Homework 1

Due in class Monday, Jan. 31

Instructions: Use whatever means you find appropriate to solve the following problems. Make sure you've read the guidelines in the Syllabus and remember to cite the use of any sources besides me and the textbook.

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1. A particle of mass m initially at rest (at $t = -\infty$) is subject to the force

$$F(t) = \frac{F_0}{1 + (\alpha t)^2}.$$

(a) What units must F_0 and α have?
(b) Sketch $F(t)$. What physical situation might $F(t)$ approximate?
(c) Find $v(t)$ and sketch it.
(d) Find $x(t)$ and sketch it.
(e) Discuss your plots in (c) and (d). What is their behavior for times long before zero (and do they reproduce your initial conditions)? Describe the motion also for times long after zero. Does this discussion support your statement in (b)?

2. A boat with initial speed v_0 is launched on a lake. The boat is slowed by the water with a force of magnitude

$$|F| = \alpha \sinh \beta v.$$

(a) Find $v(t)$ and $x(t)$ (if you can't do the integrals analytically, consider looking in a book of integrals or numerically integrating the equation of motion).
(b) Plot $v(t)$ and $x(t)$ and discuss them physically. For instance, how long does it take for the boat to stop? How far does it travel before stopping?
(c) The boat has an engine and provides its own constant thrust F_T. Find $v(t)$ and $x(t)$ for this case for an initial velocity of v_0.
(d) Find the terminal velocity v_t if one exists.
(e) Plot the $v(t)$ and $x(t)$ you found in (c) and discuss the plots physically — consider the cases $v_0 < v_t$ and $v_0 > v_t$.

3. A rigid, cylindrical bar of radius a, length ℓ, and mass M is suspended by two identical massless springs, one at each end. Assume the springs hang vertically when the bar is at rest. The system is subject to gravity.

(a) Identify appropriate generalized coordinates.
(b) Write down the Lagrangian and find the equations of motion.
(c) Identify any cyclic coordinates and state physically what the corresponding conjugate momenta mean.

4. A mass m is attached to three springs. The springs each lie at an angle of 120° to each other in a plane. The mass is free to move in the same plane. You may neglect gravity for this problem.

(a) Identify appropriate generalized coordinates.
(b) Write down the Lagrangian.
(c) Identify any cyclic coordinates and state physically what the corresponding conjugate momenta mean.

5. An electric field is turned on at $t = 0$. Mathematically, it is represented by

$$E(t) = \begin{cases} 0 & t < 0 \\ E_0 \tanh(\gamma t) & t > 0 \end{cases}$$

where E_0 and γ are real constants.

(a) Find $v(t)$ for an otherwise free electron in this field.
(b) How does $v(t)$ behave at large times (a sketch might help)? At small times? Does these make sense? Why? Also, what qualifies as a "large" and "small" time?
(c) Find $x(t)$ and sketch or plot it. Does it make sense given your $v(t)$?

6. A point particle of mass m is confined to move on the outside surface of a cone of half angle α and is subject to gravity. The cone points upward. This mass is connected by a massless string of length d to another mass M that hangs inside the cone. The string runs through the apex of the cone, and the mass M can move only vertically along the symmetry axis of the cone. There is no friction in the system.

(a) Identify generalized coordinates that completely specify the motion and any constraints on them. Write down the Lagrangian for the system.
(b) Obtain the equations of motion. Identify any cyclic variables and discuss the physical meaning of their conjugate momenta.
(c) Under what condition will the mass m leave the surface of the cone? Verify your result by directly summing the torques involved.
(d) Identify an effective potential U_{eff} and sketch it. Does it make sense? Show that the minimum requirement for M to remain stationary is

$$M \geq m \cos \alpha.$$

Discuss this condition physically and describe the motion when this condition is satisfied.
(e) When there is a stable equilibrium, find the frequency of small oscillations (HINT: Keep it in terms of the equilibrium distance). Solve the equations of motion in the small oscillation regime, finding expressions for each generalized coordinate as a function of time.

7. Someone is designing an optics device so that the path of the light through the device is

$$y(x) = y_0 \sin kx.$$

They accomplish this by having a spatially dependent index of refraction n. Note that $y(x)$ is *not* the electric or magnetic fields, but the path of the light itself.

(a) Using Fermat's principle of least time, show that the required index is

$$n(y) = n_0 \sqrt{1 + k^2 y_0^2 \left[1 - \left(\frac{y}{y_0}\right)^2\right]}$$

and explain what n_0 represents. You should assume $n = n(y)$ and only consider motion in the xy-plane. What is the range of applicability of your solution?

(b) Sketch your solution and discuss it physically. In particular, does it seem plausible that your $n(y)$ could bend light such that it follows a sinusoidal path?

(c) Your $n(y)$ was found for a particular wavelength and amplitude of the sinusoidal oscillation which implies certain initial conditions at, for instance, the entrance to the device. Describe qualitatively the path light would take if it entered the device with different initial conditions. Would it still be sinusoidal?

8. Two particles, one with mass M and charge Q and the other with m and q, are trapped by an isotropic harmonic potential with an effective spring constant k.

(a) Show, by explicit calculation, that the problem can be reduced to an equivalent one-body problem.

(b) What quantities are conserved?

(c) Identify the effective potential and sketch it. Consider both $Qq < 0$ and $Qq > 0$. Discuss the stability of circular orbits in each case.

(d) How would your answer to (b) change if there were three charges instead of two? How about for N charges?

9. Derive the conditions a force $\mathbf{F}$ must satisfy in order to define a potential energy from it. Explain the term "path independence" and specifically how it relates to your derivation. Are these the same conditions for energy to be conserved?

10. Consider the following functional

$$E[\psi] = \int_a^b dx\, \psi(x) \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right] \psi(x).$$

Minimize $E[\psi]$ with respect to $\psi(x)$ subject to the constraint

$$\int_a^b dx\, \psi(x)\psi(x) = 1.$$

You may assume $\psi(x)$ is real. Identify and discuss the equation you obtain for $\psi(x)$.

(HINT: We all know this is related with quantum mechanics... Integrate the kinetic energy term by parts and make sure to identify the conditions on $\psi(x)$ that preserve the Hermiticity of the kinetic energy term. Then apply the Euler-Lagrange equations.)