Final Exam

Dec. 11

1. Consider the helium atom in the approximation that the electron-electron interaction is neglected. Use atomic units: $e = \hbar = m_e = 1, c = \frac{1}{\alpha}$.

   a) What are the energies in this approximation?

   b) Starting in the $1s^2\ ^1S$ ground state, to what singly-excited states $1sn\ell\ ^{2S+1}L$ can electric dipole transitions occur?

   c) Write out the $1s2p_0\ ^1P$ wave function. Using Fermi's Golden Rule, what is the lifetime of this state? (Note that the dipole operator for a two-electron system is $\mathbf{D} = -(\mathbf{r}_1 + \mathbf{r}_2)$.)

   d) Write out the $1s2p_0\ ^3P$ wave function. What is the lifetime of this state? To what state (or states) can it decay?

2. a) Using the variational principle, show that the trial function

$$\psi(x) = \sum_{n=1}^{N} a_n \phi_n(x)$$

   yields the minimum energy for an arbitrary Hamiltonian when $\mathbf{a}$ satisfies an eigenvalue equation. The basis functions $\phi_n(x)$ are an arbitrary, but orthonormal, set.

   b) Apply the above result to find an approximation to the ground state of the potential

$$V(x) = \begin{cases} \infty & |x| > a \\ x & |x| \le a \end{cases}$$

   Sketch a picture of the potential with what you expect the ground state to look like. Carry out the variational calculation for N=1,2,3. That is, use only the lowest state, then the lowest two states, and so on. (HINT: Use the infinite square well solutions for the basis.) When you compare the three ground state energies, do they follow the pattern you expect for variational approximations?

3. a) Given two angular momenta $j_1 = 1$ and $j_2 = \frac{3}{2}$, what total angular momenta $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$ are possible?

   b) Write out the expansion of $|(1\tfrac{3}{2})\tfrac{1}{2}, -\tfrac{1}{2}\rangle$ in terms of tensor product states. Explicitly indicate the values of the quantum numbers that can contribute, but don't evaluate the Clebsch-Gordan coefficients.

   c) Similarly, write the expansion of $|10\rangle|\tfrac{3}{2}\tfrac{3}{2}\rangle$ in terms of coupled states.

4. Calculate the effect of the perturbation

$$V'(r) = \gamma \frac{\mathbf{L} \cdot \mathbf{S}}{r^3}$$

   on the $n = 2$ states of hydrogen where γ is a real constant.

Bonus: Calculate an approximate ground state energy for the potential

$$V(x) = \frac{1}{2}x^4$$

using the variational principle.

Possibly useful formulas

$$\begin{aligned}
\int dx x \sin ax \cos bx &= I_{ab} \\
\int dx x \sin ax \sin bx &= J_{ab} \\
\int dx x \cos ax \cos bx &= K_{ab}
\end{aligned}$$

In atomic units:

$$\begin{aligned}
R_{1s}(r) &= 2Z^{\frac{3}{2}}e^{-Zr} \\
R_{2s}(r) &= 2Z^{\frac{3}{2}}(1 - \frac{Zr}{2})e^{-\frac{Zr}{2}} \\
R_{2p}(r) &= \frac{1}{\sqrt{3}}(\frac{Z}{2})^{\frac{3}{2}}Zre^{-\frac{Zr}{2}}
\end{aligned}$$

$$\langle 1-m; 1m|00\rangle = \begin{cases} \frac{1}{\sqrt{3}} & m=1 \\ -\frac{1}{\sqrt{3}} & m=0 \\ \frac{1}{\sqrt{3}} & m=-1 \end{cases}$$

$$\int_0^\infty x^n e^{-\alpha x} = \frac{n!}{\alpha^{n+1}}$$