Midterm Exam

Due: Oct. 18, beginning of class

1. The hyperfine structure in the spectra of atoms and molecules is due to the interaction of the electron and nuclear spins. The Hamiltonian can thus be written as

$$H = -\gamma \mathbf{I} \cdot \mathbf{S}$$

where γ is a real constant, $\mathbf{I}$ is the nuclear spin, and $\mathbf{S}$ is the electron spin. The electron is spin-$\frac{1}{2}$, and we'll take the nucleus to also be spin-$\frac{1}{2}$. The operators for each component of both $\mathbf{I}$ and $\mathbf{S}$ then have the form

$$I_x, S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad I_y, S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad I_z, S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in the $\{|+\rangle_I, |-\rangle_I\}$ and $\{|+\rangle_S, |-\rangle_S\}$, respectively.

   a) Write down a basis for the direct product space.

   b) Calculate the matrix representing H in your basis from (a).

   c) Calculate the eigenvalues and eigenvectors of H and determine a C.S.C.O. for them.

   d) At $t = 0$, the system is in the state $|-\rangle_I|+\rangle_S$. Write down the eigenvector expansion of the state vector at $t = 0$, and then for an arbitrary time t later. If S_z is measured, what is the mean value of the results that can be obtained?

2. Given a one-dimensional harmonic oscillator potential in units such that the Hamiltoninan is

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2,$$

and the initial wave packet

$$\psi(x,0) = \pi^{-\frac{1}{4}} e^{-\frac{1}{2}(x-x_0)^2}$$

where x_0 is some arbitrary real constant,

   a) Describe qualitatively the motion of the wave packet for later times.

   b) Find the analytical form of the wave packet at some later time t.

   c) What is $\langle x \rangle(t)$?

HINT: You might need

$$\int_{-\infty}^{\infty} dx\; e^{-(x-y)^2} H_n(x) = \sqrt{\pi}(2y)^n$$

where $H_n(x)$ is a Hermite polynomial; also,

$$2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x).$$

3. If we now consider two non-interacting particles with the harmonic oscillator Hamiltonian from Problem 2,

   a) Determine the allowed energies of the system and their degeneracies.

   b) If the particles are identical, spin-0 bosons, repeat (a), and write down the properly symmetrized wave functions.

   c) If the particles are identical fermions, repeat (a) assuming that all of the symmetry must be accounted for by the spatial wave function, and write down the properly symmetrized wave functions.

4. a) Prove the closure relation for momentum space:

$$\delta(x - x') = \frac{1}{\pi} \int_{-\infty}^{\infty} dk \; e^{ik(x-x')}.$$

   HINT: Transform $\psi(x)$ to momentum space, then back.

   b) Show that the momentum space wave function corresponding to a normalized wave function $\psi(x)$ is also normalized.

   c) Find the momentum space wave function corresponding to the ground state of the hydrogen atom:

$$R_{1s}(r) = 2e^{-r}.$$