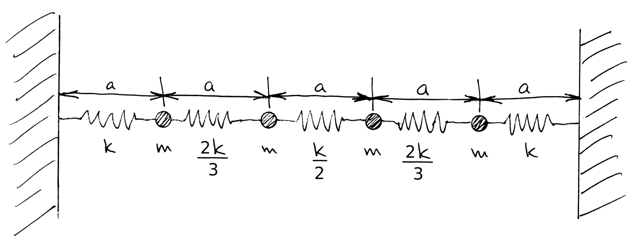
Advanced Dynamics
Final Exam
Wednesday, May 11, 2011

Instructions:

- The exam is worth a total of 120 points (100 points=100%). The points for each problem are indicated.
- If a question is unclear, ask.
- Cite anything you didn't do yourself — from notes, books, tables, etc.
- **Start each problem on a new sheet of paper.**
- If you can't find the solution because of time or difficulty, write down what you think needs to be done and what physics you expect to result.
- Your answer should be written clearly, logically, and legibly.
- Any plots, discussion, or physical insight beyond what is asked for has a good chance of becoming extra credit ...

(20 pts) **1.** The following system of masses and springs are free to move only along the line connecting them.



(a) Verify that the following vectors are proportional to the normal modes describing the system's displacements from equilibrium. If they are, properly normalize them. If they are not, find the correct normal modes.

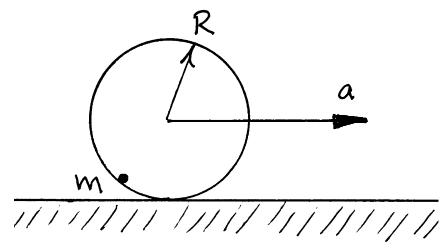
$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{v}_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \qquad \vec{v}_4 = \begin{pmatrix} 2 \\ -1 \\ -1 \\ 2 \end{pmatrix}$$

(b) What are the corresponding eigenfrequencies? Do the relative magnitudes of the eigenfrequencies make sense when you consider the corresponding normal modes? If so, discuss in what way they make sense. Sketch the normal modes in some suitable way.

(c) At $t = 0$, the left-most mass is displaced a distance λ to the right of equilibrium and released from rest. The rest of the masses are initially at rest at their equilibria. Find the subsequent motion of the system. Does the system ever return exactly to this initial configuration?

(20 pts) **2.** A circular hoop is pulled along a perfectly horizontal surface like a wheel such that its center is under constant acceleration a as shown below. The hoop rolls without slipping and is confined to a vertical plane. On the inner surface of the hoop is a small mass m that slides without friction. Assume that the dimensions of the mass are very small compared to the radius R of the hoop.



(a) Write down the Lagrangian for the system.

(b) Find the equilibrium configuration of the system. Discuss your result physically. What is its behavior in the limit $a \to 0$? $a \to \infty$? Is this behavior what you expect?

(c) Identify an effective potential and sketch it. Relate your answers from (b) to this potential.

(d) Find the frequency of small oscillations. What is its behavior in the limits of a?

(10 pts) **3.** For the following transformation in a system with two degrees of freedom,

$$Q_1 = q_1^2$$
$$Q_2 = q_1 + q_2,$$

find the most general transformation equations for P_1 and P_2 consistent with the overall transformation being canonical.

(10 pts) **4.** Is the transformation

$$Q = p + iaq \qquad P = \frac{p - iaq}{2ia}$$

canonical? If so, find a generating function for it of type F_3.

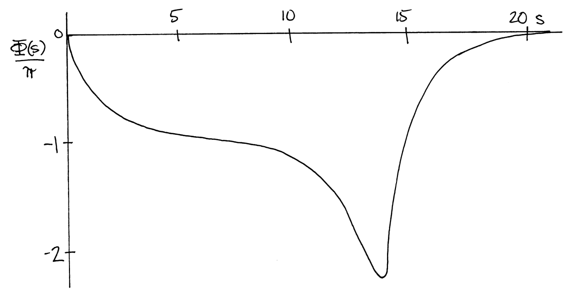
(20 pts) **5.** Given some central potential $V(r)$ describing the interaction of two particles of mass M and m, answer the following:

(a) What are the steps necessary to calculate the differential cross section $d\sigma/d\vartheta$ (or $\sigma'(\vartheta)$ in the book's notation) in the laboratory frame? Be as complete as possible — and as concise as possible.

