Homework 2
Due: Sept. 13

1. Consider a physical system whose three-dimensional state space is spanned by the orthonormal basis formed by the three kets $|u_1\rangle$, $|u_2\rangle$, and $|u_3\rangle$. In the basis of these three vectors, taken in this order, the two operators H and B are defined by:

$$H = \hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad B = b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

where ω_0 and b are real constants.

   a) Are H and B Hermitian?

   b) Show that H and B commute. Give a basis of eigenvectors common to both H and B. Write the original basis $\{|u_i\rangle\}$ in terms of this new basis. What is the transformation matrix that connects the two? Is it unitary?

   c) Of the sets of operators — $\{H\}$, $\{B\}$, $\{H,B\}$, $\{H^2,B\}$ — which form a C.S.C.O.?

2. Given the Hamiltonian (in some basis $\{|1\rangle, |2\rangle\}$)

$$H = \begin{pmatrix} E_0 & \alpha \\ \alpha & E_1 \end{pmatrix}$$

   a) Calculate the eigenvalues and eigenvectors. In the limit, $\alpha \to 0$, what should you find? Do you? Write the eigenvectors both as column vectors and in terms of $|1\rangle$, $|2\rangle$.

   b) In the limit, $E_0 = E_1$, examine the eigenvalues and eigenvectors. What do you find?

   c) If, physically, α corresponds to a coupling (interaction) between states 1 and 2, can you think of a physical situation that H describes? What are the units of α?

   d) Are the eigenvectors you find orthonormal? Should they be? Do they satisfy closure?

3. Consider a spin-$\frac{1}{2}$ particle placed in a magnetic field $\mathbf{B}$ with components:

$$\begin{aligned} B_x &= \frac{1}{\sqrt{2}}B \\ B_y &= 0 \\ B_z &= \frac{1}{\sqrt{2}}B. \end{aligned}$$

The Hamiltonian for this system is given by

$$H = -\mathbf{M}\cdot\mathbf{B} = -\gamma\mathbf{S}\cdot\mathbf{B}$$

where $\mathbf{M} = \gamma \mathbf{S}$ is the magnetic moment of the particle, γ is a constant, and $\mathbf{S}$ is the spin angular momentum operator — its components (in the $\{|+\rangle, |-\rangle\}$ basis) are given by

$$S_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S_y = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad S_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

a) Calculate the matrix representing H in the $\{|+\rangle, |-\rangle\}$ basis.

b) Calculate the eigenvalues and eigenvectors of H.

c) At $t = 0$, the system is in the state $|-\rangle$. Write down the state vector $|\psi(t)\rangle$ at an arbitrary time t later. If S_x is measured, what is the mean value of the results that can be obtained?