Homework 7

Due in class Oct. 24

1. A charged particle experiences the oscillator potential

$$V = \frac{1}{2}m\omega^2 X^2.$$

The whole system is then placed in a uniform electric field so that there is an additional potential

$$W = -q\mathcal{E}X$$

where q is the carge and $\mathcal{E}$ the magnitude of the electric field.

(a) Plot the total potential for this system with $\mathcal{E} \neq 0$.

(b) Calculate the eigenstates and energies for this system.

(c) This is a reasonable model of an electron bound to an atom placed in an electric field. A useful paramter for an atomic state is it's polarizability. Calculate the induced dipole moment for the present system for a given state $|n\rangle$.

2. Can we construct quantum states of the osillator whose expectation values mimic the classical results? The answer is yes, and they are called "quasi-classical" or "coherent" states. For a classical oscillator:

$$\begin{aligned} x(t) =& \frac{\beta}{\sqrt{2}}\left(\alpha_0 e^{-i\omega t} + \alpha_0^* e^{i\omega t}\right) \\ p(t) =& -\frac{i\hbar}{\beta\sqrt{2}}\left(\alpha_0 e^{-i\omega t} - \alpha_0^* e^{i\omega t}\right). \end{aligned}$$

(a) Using the Heisenberg equation of motion for the matrix element $\langle \mathbf{a}\rangle(t) = \langle\psi(t)|\mathbf{a}|\psi(t)\rangle$ (with $\mathbf{a}$ the lowering operator),

$$i\hbar\frac{d}{dt}\langle \mathbf{a}\rangle(t) = \langle[\mathbf{a}, H]\rangle(t),$$

find $\langle \mathbf{a}\rangle(t)$. Similarly, find $\langle \mathbf{a}^\dagger\rangle(t)$, and thus $\langle X\rangle(t)$ and $\langle P\rangle(t)$. Note that it is necessary to have $\langle \mathbf{a}\rangle(0) = \alpha_0$ to obtain agreement with the classical results.

(b) By requiring the mean value of H be the same as the classical total energy, show that

$$\langle \mathbf{a}^\dagger\mathbf{a}\rangle(0) = |\alpha_0|^2.$$

(c) Introducing the new operator

$$\mathbf{b} = \mathbf{a} - \alpha_0$$

show that we must have

$$\mathbf{a}|\psi(0)\rangle = \alpha_0|\psi(0)\rangle$$

in order to satisfy the conditions $\langle \mathbf{a}\rangle(0) = \alpha_0$ and $\langle \mathbf{a}^\dagger\mathbf{a}\rangle(0) = |\alpha_0|^2$. (HINT: Evaluate $\langle \mathbf{b}^\dagger\mathbf{b}\rangle(0)$.)

(d) Find $|\alpha\rangle = |\psi(0)\rangle$, the eigenket of $\mathbf{a}$ assuming

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n(\alpha)|n\rangle,$$

i.e. find $c_n(\alpha)$.

(e) For an oscillator in the coherent state $|\alpha\rangle$, what values of energies can be obtained and with what probability? Which energy is most likely?

(f) Calculate $|\alpha(t)\rangle$. Is this still an eigenstate of $\mathbf{a}$? If so, what is its eigenvalue?

(g) Plot $|\alpha(t)\rangle$ for a few times in the interval $t \in [0, \frac{6\pi}{\omega}]$. Note any features.

3. Now place our charged oscillator from Prob. 2 in a time-varying electric field $\mathcal{E}(t)$. The total potential is

$$V(X,t) = \frac{1}{2}m\omega^2 X^2 - q\mathcal{E}(t)X.$$

(a) The number $\alpha(t) = \langle\psi(t)|\mathbf{a}|\psi(t)\rangle$ evolves according to (see Prob. 3)

$$\frac{d}{dt}\alpha(t) = -i\omega\alpha(t) + i\lambda(t)$$

with

$$\lambda(t) = \frac{q}{\sqrt{2m\hbar\omega}}\mathcal{E}(t).$$

Integrate this equation to find $\alpha(t)$.

(b) Assume that at $t = 0$, $|\psi(t)\rangle = |n = 0\rangle$. Take

$$\mathcal{E}(t) = \mathcal{E}_0(t)\sin\omega' t$$

for $0 \le t \le T$ and zero otherwise. If the energy is measured for some time $t > T$, what results can be found and with what probabilities? Consider the case when $\omega' = \omega$ and when $\omega' \neq \omega$. Note that the coherent state $|\alpha\rangle(0)$ is related to $|n = 0\rangle$ by

$$|\alpha(0)\rangle = e^{-\frac{|\alpha_0|^2}{2}} e^{\alpha_0\mathbf{a}^\dagger} e^{-\alpha_0^*\mathbf{a}}|0\rangle.$$