(b) If, in the lab frame, mass M is incident on mass m (*i.e.* m is initially at rest) and $M = 100m$, say everything you can about the lab frame differential cross section. A sketch is encouraged.

(c) Under what conditions do orbiting trajectories result? How about spiraling trajectories? What is the difference between the two? Be as quantitative as you can, but illustrate your answers with appropriate sketches. Be sure to sketch the two kinds of trajectories, too.

(d) In the sketch of $\Phi(s)$ below, identify the features that correspond to orbiting trajectories, spiraling trajectories, rainbow scattering, glory scattering, and any other feature we have discussed.



(e) Sketch the center-of-mass-frame differential cross section corresponding to $\Phi(s)$ from (d). Be sure to identify all of the features you found to be present in (d).

(20 pts) **6.** A membrane is stretched taut between two co-planar, concentric circular hoops. The membrane thus forms an annulus (ring) with inner radius a and outer radius $2a$. The membrane is securely fastened to the rigid hoops so that its edges cannot move. Take the areal mass density σ to be constant and the tension τ to be constant.

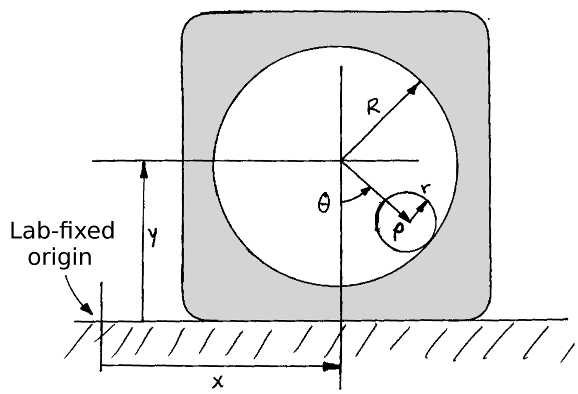
(a) Find the condition(s) satisfied by the eigenfrequencies for small oscillations of the membrane. Be sure to discuss any degeneracies.

(b) Find the normal modes. Sketch the displacement for the lowest mode and for the first excitation in each degree of freedom. (Three modes total.) Clearly indicate any nodes and the sign of the displacement.

(c) At $t = 0$, the membrane has some initial conditions $u(t = 0)$ and $\dot{u}(t = 0)$ where u is the displacement. Write down the displacement for any later time t. Be explicit and leave only integrals unevaluated.

(d) At $t = 0$, the membrane is at rest at its equilibrium (its ground state). If you can only apply a perturbation that is a combination of two delta functions, describe what their configuration should be to excite each of the three modes you identified in (b). (Assume the perturbation is applied for a short time only.) In each case, justify your chosen configuration physically in the context of your answer to (c). Also, identify what the largest contaminating mode(s) is going to be. State whether the contamination is likely to be large or small.

(20 pts) **7.** A hoop of mass m is placed on the inner surface of the block as shown below. It rolls without slipping in the plane of the figure below. The block can slide without friction on the horizontal surface, also only in the plane of the figure. Take the mass of the block to be M, and neglect any rotation it might have.



(a) Using the coordinates x, y, ρ, and θ defined in the figure to be the dynamical variables, what are the constraints? (They may involve other coordinates, too.)

(b) Write down the Lagrangian $L(x, y, \rho, \theta)$ for the system. If you have any coordinates besides these, they should be eliminated using your constraints if possible.

(c) Find the equations of motion.

(d) What, if anything, is conserved for this problem? Give the justification in each case.

(e) Assume that the hoop is initially placed at $\theta = \pi/2$ and the block is stationary. Describe in words what the subsequent motion will be.