

Final Exam

Due by 5:00 PM Tuesday, Dec. 17

Instructions:

- Unlike the homework, this exam is not to be discussed with your classmates (or other professors, students, postdocs, etc.). If you have questions, feel free to bring them to me, and I will do my best to answer them. You may use any other resource you like, but cite anything you didn't do yourself. If you use Mathematica, say so; if you pull an answer off the web, say so; if you use a result from a book or class, say so.
 - Remember to write down everything you can about a problem! Even if you can't find the solution because of time or difficulty, you can write down what you think needs to be done and what physics you expect to result.
 - Any plots, discussion, or physical insight beyond what is asked for has a good chance of becoming extra credit ...
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(15 pts) 1. Consider two *different* spin- $\frac{1}{2}$ particles whose Hamiltonian is completely specified as $H = c\mathbf{S}_1 \cdot \mathbf{S}_2$ where c is a real constant.

- (a) What are the constants of motion?
- (b) Calculate the eigenvalues and eigenvectors of the Hamiltonian.
- (c) Consider that at time $t = 0$ the spin of particle (1) along the z -axis is up and the spin of particle (2) along the z -axis is down. What is the wave function of the system at a later time t ?
- (d) Calculate the probability that at a later time t the spins of both particles are aligned as they were at $t = 0$.

(10 pts) 2. Consider three noninteracting particles in a 1D simple harmonic oscillator.

- (a) Assuming the particles have spin 0, find the 4 lowest eigenenergies, their degeneracy, and write down the corresponding eigenstates (in terms of $\phi_n(x)$ is fine).
- (b) Repeat (a), but now assume that the particles have spin $\frac{3}{2}$. Don't worry about constructing eigenstates of the total spin, *i.e.* leave the spin part in the uncoupled basis.

(10 pts) 3. Consider a hydrogen atom in its ground state. An electric field, $\mathcal{E}(t)$, is applied in the z -direction.

$$\mathcal{E}(t) = \begin{cases} 0 & t < 0 \\ \mathcal{E}_0 e^{-\frac{t}{\tau}} & t > 0. \end{cases}$$

- (a) Apply first-order time-dependent perturbation theory to calculate an expression for the probability that the hydrogen atom is in an excited state.
- (b) What is the probability if the excited state is (i) $2s$ and (ii) $2p$? What about $3d$? Does your answer make sense?

(15 pts) 4. Variational principle.

- (a) We said in class that a trial function of the form

$$\psi(x) = \sum_{n=1}^N a_n \phi_n(x)$$

yields the minimum energy for an arbitrary Hamiltonian when \mathbf{a} satisfies an eigenvalue equation after applying the variational principle. The basis functions $\phi_n(x)$ are an arbitrary, but orthonormal, set. Prove this statement.

- (b) Apply the above result to find an approximation to the ground state of the potential

$$V(x) = \begin{cases} \infty & |x| > a \\ |x| & |x| \leq a \end{cases}$$

First, sketch a picture of the potential with what you expect the ground state to look like. Then, carry out the variational calculation for $N=1,2,3$. That is, use only the lowest state, then the lowest two states, and so on. (HINT: Use the infinite square well solutions for the basis.) When you compare the three ground state energies, do they follow the pattern you expect for variational approximations?

- (c) Treat the above potential in first order perturbation theory and compare to your results from (b). Explain whether your results make sense.

(10 pts) 5. If $j_1 = 4$ and $j_2 = \frac{5}{2}$, then

- (a) list the possible values of the total angular momentum j where $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$;
- (b) write the explicit expansion of $|(4\frac{5}{2})\frac{7}{2}\frac{7}{2}\rangle$ in terms of uncoupled states;
- (c) repeat (b) for $|(4\frac{5}{2})\frac{13}{2}\frac{13}{2}\rangle$ and $|(4\frac{5}{2})\frac{9}{2}\frac{3}{2}\rangle$;
- (d) write the explicit expansion of $|4-3\rangle|\frac{5}{2}-\frac{3}{2}\rangle$ in terms of coupled states.

(15 pts) 6. Let's apply the variational principle to helium ... again. This time, let's use a little more sophisticated trial function with two parameters. The primitive term will thus look like

$$e^{-\alpha r_1} e^{-\beta r_2} Y_{00}(\Omega_1) Y_{00}(\Omega_2).$$

You will need to normalize, symmetrize, etc. on your own.

- (a) Write down the ground singlet state (properly symmetrized and angular momentum eigenstates!).
- (b) Apply the variational principle to find the optimal values of α and β , and thus the best energy for this trial function. You will have to find α and β numerically, but some physical reasoning will limit the values they can have. A little more fiddling on your calculator or with Mathematica should get you results that are accurate enough.
- (c) Interpret the values of α and β from (b) physically. Also, compare your energy from (b) to the other approximate energies you have obtained for helium. Does the present result compare as you expect?
- (d) Repeat (b) for the H^- ion. Is H^- stable in this approximation, or is it more favorable to break up into $\text{H}+e$? What are α and β telling you now?

(15 pts) 7. Consider now a spin-1 particle in an external magnetic field. The Hamiltonian is simply the magnetic dipole interaction

$$H = -\mathbf{M} \cdot \mathbf{B} = -\gamma \mathbf{S} \cdot \mathbf{B}$$

where γ is a positive, real constant. We take the magnetic field to be $\mathbf{B} = B_0 \hat{\mathbf{y}}$.

- (a) Calculate the matrix representing H .
- (b) Calculate the eigenvalues and eigenstates.
- (c) At $t = 0$, the system is in the state $|0\rangle$. Write down the state vector $|\psi(t)\rangle$ at an arbitrary time t later.
- (d) If S_z is measured at t , what values can be obtained and with what probabilities?
- (e) What is the expectation value of S_z and what does this quantity represent physically?

(10 pts) 8. Consider a particle in a 2D simple harmonic oscillator.

- (a) What are the three lowest energies?
- (b) Write down the wave functions corresponding to these energies.
- (c) The perturbation

$$H_1 = \lambda XY$$

is now applied to the system. Calculate the resulting energy shifts to each of the states in (b) to first order in the perturbation.

(15 pts) **Extra Credit.** Apply the variational principle to the potential

$$V(x) = \frac{1}{2}x^4$$

to obtain the ground and first excited state energies to two digits.