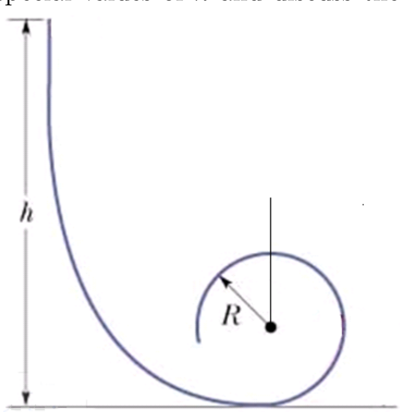
Exam 1

Due at the beginning of class, Monday, February 28

Instructions:

- The exam is worth a total of 100 points, with the points for each problem indicated.
- Each extra credit items are labeled "EC".
- If a question is unclear, ask.
- Cite anything you didn't do yourself — from notes, books, tables, etc.
- Remember to write down everything you can about a problem — including a figure! Even if you can't find the solution because of time or difficulty, you can write down what you think needs to be done and what physics you expect to result.
- Any plots, discussion, or physical insight beyond what is asked for has a good chance of becoming extra credit ...

(20 pts) 1. A solid sphere of radius r and mass m rolls without slipping on the inside of the track shown below. Beginning at the bottom, the loop at the end of the track follows a circular path of radius R as indicated. The sphere is released from a point on the track such that it falls through a height h before entering the loop. Using Lagrange multipliers, calculate the angle at which the sphere falls away from the track. You may assume that $h > R$, and measure the angle from the vertical line shown. Be sure to identify any special values of h and discuss them physically.



(20 pts) 2. A particle with charge q and mass m is subject to the electric field

$$\mathcal{E}(t) = \mathcal{E}_0 e^{-\gamma|t|}.$$

The particle can only move in one dimension, along the direction of the field. Taking the particle to be initially (i.e. $t \to -\infty$) at rest, answer the following:

(a) Find $v(t)$ and plot it. Discuss it physically.

(b) Find $x(t)$ and plot it. Discuss it physically.

(c) Do $x(t)$ and $v(t)$ behave at large times as you expect them to? Make sure to define "large" times.

(20 pts) 3. A particle with mass m_1 is incident upon a stationary particle with mass m_2, and they collide elastically. Their interaction is described by

$$V(r) = De^{-\alpha r}.$$

The numerical values of the various quantities are:

$$\begin{aligned} m_1 &= 100 \\ m_2 &= 1 \\ D &= -27.2 \\ \alpha &= 0.25, \end{aligned}$$

and you can take them to be in SI units. Make sure all calculations below are accurate to at least 0.1%.

(a) Calculate the scattering angle $\Theta(s)$ and differential cross section $d\sigma/d\Theta$ at energies E=1, 10, and 100. Plot both $\Theta(s)$ and $d\sigma/d\Theta$ for all energies. Discuss them physically, identifying any characteristic features of the types we've discussed in class. Also be sure to compare the cross sections at the different energies, and discuss the comparison physically. For instance, are the cross sections behaving as you would expect for $E \to \infty$? $E \to 0$?

(b) Calculate the differential cross section $d\sigma/d\vartheta$ at the same energies as in (a). Plot and discuss them in relation to the cross sections in (a). How would the results change if the values of m_1 and m_2 were interchanged? If they do change, why do they change?

(EC) Calculate and plot $\Theta(s)$ and $d\sigma/d\Theta$ at E=0.1. Discuss its behavior in the context of your answers to (a).

(20 pts) 4. Since the Earth is not a perfect sphere, its gravitational potential is not perfectly $-GMm/r$. Given that there is no negative mass, the first multipole term that will enter into the description of the mass distribution is the quadrupole. So, let's explore the effect of a quadrupole addition to the potential, assuming it is still strictly central. That is, take the Earth's gravitational potential felt by a satellite of mass m to be

$$V(r) = -G\frac{Mm}{r} + h\left(\frac{R}{r}\right)^3, \qquad r > R.$$

(M is the Earth's mass; and R, its radius.) The potential energy contributed by the quadrupole term is very small compared to the monopole term.

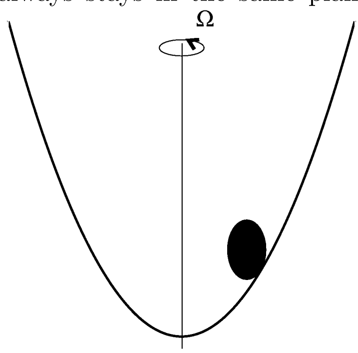
(a) Find the radius of a circular orbit to at least first order in h in terms of the basic parameters of the problem and any conserved quantities. Calculate its period.

(b) Calculate the period of radial oscillations for slight perturbations from this circular orbit.

(c) Plot the orbit. Can it be approximated by a precessing ellipse? If so, find the precession frequency and state whether it's in the same direction as the orbital motion or opposite to it.

(d) Discuss any consequences for your answers above of the quadrupole contribution being attractive or repulsive.

(20 pts) 5. As shown in the figure below, a vertical parabolic track is constructed so that an object can roll along its inside surface without slipping. With some external force, the track is rotated at a constant angular frequency Ω about the vertical axis as indicated. The rolling object always stays in the same plane as the track.



In the plane of the track, with the vertical direction labeled y and the horizontial x, the equation describing the shape of the track is $y = \frac{1}{2}x^2$. The system is subject to gravity.

(a) Assuming that the rolling object has mass m, radius r, and moment of inertia I, obtain the equations of motion using the Lagrangian formalism. Use any first integrals to simplify your equations of motion. Identify any such quantities physically.

(b) Identify an effective potential and sketch it. Discuss it physically. In your discussion, be sure to calculate the equilibrium position(s) and discuss whether they are stable or not. Also, be sure to indicate how they depend on the parameters of the problem (m, r, I, Ω) and if there are physically significant features associated with this dependence.