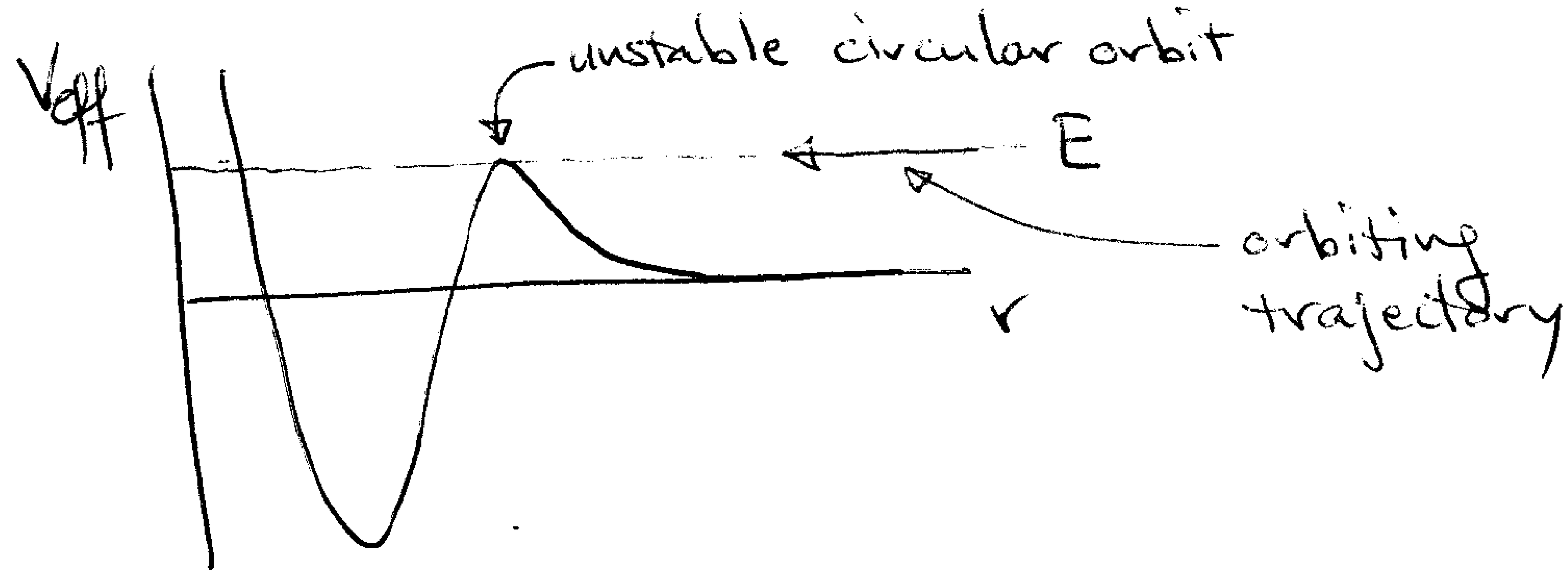
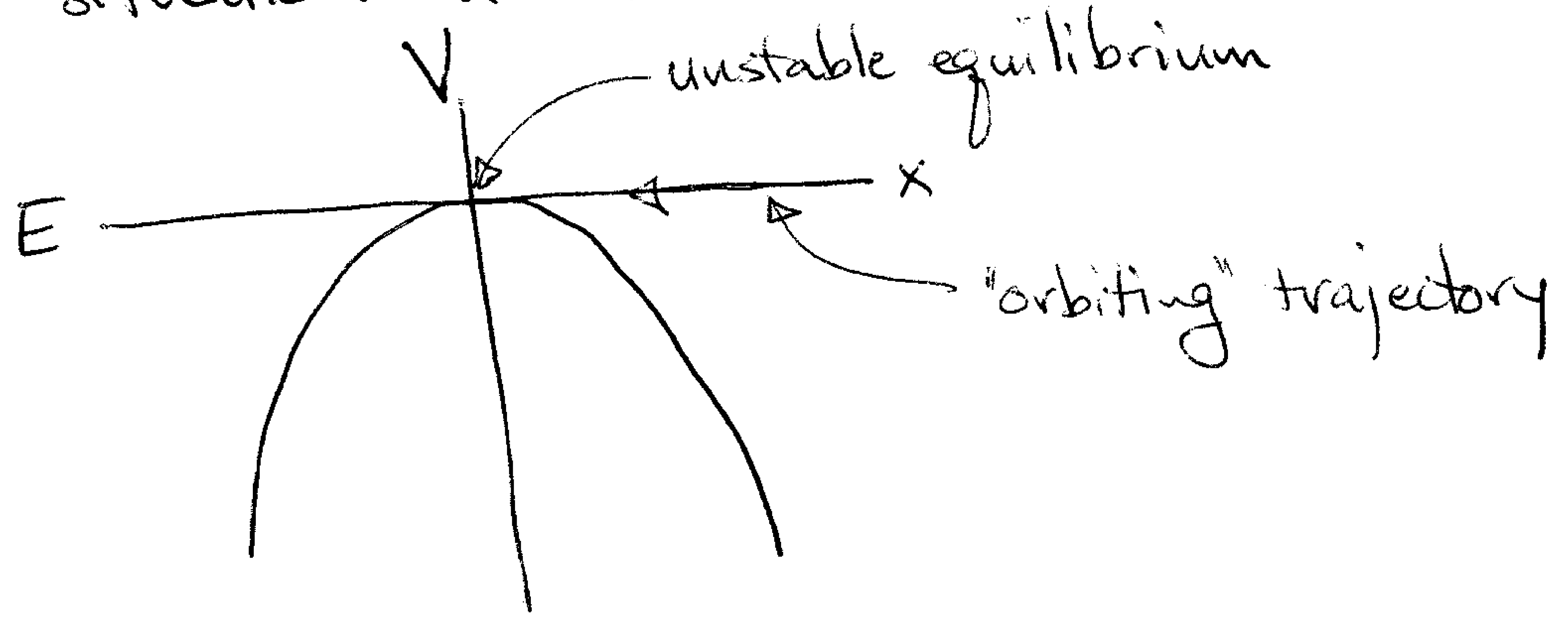
"Orbiting" trajectories

