Mid-term Exam

Due at the beginning of class Oct. 29

Instructions:

- 140 points are possible on this exam, but 120 pts≡100%
- Unlike the homework, this exam is not to be discussed with your classmates (or other professors, students, postdocs, etc.). If you have questions, feel free to bring them to me, and I will do my best to answer them. You may use any other resource you like, but cite anything you didn't do yourself. If you use Mathematica, say so; if you pull an answer off the web, say so; if you use a result from a book or class, say so.
- Remember to write down everything you can about a problem! Even if you can't find the solution because of time or difficulty, you can write down what you think needs to be done and what physics you expect to result.
- Any plots, discussion, or physical insight beyond what is asked for has a good chance of becoming extra credit ...

(5 pts) 1. Suppose $|i\rangle$ and $|j\rangle$ are eigenkets of some Hermitian operator A. Under what condition(s) can we conclude that $|i\rangle + |j\rangle$ is also an eigenket of A? Justify your answer.

(10 pts) 2. The Hamiltonian operator for a two-state system is given by

$$H = a\left(|1\rangle\langle 1| - |2\rangle\langle 2|\right) + b\left(|2\rangle\langle 1| + |1\rangle\langle 2|\right)$$

where a and b are constants with units of energy.

(a) Calculate the matrix elements of H in the basis $\{|1\rangle, |2\rangle\}$.

(b) Find the energy eigenvalues and the corresponding eigenkets.

(c) Suppose there was a typo, and the Hamiltonian instead read

$$H = a\left(|1\rangle\langle 1| - |2\rangle\langle 2|\right) + b|1\rangle\langle 2|.$$

What principle is now violated? Illustrate your point explicitly by attempting to solve the most general time-dependent problem using this Hamiltonian.

(10 pts) 3. Consider a three-dimensional ket space. If a certain set of orthonormal kets — say $|1\rangle$, $|2\rangle$, and $|3\rangle$ — are used as the basis kets, the operators A and B are represented by

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \quad B = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}$$

with both a and b real.

(a) Obviously A exhibits a degenerate spectrum. Does B also have a degenerate spectrum?

(b) Show that A and B commute.

(c) Find a new set of orthonormal kets that are simultaneous eigenkets of both A and B. Specify the eigenvalues of A and B associated with each eigenket. Does your specification completely characterize each ket, *i.e.* is there a unique label for each ket?

(15 pts) 4. Consider a one-dimensional simple harmonic oscillator.

(a) Construct a linear combination of the states $|0\rangle$ and $|1\rangle$ such that $\langle X\rangle$ is as large as possible.

(b) You know that $\langle n|X|n\rangle = 0$, so explain how it is possible to construct a superposition state with $\langle X\rangle \neq 0$. Feel free to use a sketch, words, or equations.

(c) Suppose the oscillator is in the state constructed in (a) at $t = 0$. What is the state vector for $t > 0$? Describe its behavior.

(d) Evaluate $\langle X\rangle(t)$.

(e) Evaluate ΔX as a function of time.

(15 pts) 5. At $t = 0$, the position of a particle of mass m exhibiting simple harmonic motion with frequency ω is measured to be $x = 0$.

(a) Write down the wave function for $t > 0$, but don't worry about evaluating the overall normalization constant or any other mathematical sublleties. (HINT: Think about the completeness relation.)

(b) If energy is measured at some time $t > 0$, what values can be obtained and with what probabilities?

(c) Explain whether or not your answers to (a) and (b) make sense and why.

(10 pts) 6. A particle is in the ground state of the infinite square well

$$V(x) = \begin{cases} 0 & 0 < x < a \\ \infty & \text{otherwise.} \end{cases}$$

At $t = 0$ the wall at $x = a$ is suddenly moved to $x = 2a$. This process happens very fast — approximately instantaneously.

(a) Calculate the probability that long after $t = 0$ the system is in the ground state of the new potential. What is the earliest time t for which your result is valid?

(b) How fast must the change take place for this "instantaneous" assumption to be valid?

(15 pts) 7. Neutral K-mesons are created in one of two states: $|K^0\rangle$ or $|\bar{K}^0\rangle$. These states are not eigenstates of the Hamiltonian that describes the system. Instead,

$$H = m\left(|K^0\rangle\langle K^0| + |\bar{K}^0\rangle\langle\bar{K}^0|\right) + \varepsilon\left(|\bar{K}^0\rangle\langle K^0| + |K^0\rangle\langle\bar{K}^0|\right).$$

(a) Find the representation of the operator H in the $\{|K^0\rangle, |\bar{K}^0\rangle\}$ basis.

