for the lowest energy singlet state.

(c) Calculate the energy shift due to the electron-electron interaction for the lowest energy triplet state.

(d) If we make this oscillator a quantum dot as in the previous two problems, then is the electron-electron interaction really a perturbation? Besides the change in the mass of the electron, you should take into account the screening of the Coulomb potential by the medium. This is most easily accomplished by changing the permittivity of space from ε_0 to ε. For GaAs, $\varepsilon = 20\varepsilon_0$.

(e) Repeat part (a) if the two particles are deuterons instead of electrons. Assume the deuterons have a spin of 1.