The question was raised in class whether an orbiting scattering trajectory is distinguishable from the unstable! circular orbit possible at the top of the barrier. That is, for an effective potential like



This is a question we can easily answer using the tools we've developed. For simplicity, let's model the situation in 1D as



We take

$$V = -\alpha x^2 \quad \text{and} \quad E = 0.$$

Then, from the total energy

$$\int_{t_0}^{t} dt' = \int_{x_0}^{x} dx' \, \frac{1}{\pm\sqrt{\frac{2}{m}(E-V)}}$$

Substituting E & V, we have

$$t - t_0 = \int_{x_0}^{x} \frac{dx'}{\pm\sqrt{\frac{2}{m}\alpha x'^2}} = \pm\sqrt{\frac{m}{2\alpha}}\ln\frac{x}{x_0}$$

$$\Rightarrow x = x_0 e^{\pm\sqrt{\frac{2\alpha}{m}}(t-t_0)}.$$

We have a few cases to consider:

$\underline{x_0 = 0}$ $x = 0$ → particle is at unstable equilibrium (circular orbit in original question) and never moves away

$\underline{|x_0| \neq 0}$, $\underline{v_0 \text{ towards } x=0}$ } $x = x_0 e^{-\sqrt{\frac{2\alpha}{m}}(t-t_0)}$ → particle heading towards barrier (orbiting trajectory in original question); it only reaches $x=0$ for $t \to \infty$; gives a trajectory in original problem that spirals in to limiting circular orbit

$\underline{|x_0| \neq 0}$, $\underline{v_0 \text{ away from } x=0}$ } $x = x_0 e^{+\sqrt{\frac{2\alpha}{m}}(t-t_0)}$ → particle leaves ~~the~~ region of unstable equilibrium; distance grows <u>exponentially</u> with time