(b) Calculate the energy eigenvalues and corresponding eigenvectors.

(c) Show that if the state $|K^0\rangle$ is created at time $t = 0$, then the probability that the state is $|K^0\rangle$ at a later time t oscillates between 0 and 1.

(d) Evaluate the oscillation time if m=495 MeV and ε=5 eV.

(15 pts) 8. A particle of mass m moves in an infinite square well of width a:

$$V(x) = \begin{cases} 0 & 0 < x < a \\ \infty & \text{otherwise.} \end{cases}$$

The system is known to be in a state consisting of an equal admixture of the two lowest energy eigenstates.

(a) Calculate the eigenvalues and normalized eigenfunctions for the two lowest states. Do these states have the same parity?

(b) Write down the most general total wave function consistent with the information known about the system.

(c) Find the probability as a function of time that the particle will be found in the right half of the well.

(15 pts) 9. A particle of mass m is confined to a one-dimensional potential

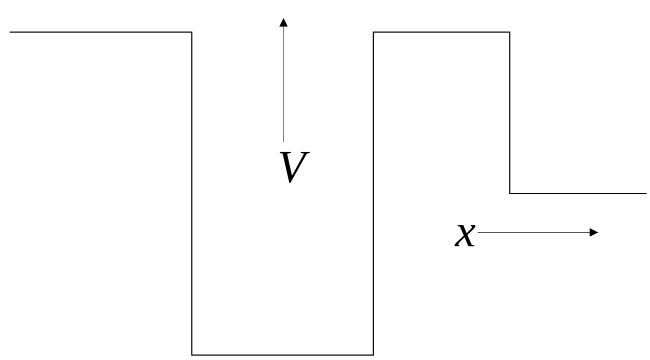
$$V(x) = -\gamma a \delta(x).$$

(a) Show that there is one bound state. Find its energy and wave function.

(b) Calculate the transmission and reflection coefficients for scattering from this potential. Plot them and point out as much physics in them as you can. You might find it useful to compare with the results you know for other potentials.

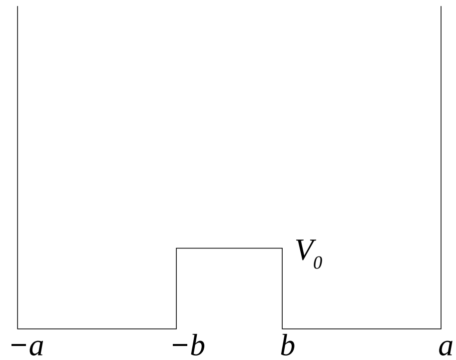
(c) Repeat part (a), but using momentum space. That is, rewrite the Schrödinger equation in momentum space and solve it in terms of E, γ, a, $\psi(0)$, and the momentum p. The constant $\psi(0)$ is the value of the coordinate space wave function at $x = 0$. Transform your momentum space solution $\bar{\psi}(p)$ back to coordinate space to show that it agrees with your solution from (a).

(15 pts) 10. Write down everything thing you can say about the motion of a particle of mass m in the potential



Some things to consider might be the wave functions, scattering, and bound states, but there are certainly more. Where you can, give semi-quantitative estimates based on simple arguments — the number of bound states, if any, for instance. Sketches or plots could be useful.

(15 pts) 11. Consider an infinite square well with a barrier:



(a) Calculate the lowest two energy eigenstates and energies. You need only graphically indicate the solution of the transcendental equations you obtain. Sketch or plot both wave functions.

(b) How do their energies behave as a function of V_0? You can be qualitative here, but indicate the basic steps of any calculation or solution, and be clear on the physical explanation. What happens in the limit $V_0 \to \infty$? $V_0 \to 0$? For this explanation, you may want to approximate the eigenstates with a linear combination of ψ_L and ψ_R as we did in class and in homework.

(c) We know that at $t = 0$, the particle is localized on the left side of the barrier. Using just the lowest two states from (a), write down a total wave function $\psi(x, 0)$ that most closely describes this case. Plot your resulting wave function.

(d) Write down $\psi(x, t)$ and describe the behavior of the system as a function of time. How long does the particle take to become maximally localized to the right side of the barrier?